Part D

E-Recursion

E-recursion theory extends the notion of computation from hereditarily finite sets to sets of arbitrary rank. Selection theorems, forcing arguments and priority constructions are developed in the setting of *E*-recursive enumerability.

Chapter X
E-Closed Structures

The class of partial E-recursive functions is defined. A connection is made between E-closure and Gödel's L. Gandy selection is proved. Moschovachis witnesses are introduced and used to characterize divergence of computations in terms of reflection phenomena.

1. Partial E-Recursive Functions

The intent of E-recursion theory is to assign a meaning to $\{e\}(x)$ for every set x via an appropriate notion of computation. The Σ_1 admissibility approach of Part C is deemed inappropriate since it equates "computation" with "existential witness". Let $f: V \to V$ be a partial Σ_1 function; that is the graph of f is definable over V, the class of all sets, by some formula $(\mathrm{Ew}) D(x, y, w)$, where D is a Δ_0 formula of ZF. It makes sense to speak of computing $f(x)$ by searching for a pair $\langle y, w \rangle$ that satisfies $D(x, y, w)$ in V, but it does not seem sensible to construe a wide open search through V as a computation procedure. Even if V is replaced by L, thereby allowing the search to proceed methodically through the ordinals, it is still difficult to view the search as the execution of a rule of computation.

Computation rules permit a computer to move in a direct way from argument to value by following an initial instruction and then other instructions developed along the way. Following the rules generates a tree-like object called a computation. An element of self-reference is to be expected since rules may talk about rules.

Kleene [1959] was the first to give schemes that assign a meaning to $\{e\}(x)$ when x is an object of finite type. (Integers are of type 0, and an object of type n is an arbitrary collection of objects of type less than n.) E-recursion generalizes the so-called normal version of his theory; "normal" indicates that equality is a recursive predicate. The Normann schemes for E-recursion [1978] extend the normal Kleene schemes to objects of arbitrary type, namely sets. They were devised by Normann, and subsequently by Moschovakis.

1.1 The Normann Schemes. Assume some standard method of encoding finite sequences. Let $\langle a_1, \ldots, a_n \rangle$ be the code for $a_1, \ldots, a_n$. Each Normann scheme is a closure condition in the positive inductive definition of E, the class of all tuples $\langle e, \langle x_1, \ldots, x_n\rangle, y\rangle$ such that $\{e\}(x_1, \ldots, x_n)$ is defined and equal to y.

(1) $\{e\}(x_1, \ldots, x_n) = x_i$ if $e = \langle 1, n, i\rangle$ (projection).

(2) $\{e\}(x_1, \ldots, x_n) = x_i - x_j$ if $e = \langle 2, n, i, j\rangle$ (difference).

(3) $\{e\}(x_1, \ldots, x_n) = \{x_i, x_j\}$ if $e = \langle 3, n, i, j\rangle$ (pairing).

(4) $\{e\}(x_1, \ldots, x_n) \simeq \cup\{\{c\}(y, x_2, \ldots, x_n) \mid y \in x_1\}$ if $e = \langle 4, n, c\rangle$ (bounding with union).

The left side of (4) is not defined unless $\{c\}(y, x_2, \ldots, x_n)$ is defined for all $y \in x_1$.

(5) $\{e\}(x_1, \ldots, x_n) \simeq \{c\}(\{d_1\}(x_1, \ldots, x_n), \ldots, \{d_m\}(x_1, \ldots, x_n))$ if $e = \langle 5, n, m, c, d_1, \ldots, d_m\rangle$ (composition).

(6) $\{e\}(c, x_1, \ldots, x_n, y_1, \ldots, y_m) \simeq \{c\}(x_1, \ldots, x_n)$ if $e = \langle 6, n, m\rangle$ (enumeration).

$\simeq$ is Kleene's symbol for strong equality. If f and g are partial functions, then $f(x) \simeq g(x)$ iff neither $f(x)$ nor $g(x)$ is defined, or $f(x)$ and $g(x)$ are defined and equal.

Let A be a class of sets. A is closed with respect to scheme (1) if $\langle e, \langle x_1, \ldots, x_n\rangle, x_i\rangle \in A$ whenever $x_1, \ldots, x_n \in V$ and $e = \langle 1, n, i\rangle$. The closure of A with respect to schemes (2) and (3) is defined similarly. A is closed with respect to scheme (4) if $\langle e, \langle x_1, \ldots, x_n\rangle, z\rangle \in A$ whenever

$$e = \langle 4, n, c\rangle \mathbin{\&} x_1, \ldots, x_n \in V,$$
$$(y)_{y \in x_1} (\mathrm{E}w)\,[\langle c, \langle y, x_2, \ldots, x_n\rangle, w\rangle \in A], \text{ and}$$
$$z = \{w \mid (\mathrm{E}y)_{y \in x_1}\,[\langle c, \langle y, x_2, \ldots, x_n\rangle, w\rangle \in A]\}.$$

The closure of A with respect to schemes (5) and (6) is defined similarly.

E is defined to be the least class closed with respect to schemes (1)–(6). E is viewed most simply as the intersection of all classes closed under (1)–(6). This view makes good sense, since each scheme is equivalent to a positive closure condition. Something is put in E if something else was put in earlier. The "intersection" proof of the existence of E can be formalized in ZF by thinking of V as the union of many Σ_1 admissible sets and doing the "intersection" proof within each Σ_1 admissible set. Another approach to the existence of E is that of natural enumeration. First some definitions.

$$\{e\}(x_1, \ldots, x_n) \text{ is defined and equal to } y$$

if $\langle e, \langle x_1, \ldots, x_n\rangle, y\rangle \in E$. A partial function f from V into V is partial E-recursive if there exists an e such that $f(x) \simeq \{e\}(x)$ for all $x \in V$. $\{e\}(x)$ *converges* (in symbols $\{e\}(x)\downarrow$) if $\{e\}(x)$ is defined. Otherwise $\{e\}(x)$ diverges (in symbols $\{e\}(x)\uparrow$).

1.2 Natural Enumeration of E. σ is an arbitrary ordinal number. Proceed at stage σ as follows.

$\sigma = 0$: $\langle e, \langle x_1, \ldots, x_n \rangle, x_i \rangle$ is put in E if $e = \langle 1, n, i \rangle$. Schemes (2) and (3) are treated similarly.

$\sigma > 0$: $\langle e, \langle x_1, \ldots, x_n \rangle, z \rangle$ is put in E if

$e = \langle 4, n, c \rangle$,

$(y)_{y \in x_1}$ (Ew)$[\langle c, \langle y, x_2, \ldots, x_n \rangle\; w \rangle$ put in E prior to stage $\sigma]$,

and $z = \{w \mid (\mathrm{Ey})_{y \in x_1} [\langle c, \langle y, x_2, \ldots, x_n \rangle, w \rangle$ is already in $E]\}$.

Schemes (5) and (6) are treated similarly.

Note that scheme (4) is the only scheme that pushes the natural enumeration into infinite stages.

The natural enumeration makes it easy to check certain points about E. For example, if $\{e\}(x)$ is defined, then it has only one value (cf. Exercise 1.5). Thus for all e, $\lambda x \mid \{e\}(x)$ is a partial *function*. The natural enumeration is defined by a Σ_1 recursion, and so E is a Σ_1 class (cf. VII.1.6). To be precise there is a Σ_1 formula $\mathscr{F}(x)$ of ZF such that for all $a \in V$,

$$a \in E \leftrightarrow V \models \mathscr{F}(\underline{a}).$$

It follows that the graph of each partial E-recursive function is Σ_1 definable over V. The converse is false. Consider $\mathcal{O}(x)$, Gödel's order of constructibility function

$$\mathcal{O}(x) \simeq \mu\gamma[x \in L(\gamma + 1) - L(\gamma)].$$

$\mathcal{O}(x)$ is partial Σ_1 but not partial E-recursive. Roughly speaking, if $x \in L$, then $\mathcal{O}(x)$ can be found by an unbounded search but not by any computation procedure that moves in a direct manner from x to $\mathcal{O}(x)$. Some forcing arguments will be applied in Chapter XI to show the non-E-recursiveness of certain predicates. On the other hand many Σ_1 functions are E-recursive, and the reason for this is best expressed by van de Wiele's theorem in Chapter XIII.

Let f: $\omega \to \omega$ be a partial function. Then f is partial recursive in the sense of classical recursive theory iff f is partial E-recursive iff f is Σ_1 over HF, the hereditarily finite sets (cf. Exercise 1.8). Agreements between E-recursion and Σ_1 admissibility will figure prominently up ahead. They are often consequences of selection theorems that make it possible to E-recursively compute existential witnesses.

Let $P(x)$ be a predicate and $f(x)$ its representing function:

$$P(x) \to f(x) = 0; \ \sim P(x) \to f(x) = 1.$$

$P(x)$ is said to be E-recursive if f is E-recursive.

1.3 Proposition. *If $P(x)$ is Δ_0^{ZF}, then $P(x)$ is E-recursive.*

Proof (a sketch). By induction on the logical complexity of $P(x)$. Observe that

$$x \in y \leftrightarrow \{x\} - y = \varnothing.$$

Suppose $g(x, y)$ is the representing function of $Q(x, y)$. Then

$$\begin{aligned}(x)_{x \in z}\, Q(x, y) &\leftrightarrow (x)_{x \in z}\, [g(x, y) = 0] \\ &\leftrightarrow \cup\{g(x, y) \mid x \in z\} = \varnothing. \qquad \square\end{aligned}$$

A predicate $P(x)$ is said to be *E-recursively enumerable* if there exists a partial E-recursive function f such that for all x,

$$P(x) \leftrightarrow f(x)\downarrow.$$

A is E-recursively enumerable if $x \in A$ is E-recursively enumerable.

Warning: The graph of a function can be E-recursively enumerable, or even E-recursive, without the function being partial E-recursive. One example is Gödel's order of constructibility function $\mathcal{O}(x)$. The graph of $\mathcal{O}$ is E-recursive, but $\lambda x \mid \mathcal{O}(x)$ is not partial E-recursive. Another example is the predicate, "x is an ordinal but not a cardinal". It is Σ_1 but not E-recursively enumerable.

1.4 *E*-Closed Structures. Let A be a transitive set (that is, $x \in A \;\&\; y \in x \to y \in A$). A is said to be E-closed if

$$\vec{x} \in A \;\&\; f(\vec{x})\downarrow \to f(\vec{x}) \in A \quad (\vec{x} \text{ is } x_1, \ldots, x_n)$$

for every partial E-recursive f. It will be seen that every Σ_1 admissible set is E-closed, but not conversely. Part D is devoted largely to the study of forcing and priority arguments over E-closed structures. What does it mean to force a computation to converge?, to diverge? How is Post's problem formulated and solved in E-recursion? The answers bear little resemblance to their counterparts in α-recursion theory.

The E-closure of x, in symbols $E(x)$, is the least E-closed $y \supseteq TC(\{x\})$. Recall that $TC(z)$ is the least transitive $y \supseteq z$. $E(x)$ is transitive by an induction that parallels the natural enumeration of E. Suppose $z \in E(x)$ to see $z \subseteq E(x)$. Assume z is $\cup\{\{c\}(y) \mid y \in x\}$. Let e be $\langle 4, 1, c\rangle$. Suppose $\langle e, x, z\rangle$ is put in E at stage σ. But then

$$\langle c, \langle y\rangle, \{c\}(y)\rangle$$

is put in E prior to stage σ for all $y \in x$. By induction $\{c\}(y) \subseteq E(x)$ for all $y \in x$, and so $z \subseteq x$. Note that the natural enumeration of E induces a natural enumeration of $E(x)$.

$E(\varnothing)$ is HF. $E(\omega)$ is $L(\omega_1^{\text{CK}})$. Let $A_1(x)$ be the least Σ_1 admissible $y \supseteq TC(\{x\})$. Then $E(x) \subseteq A_1(x)$ (cf. Exercise 1.7). It will be shown that $E(2^\omega)$ is not Σ_1 admissible. Thus $E(2^\omega)$ is a proper subset of $A_1(2^\omega)$.

1.5–1.8 Exercises

1.5. Show: if $\{e\}(x)\downarrow$, then the value of $\{e\}(x)$ is unique.

1.6. Complete the proof of Proposition 1.3.

1.7. For any set x, let α^x be the least γ such that $L(\gamma, TC(\{x\}))$ is Σ_1 admissible. Show $L(\alpha^x, TC(\{x\}))$ is the least Σ_1 admissible $y \supseteq TC(\{x\})$, namely $A_1(x)$. Show $E(x) \subseteq A_1(x)$. ($L(\gamma, x)$ is defined just before 2.8.)

1.8. Let $f\colon \omega \to \omega$ be partial. Show f is partial recursive iff f is partial E-recursive.

2. Computations

A computation instruction is any $(n+1)$-tuple of the form $\langle e, x_1, \ldots, x_n\rangle$, or more simply $\langle e, x\rangle$. $\langle e, x\rangle$ is to be thought of as instruction to compute $\{e\}(x)$. Associated with computation instruction $\langle e, x\rangle$ is a tree $T_{\langle e,x\rangle}$. Every node of the tree is a computation instruction. The tree is to be visualized as starting at the top of the page and then branching downward. All the Normann schemes, save (4), give rise to finite branching. The process of forming $T_{\langle e,x\rangle}$ can be regarded as a reversal of the natural enumeration of E (cf. subsection 1.2). If $\{e\}(x)\downarrow$ and $\langle e, x, \{e\}(x)\rangle$ is put in E at stage σ, then $T_{\langle e,x\rangle}$ traces $\{e\}(x)$ back to its roots below stage σ. If $\{e\}(x)\uparrow$, then $T_{\langle e,x\rangle}$ witnesses the fact that $\langle e, x, w\rangle \notin E$ for any w.

2.1 The Universal Computation Tree. The nodes of U, the universal computation tree, are computation instructions. b is a subcomputation instruction of a, in symbols $a >_U b$, if there exists a finite sequence $b_0, \ldots, b_n$ such that $a = b_0$, $b_n = b$, and b_{i+1} is an *immediate* subcomputation instruction of b_i $(i < n)$. The definition of immediate subcomputation instruction (immed. subcomp. instruc.) has six clauses corresponding to the Normann schemes and a seventh clause technical in nature.

(i) If $e = \langle 1, n, i\rangle$ $(1 \le i \le n)$, then $\langle e, x_1, \ldots, x_n\rangle$ has no immed. subcomp. instrucs. In other words $\langle e, x_1, \ldots, x_n\rangle$ is a terminal, or minimal, node of U.

(ii) Similar to (i) with $e = \langle 2, n, i, j\rangle$.

(iii) Similar to (ii) with $e = \langle 3, n, i, j\rangle$.

(iv) If $e = \langle 4, n, c\rangle$, then $\langle c, y, x_2, \ldots, x_n\rangle$ is an immed. subcomp. instruc. of $\langle e, x_1, \ldots, x_n\rangle$ for every $y \in x_1$.

(v) (a) If $e = \langle 5, n, m, c, d_1, \ldots, d_m\rangle$, then $\langle d_i, x_1, \ldots, x_n\rangle$ is an immed. subcomp. instruc. of $\langle e, x_1, \ldots, x_n\rangle$ for $1 \le i \le m$.

(v) (b) If $e = \langle 5, n, m, c, d_1, \ldots, d_m\rangle$ and $\{d_i\}$ $(x_1, \ldots, x_n)$ converges and equals y_i $(1 \le i \le m)$, then $\langle c, y_1, \ldots, y_m\rangle$ is an immed. subcomp. instruc. of $\langle e, x_1, \ldots, x_n\rangle$.

(vi) If $e = \langle 6, n, m\rangle$, then $\langle c, x_1, \ldots, x_n\rangle$ is an immed. subcomp. instruc. of $\langle e, c, x_1, \ldots, x_n, y_1, \ldots, y_m\rangle$.

(vii) If e is not an index of a scheme, or n is not the correct number of arguments, then $\langle e, x_1, \ldots, x_n\rangle$ is an immed. subcomp. instruc. of $\langle e, x_1, \ldots, x_n\rangle$.

Clause (vii) is needed in the proof of Proposition 2.2.

Clause (v)(b) is worthy of a moment's attention. It differs from the other clauses by requiring convergence information for the establishment of subcomputation instructions. If (v)(b) is overlooked, then it is easy to think (mistakenly) that the immediate subcomputation relation is E-recursive. In fact it is not E-recursive, but is E-recursively enumerable thanks to scheme (6), the enumeration scheme. Another trap is to think $>_U$ is E-recursively enumerable. Not so, it is merely Σ_1.

The tree-like object consisting of $\langle e, x\rangle$ and that part of $>_U$ below $\langle e, x\rangle$ is called the *computation* of $\{e\}(x)$, and is denoted by $T_{\langle e, x\rangle}$.

2.2 Proposition. *$\{e\}(x)\downarrow \leftrightarrow T_{\langle e, x\rangle}$ is wellfounded.*

Proof. First suppose $T_{\langle e, x\rangle}$ is wellfounded. By clause (vii) of the definition of immediate subcomputation, e is the index of a Normann scheme. A straightforward transfinite induction on $>_U$ restricted to $T_{\langle e, x\rangle}$ shows if $\langle c, z\rangle$ is a node on $T_{\langle e, x\rangle}$, then $\{c\}(z)\downarrow$.

Now suppose $\{e\}(x)\downarrow$. Then $\langle e, x, \{e\}(x)\rangle$ was put in E at some stage σ, as described in subsection 1.2. If $\langle c, u\rangle$ is an immed. subcomp. instruc. of $\langle e, x\rangle$, then $\langle c, u, \{c\}(u)\rangle$ was put in E prior to stage σ. By induction on σ, $T_{\langle c, u\rangle}$ is well-founded. But then $T_{\langle e, x\rangle}$ is well-founded. $\square$

If $\{e\}(x)\downarrow$, then travelling down the tree $T_{\langle e, x\rangle}$ retraces the steps by which $\langle e, x, \{e\}(x)\rangle$ was put in E.

2.3 Effective Transfinite Recursion (ETR). ETR is an essential tool in the study of E-recursion. It originated in the work of Kleene and Church on notations for ordinals, and was clarified by Rogers. In this book it is presented as Theorem 3.2.I. It is a consequence of Theorem 3.1.I, Kleene's fixed point theorem. The proof of 3.1.I remains valid when $\{d\}$ is interpreted as the $\{d\}$-th partial E-recursive function. Consequently 3.2.1 is valid under a similar interpretation. The next proposition, 2.4, is a useful recursion principle, more comprehensible than ETR, but weaker.

2.4 Proposition. *Suppose H is a total E-recursive function. Then the unique f such that*

$$f(\gamma) = H(f \restriction \gamma) \text{ for every ordinal } \gamma$$

is E-recursive.

Proof. Let $I: \omega \to \omega$ be a recursive function such that for all $e < \omega$ and all γ,

$$\{I(e)\}(\gamma) \simeq H(\{e\} \restriction \gamma).$$

By the fixed point theorem (for partial E-recursive functions) there is a c such that $\{I(c)\} \simeq \{c\}$. Then f is $\{c\}$.

2.5 Lemma. *There exists a partial E-recursive function g such that for all $d < \omega$ and all x:*

(i) $\{d\}(x)\downarrow \leftrightarrow g(d,x)\downarrow$
(ii) $\{d\}(x)\downarrow \rightarrow g(d,x) = T_{\langle d,x\rangle}$.

Proof. By effective transfinite recursion. The definition of the recursive iterator I has several cases, one for each Normann scheme. From now on it will be convenient to restrict attention to an artificial Normannesque scheme that combines the important features of the Normann schemes. Regard scheme

$$\text{(T)} \qquad \{2^m \cdot 3^n\}(x) \simeq \{\{n\}(y) \mid y \in \{m\}(x)\}.$$

It is understood in (T) that $\{2^m \cdot 3^n\}(x)\downarrow$ iff $\{m\}(x)\downarrow$ and $\{n\}(y)\downarrow$ for all $y \in \{m\}(x)$.The immediate subcomputations of $\langle 2^m \cdot 3^n, x\rangle$ are:

$$\text{(Ta)} \qquad \langle m, x\rangle;$$

$$\text{(Tb)} \qquad \langle n, y\rangle \quad \text{if} \quad \{m\}(x)\downarrow \quad \text{and} \quad y \in \{m\}(x).$$

A recursion or induction based on the Normann schemes has six cases, one for each scheme. From now on only one case will be considered, that of scheme T. (T stands for typical.)

Thus the recursive iterator $I(e)$ is defined explicitly only for case T. If $\{2^m \cdot 3^n\}(x)\downarrow$, then

$$(1) \qquad T_{\langle 2^m \cdot 3^n, x\rangle} = T_{\langle m,x\rangle} \cup \bigcup\{T_{\langle n,y\rangle} \mid y \in \{m\}(x)\}.$$

(The $\bigcup$ operation of (1) is the union operation of set theory slightly modified so as to produce trees from subtrees in an appropriate fashion.) Define $I(e)$ so that

$$(2). \qquad \{I(e)\}(2^m \cdot 3^n, x) \simeq \{e\}(m,x) \cup \bigcup\{\{e\}(n,y) \mid y \in \{m\}(x)\}.$$

Observe that $I(e)$ is defined even when $\{2^m \cdot 3^n\}(x)\uparrow$, because $I(e)$ is merely an instruction.

By the fixed point theorem, there is a c such that $\{I(c)\} \simeq \{c\}$. Let $g(d,x)$ be $\{c\}(d,x)$. If $\{d\}(x)\downarrow$, then $T_{\langle d,x\rangle}$ is well-founded according to Proposition 2.2. In that event an induction on $>_U$ restricted to $T_{\langle d,x\rangle}$ shows $g(b,u) = T_{\langle b,u\rangle}$ for all $\langle b,u\rangle \leq_U \langle d,x\rangle$. The induction succeeds because $\{I(c)\} \simeq \{c\}$. □

Let $\mathscr{E}$ be an E-closed structure. Lemma 2.5 implies $\mathscr{E}$ is closed with respect to the formation of convergent computation trees: if $x \in \mathscr{E}$ and $\{e\}(x)\downarrow$, then $T_{\langle e,x\rangle} \in \mathscr{E}$. A major question to be considered shortly is: Suppose $x \in \mathscr{E}$ and $\{e\}(x)\uparrow$; does $T_{\langle \varepsilon,x\rangle}$, or some significant part of $T_{\langle e,x\rangle}$, belong to $\mathscr{E}$? The answer is yes in many cases. For example, if $L(\kappa)$ is E-closed but not Σ_1 admissible, $x \in L(\kappa)$ and $\{e\}(x)\uparrow$, then some infinite descending path through $T_{\langle e,x\rangle}$ is an element of $L(\kappa)$ (cf. Theorem 5.7.)

Every computation tree $T_{\langle e,x\rangle}$ has a height denoted by $|T_{\langle e,x\rangle}|$. If $T_{\langle e,x\rangle}$ is not wellfounded, then $|T_{\langle e,x\rangle}|$ is defined to be ∞ with the understanding that $\infty > \gamma$ for every ordinal γ. If $T_{\langle e,x\rangle}$ is wellfounded, then $|T_{\langle e,x\rangle}|$ is $\sup^+$ (strict least upper bound) of

$$\{|T_{\langle c,y\rangle}| \mid \langle c,y\rangle \text{ is an immed. subcomp. instruc. of } \langle e,x\rangle\}.$$

Also $|\{e\}(x)|$, by definition, is $|T_{\langle e,x\rangle}|$ and is called the length of the computation of $\langle e,x\rangle$.

2.6 Lemma. *There exists a partial E-recursive function h such that for all e and x:*

(i) $\{e\}(x)\downarrow \leftrightarrow h(e,x)\downarrow$

(ii) $\{e\}(x)\downarrow \rightarrow h(e,x) = |\{e\}(x)|$.

Proof. By effective transfinite recursion. Only scheme T is considered, as in the proof of Lemma 2.5. Thus $e = 2^m \cdot 3^n$, and $|\{e\}(x)|$ is the strict least upper bound of

$$\{|\{m\}(x)|\} \cup \{|\{n\}(y)| \mid y \in \{m\}(x)\}. \tag{1}$$

The recursive iterator $I\colon \omega \to \omega$ is such that for all $c < \omega$ and all x,

$$\{I(c)\}(e,x) \simeq \sup{}^+((1)).$$

The supremum operation is effective thanks to Normann scheme (4) of 1.1. h is $\{d\}$, where $\{I(d)\} \simeq \{d\}$. $\square$

2.7 Lemma. *The predicates,* $|\{e\}(x)| < \gamma$ *and* $|\{e\}(x)| = \gamma$, *are E-recursive.*

Proof. By effective transfinite recursion on γ. Similar to the proof of Lemma 2.6. $\square$

Let x be any set. Define:

$L(0,x) = x$;

$L(\delta+1,x) =$ set of all first order definable subsets of $L(\delta,x)$;

$L(\lambda,x) = \bigcup\{L(\delta,x) \mid \delta < \lambda\}$ (limit λ).

If y is transitive, then $L(\delta,y)$ is transitive for every ordinal δ. $TC(\{x\})$ is the least transitive set with x as a member.

The proof of Proposition 2.8 relies on a *flat*, ordered pairing function f, an idea of Quine.

$$f(x,y) = \begin{matrix} \langle x,y\rangle \quad \text{if} \quad x,y \in L(\omega), \\ \{f(a,b) \mid a \in x \quad \& \quad b \in y\} \quad \text{otherwise.} \end{matrix}$$

If x or y has infinite rank, then rank $(f(x, y)) = \max(\text{rank}(x), \text{rank}(y))$. This last says f is flat. f is relativized to x by replacing $L(\omega)$ by $L(\omega, TC(\{x\}))$.

2.8 Proposition. *If* $\{e\}(x)\downarrow$ *and* $|\{e\}(x)| \geq \omega$, *then* $\{e\}(x)$ *and* $T_{\langle e, x\rangle}$ *are first order definable over* $L(|\{e\}(x)|, TC(\{x\}))$.

Proof. Let e be $2^m \cdot 3^n$ (scheme (T) of Lemma 2.5). By induction on $|\{e\}(x)|$, $\{m\}(x)$ and $\{n\}(y)$ $(y \in \{m\}(x))$ belong to $L(|\{e\}(x)|, TC(\{x\}))$.

$$\{e\}(x) \quad \text{is} \quad \{\{n\}(y) \mid y \in \{m\}(x)\}.$$

View $T_{\langle e, x\rangle}$ as a set of ordered pairs defined by the flat function f immediately above. $\square$

2.9 Recursive Ordinals. $y, x_1, \ldots, x_n$ are sets. y is said to be E-recursive in $x_1, \ldots, x_n$ (in symbols, $y \leq_E x_1, \ldots, x_n$) if $y = \{e\}(x_1, \ldots, x_n)$ for some e. If f is an E-recursive function, then $f(x) \leq_E x$ for all x. More precisely, $f(x) \leq_E x$ uniformly in x. Note that $TC(x) \leq_E x$ uniformly. $\leq_E$ is not a generalization of Turing reducibility. That will come later. $\leq_E$ can be viewed as a generalization of hyperarithmetic reducibility. If $x, y \subseteq \omega$, then

$$y \leq_h x \leftrightarrow y \leq_E x, \omega.$$

An ordinal γ is said to be E-recursive in x if $\gamma \leq_E x$. Thus the recursive ordinals of Part A are simply the ordinals E-recursive in ω. Define

$$\kappa_0^x = \sup\{\gamma \mid \gamma \leq_E x\}.$$

κ_0^x is always a limit ordinal of countable cofinality. In general the ordinals recursive in x do not constitute an initial segment. If $x \subseteq \omega$, then Church–Kleene ω_1^x is $\kappa_0^{x,\omega}$.

κ_0^x need not be a Σ_1 admissible ordinal. Consider $x = \omega_1$. $\omega_1 < \kappa_0^{\omega_1}$ since $\omega_1 \leq_E \omega_1$. But $\kappa_0^{\omega_1}$ is less than the first Σ_1 admissible greater than ω_1, as will be seen below (cf. Exercise 5.15.)

κ_0^x has a natural development akin to the enumeration of a complete Postian set. A is said to be *E-recursively enumerable in* x if

$$A = \{y \mid \{e\}(y, x)\downarrow\}.$$

The complete E-recursively enumerable-in-x subset of ω is

$$K^x = \{n \mid n \in \omega \quad \& \quad \{(n)_0\}\ ((n)_1, x)\downarrow\}.$$

It follows from Lemma 2.6 that

$$\kappa_0^x = \sup\{\,|\{(n)_0\}((n)_1, x)|\ \mid n \in \kappa^x\}. \tag{1}$$

The ordinals occurring in the set on the right side of (1) are, by definition, the ordinals constructive in x. For each γ constructive in x, define

$$K_\gamma^x = \{n \mid n \in \omega \quad \& \quad |\{(n)_0\}\,((n)_1)| < \gamma\}.$$

Then $K^x = \bigcup\{K_\gamma^x \mid \gamma$ constructive in $x\}$. Let λ_0^x be the ordertype, necessarily countable, of the set of ordinals constructive in x. Then K^x can be enumerated in λ_0^x steps. Step γ produces K_γ^x, a set E-recursive in x. The natural development of K^x is sometimes described as an "enumeration with gaps", since consecutive, E-constructive-in-x ordinals can be quite far apart.

Warning: an ordinal can be E-recursive in x without being E-constructive in x (cf. Exercise 2.13).

Recall that the E-closure of x (in symbols, $E(x)$) is the least E-closed $y \supseteq TC(\{x\})$. Thus $x \in E(x)$ and $E(x)$ is transitive. In fact $E(x)$ is the least transitive, E-closed set with x as an element. Each member of $E(x)$ is of the form $\{e\}\,(x, a_0, \ldots, a_n)$, where $a_i \in TC(x)$. It follows that κ^x, the supremum of all ordinals in $E(x)$, is

$$\sup\{\gamma \leq_E x, a_0, \ldots, a_n \text{ for some } a_0, \ldots, a_n \in TC(x)\}.$$

With the aid of Proposition 2.8 a precise connection can be made between $E(x)$ and $L(x)$. First observe that $\kappa^\omega = \omega_1^{CK}$ and $E(\omega) = L(\omega_1^{CK})$.

2.10 Proposition. $E(x) = L(\kappa^x, TC(\{x\}))$.

Proof. By Proposition 2.8, $E(x) \subseteq L(\kappa^x, TC(\{x\}))$. Suppose $\gamma < \kappa^x$ and y is first order definable over $L(\gamma, TC(\{x\}))$ via parameter $p \in L(\gamma, TC(\{x\}))$. Then $y \leq_E L(\gamma, TC(\{x\})), p$. By induction on γ, $p \in E(x)$, so it suffices to show $L(\gamma, TC(\{x\})) \leq_E \gamma, x$. The usual set theoretic definition of $L(\gamma, TC(\{x\}))$ by transfinite recursion is in fact effective, hence $\lambda\gamma \mid L(\gamma, TC(\{x\}))$ is E-recursive in x. $\square$

2.11–2.12 Exercises

2.11. What is the role of flat pairing in Proposition 2.8?

2.12. Verify that $\kappa^\omega = \omega_1^{CK}$.

2.13. Let λ be the limit of all countable ordinals E-constructive in ω_1. Show λ is E-recursive in ω_1.

3. *Reflection*

The notion of reflection is needed in the study of divergence in Section 4. If $\{e\}\,(x)\uparrow$, then any infinite, descending path through $T_{\langle e, x\rangle}$ is termed a Moschovakis witness to the divergence of $\{e\}(x)$. The least ordinal that suffices to construct such a witness is, under favorable circumstances, the greatest x-reflecting ordinal. A precise connection between divergence and reflection was first made by Harrington [1973] in the setting of Kleene's theory of recursion in normal objects of finite type.

An ordinal δ is said to be x-reflecting if

$$L(\delta, TC(\{x\}) \models \mathscr{F} \rightarrow L(\kappa_0^x, TC(\{x\})) \models \mathscr{F}$$

for every Σ_1^{ZF} sentence $\mathscr{F}$ whose only parameter is x. Clearly the union of x-reflecting ordinals is x-reflecting. Define

$$\kappa_r^x = \text{the greatest } x\text{-reflecting ordinal.}$$

$\kappa_0^x \leq \kappa_r^x$, and in a moment it will be seen that $\kappa_r^x \leq \kappa^x$. The situation of greatest interest occurs when $\kappa_0^x < \kappa_r^x < \kappa^x$, because it is then that non-trivial reflection phenomena occur within $E(x)$. An important early result of Harrington [1973],

$$\kappa_r^{2^\omega} > \kappa_0^{2^\omega} \ ,$$

had key applications in forcing and degree-theoretic arguments. It was preceded by a result of Sacks [1974] needed for a forcing proof,

$$\kappa_r^{2^{2^\omega}} > \kappa_0^{2^{2^\omega}} \ .$$

For a dynamic view of κ_r let $A \subseteq TC(\{x\})$ be E-recursively enumerable in x. Suppose $b \in A$ and the height of a computation c that puts b in A is γ for some $\gamma < \kappa_r^x$. Thus

$$L(\gamma + 1, TC(\{x\})) \models (\mathrm{Ec})(\mathrm{Eb})\,[c \text{ puts } b \text{ in } A].$$

By reflection there is a c_0 and b_0 such that $|c_0| < \kappa_0^x$ and c_0 puts b_0 in A. Since κ_0^x is the limit of ordinals recursive in x, there is an infinite $\beta \leq_E x$ such that $|c_0| < \beta$. By Proposition 2.8, K_β, the set of all computations from x of height less than β, is first order definable over $L(\beta, TC\{x\}))$. Let γ_0 be the length of the shortest computation that puts some γ into A. $\gamma_0 < \beta$, and γ_0 is first order definable over $L(\beta, TC(\{x\}))$. Hence $\gamma_0 \leq_E x$. Let A_0 be the set of all y put in A via computations of height at most γ_0. Then A_0 is a non-empty subset of A E-recursive in x.

To sum up, if a Σ_1 fact about x is true in $L(\gamma, TC(\{x\}))$ for some $\gamma < \kappa_r^x$, then it is true in $L(\gamma_0, TC(\{x\}))$ for some $\gamma_0 \leq_E x$.

3.1 Proposition. *There exists a* Π_3^{ZF} sentence $\mathscr{F}$ such that for all transitive A,

$$A \text{ is } E\text{-closed} \leftrightarrow A \models \mathscr{F}.$$

Proof. Let $\mathscr{F}$ have Π_2^{ZF} clauses that require closure under pairing, difference and union operations. The remainder of $\mathscr{F}$ insures:

$$(x)\,[(y)_{y\in x}(\mathrm{E}\gamma)P(c, y, \gamma) \to (\mathrm{E}\gamma)(y)_{y\in x}P(c, y, \gamma)], \tag{1}$$

where $P(c, y, \gamma)$ is

$$T_{\langle c,\, y\rangle} \text{ is first ord. def. via } M \text{ over } L(\gamma, TC(\{x\})). \tag{2}$$

M is a recursion, hidden in the proof of Proposition 2.8, that tries to define $T_{\langle c,\, y\rangle}$ over $L(\gamma, TC(\{x\}))$. M succeeds iff $y\in x$ and $\gamma \geq \max(\omega, |\{c\}(y)|)$. It follows that (1) is Π_3^{ZF}. $\square$

3.2 Proposition. *$\kappa_r^{x,a} \leq \kappa^x$ for all $a\in TC(x)$.*

Proof. Suppose $a\in TC(x)$ and $\kappa_r^{x,a} > \kappa^x$. Since $E(\langle a,x\rangle) = E(x)$, it follows from Proposition 2.10 that

$$E(\langle a,x\rangle)\in L(\kappa_r^{x,a},\ TC(\{x\})). \tag{1}$$

With the aid of Proposition 3.1, (1) can be "reflected down" to $\kappa_0^{x,a}$. Thus

$$E(\langle a,x\rangle)\in L(\kappa_0^{x,a},\ TC(\{x\})).$$

But then $\kappa^{x,a} < \kappa_0^{x,a}$, an absurdity. $\square$

In Section 4 it will be shown that $\kappa_r^{a,x} = \kappa^x$ for all $a\in TC(x)$ only if $E(x)$ is Σ_1 admissible.

4. Gandy Selection

Selection principles in E-recursion theory have the following form. If A is E-recursively enumerable in x and nonempty, then there exists a nonempty $b \leq_E x$ such that $b \subseteq A$. In addition there is a uniform method for computing b from x and an index for A. If $A \subseteq \omega$ and $x\in\omega$, then A is a classical recursively enumerable set and selection is proved by enumerating A until some element appears. This trivial proof is worth considering more closely. Let K_σ^x be the set of all computations from x of length less than σ for σ a limit ordinal. By Proposition 2.8, $\mathrm{K}_\sigma^x \leq_E \sigma, x$ uniformly in σ and x. Let σ_A be the least σ such that some computation in K_σ^x puts an element in A. Let b be the set of all such elements of A.

Then $b \leq_E \sigma_A, x$. So to complete the proof of selection it need only be shown that $\sigma_A \leq_E x$. In general, this last is false. It is true if $A \subseteq \omega$ and $x \in \omega$ by arguments of classical recursion theory. If $A \subseteq \omega$ and x is an arbitrary set, then the arguments of this section show $\sigma_A \leq_E x$. Later it will be seen that there exists a nonempty $A \subseteq 2^\omega$, E-recursively enumerable in 2^ω, such that $\sigma_A \not\leq_E 2^\omega$. The study of forcing over E-closed structures in the next chapter owes some of its technical interest to the lack of general selection principles.

Gandy [1967] originally proved selection for $A \subseteq \omega$ and $x = \omega$. Moschovakis extended the result to x's of higher type. The proof given here is derived largely from Moldstad [1977].

4.1 Theorem (Gandy Selection). *There exists a partial E-recursive function $\phi(e, x)$ such that for all $e < \omega$ and all x:*

(*i*) $(\mathrm{E}n)_{n<\omega}\,[\{e\}(n,x)\downarrow] \leftrightarrow \phi(e,x)\downarrow$;

(*ii*) $\phi(e,x)\downarrow \rightarrow \{e\}(\phi(e,x),x)\downarrow$.

The proof of Theorem 4.1 requires a preparatory lemma originated, and dubbed "stage comparison", by Moschovakis.

4.2 Lemma. *Suppose $\{d\}(x)\downarrow$ or $\{e\}(y)\downarrow$. Then*

$$\min(|\{d\}(x)|, |\{e\}(y)|) \leq_E x, y \textit{ (uniformly)}.$$

Proof. By effective transfinite recursion on min. The rough idea behind the recursion step is:

$$\min(|u|, |v|) = \max\{\min(|a|, |b|) \mid a <_U u \;\&\; b <_U v\},$$

where $<_U$ is the universal computation tree of Section 2. Some smoothing is needed to make this idea work, because if $u\uparrow$, then $\{a \mid a <_U u\rangle$ may not be E-recursive in u. Assume $d = 2^m \cdot 3^n$ and $e = 2^p \cdot 3^q$. (As in the proof of 2.5, only scheme T is considered.) Define:

$$\alpha = \min(|\{2^m \cdot 3^n\}(x)|, |\{2^p \cdot 3^q\}(y)|);$$
$$\alpha_1 = \min(|\{m\}(x)|, |\{p\}(y)|).$$

$\alpha_1 < \alpha$, so by recursion $\alpha_1 \leq_E x, y$. With the aid of Lemma 2.7, an $s \in \{\{m\}(x), \{p\}(y)\}$ can be selected so that $|s| = \alpha_1$.

Suppose s is $\{m\}(x)$. Define:

$$\beta_b = \min(|\{n\}(b)|, |\{p\}(y)|);$$
$$\beta = \sup{}^+\{\beta_b \mid b \in \{m\}(x)\}.$$

By recursion $\beta_b \leq_E x, y, b$ (uniformly in b), hence $\beta \leq_E x, y$. Now use 2.7 to see if

$$\alpha \leq \max(|\{m\}(x)|, \beta) + 1. \tag{1}$$

If (1) is false, then $\{n\}(b)\uparrow$ for some $b \in \{m\}(x)$, and so $\alpha = |\{2^p \cdot 3^q\}(y)|$.
If s is $\{p\}(y)$, then proceed as in the above paragraph with q, m in place of n, p. $\square$

4.3 Proof of Gandy Selection. For simplicity drop x. By Lemmas 2.7 and 4.2 there exists a partial E-recursive ψ such that

$$\psi(r,e,k) \simeq \begin{array}{ll} \{r\}(e,k+1)+1 & \text{if } \{r\}(e,k+1)\downarrow \;\&\; |\{r\}(e,k+1)| \leq |\{e\}(k)|, \\ 0 & \text{if } |\{e\}(k)| < |\{r\}(e,k+1)|. \end{array}$$

The instructions for computing ψ are as follows. Use Lemma 4.2 to compute the min of $\{r\}(e,k+1)$ and $\{e\}(k)$. If the min is undefined, then $\psi(r,e,k)$ is undefined. If the min is defined, then use Lemma 2.7 to decide which case of the definition of ψ applies.

Kleene's fixed point theorem provides a c such that

$$\psi(c,e,k) \simeq \{c\}(e,k)$$

for all e and k. Observe that the computation of $\{c\}(e,0)$ compares $|\{c\}(e,1)|$ with $|\{e\}(0)|$; the computation of $\{c\}(e,1)$ compares $|\{c\}(e,2)|$ with $|\{e\}(1)|$; and so on. $\{c\}$ is designed to seek the least k such that $|\{e\}(k)|$ is the minimum of $\{|\{e\}(m)| \mid m < \infty\}$.

First note that

$$(k)[\{c\}(e,k+1)\downarrow \to \{c\}(e,k)\downarrow], \tag{1}$$

$$(k)[\{e\}(k)\downarrow \to \{c\}(e,k)\downarrow]. \tag{2}$$

Hence

$$(k)[\{e\}(k)\downarrow \to \{c\}(e,0)\downarrow]. \tag{3}$$

Now observe that

$$\{c\}(e,0)\downarrow \to (\mathrm{E}k)[|\{e\}(k)| < |\{c\}(e,k+1)|]. \tag{4}$$

Otherwise $\{c\}(e,k) > \{c\}(e,k+1)$ for all k, an absurdity.

It remains only to show

$$(\mathrm{E}k)[\{e\}(k)\downarrow] \to \{e\}(\{c\}(e,0))\downarrow. \tag{5}$$

Suppose the left side of (5) holds. By (3) and (4),

$$\{c\}(e,0)\downarrow, \quad \text{and} \quad |\{e\}(k)| < |\{c\}(e,k+1)|$$

for some k; let k_0 be the least such k. Then

$$\{c\}(e,k_0)=0, \quad \text{and} \quad \{c\}(e,j)=\{c\}(e,j+1)+1$$

for all $j<k_0$. Hence $\{c\}(e,0)=k_0$. $\square$

4.4 Corollary. *The class of E-recursively enumerable predicates is closed under finite disjunction and existential number quantification.*

Proof. Suppose n is a number variable and $P(n,x)$ is E-recursively enumerable. By Gandy selection there is a partial E-recursive $\phi(x)$ such that

$$(\mathrm{E}n)\ P(n,x) \leftrightarrow \phi(x)\downarrow. \quad \square$$

4.5 Corollary. *Suppose $P(x,y)$ is E-recursively enumerable and*

$$(x)_{x\in z}(\mathrm{E}y)\,[y \leq_E x \quad \& \quad P(x,y)].$$

Then there exists a partial E-recursive function f such that

$$(x)_{x\in z}\,[f(x)\downarrow \quad \& \quad P(x,f(x))].$$

Proof. The predicate, $\{e\}(x)\downarrow\ \&\ P(x,\{e\}(x))$, is E-recursively enumerable. By Gandy selection there is a partial E-recursive ϕ such that

$$(\mathrm{E}e)\,[\{e\}(x)\downarrow \quad \& \quad P(x,\{e\}(x))] \leftrightarrow \phi(x)\downarrow, \text{ and}$$
$$\phi(x)\downarrow \rightarrow \{\phi(x)\}(x)\downarrow \quad \& \quad P(x,\{\phi(x)\}(x)).$$

Let $f(x) \simeq \{\phi(x)\}(x)$. $\square$

The proof of Corollary 4.5 deserves a gloss. Suppose $R(x,y)$ is an E-recursively enumerable predicate and there is a y such that

$$(1) \qquad y \leq_E x \quad \text{and} \quad R(x,y).$$

Then Gandy selection legitimizes a search through the y's E-recursive in x until one is found that satisfies (1). In addition the search procedure is uniform in x and R. Other selection principles will be proved in later sections. They too can be viewed as legitimizing certain searches.

4.6 Corollary. *If $a \subseteq \omega$, then $E(a)$ is Σ_1 admissible.*

Proof. Let $D(x,y,z)$ be a Δ_0^{ZF} formula. Suppose

$$E(a)\ (x)_{x\in p}\,(\mathrm{E}y)\,D(x,y,q)$$

for some $p, q \in E(a)$. Since $a \subseteq \omega$, every $y \in E(a)$ is E-recursive in a. It follows from Corollary 4.5 that there exists a partial E-recursive function f such that

$$(x)_{x\in p}\,[f(x,q)\downarrow \quad \& \quad D(x,f(x,q))].$$

Let $r = \{f(x,q) \mid x \in p\}$. Then $r \leq_E p, q$ and

$$E(a) \models (x)_{x\in p}(\mathrm{E}y)_{y\in r} D(x,y,q). \qquad \square$$

Let $\mathrm{Ad}_1(x)$ be the least Σ_1 admissible set with x as a member. $\mathrm{Ad}_1(x)$ is $L(\alpha, TC(\{x\}))$ for some Σ_1 admissible α determined by x. $\mathrm{Ad}_1(x)$ is E-closed, hence $E(x) \subseteq \mathrm{Ad}_1(x)$. It follows from Corollary 4.6 that $E(a)$, for $a \subseteq \omega$, equals $\mathrm{Ad}_1(a)$. According to Exercise 4.9, subsets of $E(a)$ are Σ_1 definable over $E(a)$ iff they are E-recursively enumerable in a.

The situation with respect to $E(2^\omega)$ is more complex. It will be seen that $E(2^\omega)$ is not Σ_1 admissible because it admits Moschovakis witnesses.

4.7 Corollary. *Suppose $\kappa_r^{x,a} = \kappa^x$ for all $a \in TC(x)$. Then $E(x)$ is Σ_1 admissible.*

Proof. Suppose $D(u,v,w)$ is a Δ_0^{ZF} formula, and

$$E(x) \models (u)_{u\in c}(\mathrm{E}v)\,D(u,v,p)$$

for some $c, p \in E(x)$. For the moment assume $c \subseteq TC(x)$ and $p = x$. By Proposition 2.10

$$E(x) = L(\kappa_r^{x,a},\ TC(\{x\}))$$

for all $a \in TC(x)$. By reflection, for each $a \in c$, there is a $\delta \leq_E x, a$ such that

$$L(\delta, TC(\{x\})) \models (\mathrm{E}v)\,D(a,v,p).$$

It follows from Corollary 4.5 that δ can be construed as a partial E-recursive function f of x and a, defined for all $a \in c$. Normann's bounding scheme, (4) of subsection 1.1, implies that $\{f(x,a) \mid a \in c\}$ is bounded above by some $\gamma \leq_E x, c$. Hence there is a $d \in E(x)$ such that

$$E(x)\ (u)_{u\in c}(\mathrm{E}v)_{v\in d}\,D(u,v,p);$$

d is $L(\gamma, TC(\{x\}))$. Thus $E(x)$ is Σ_1 admissible.

The assumptions, $c \subseteq TC(x)$ and $p = x$, have to be discharged. $TC(x)$ is infinite; otherwise $E(x)$ is HF, hence Σ_1 admissible. According to Exercise 4.11 there exists a $g \in E(x)$ such that g is a one-one map of c into $TC(x)$. Then

$$(u)_{u\in c}(\mathrm{E}v)\ D(u,v,p)$$

is equivalent to

$$(u)_{u\in g[c]}\,(\mathrm{E}v)\ D(g^{-1}(u),v,p).$$

Thus it is safe to assume $c \subseteq TC(x)$.

Recall that $p = \{e\}(x, b)$ for some $b \in TC(x)$. Then $D(u, v, p)$ is equivalent to $(E\beta)(\mathrm{Eb})_{b \in TC(x)}[\,|\{e\}(x, b)| = \beta \;\&\; D(u, v, \{e\}(x, b))]$. $\square$

4.8–4.12 Exercises.

4.8. Prove the class of E-recursively enumerable predicates is closed under finite disjunction.

4.9. Assume $a \subseteq \omega$ and $x \subseteq E(a)$. Show x is E-recursively enumerable in a iff x is Σ_1 definable over $E(a)$.

4.10. Assume $a \subseteq \omega$. Show a is E-recursively enumerable in ω iff a is Π_1^1.

4.11. Assume x is infinite and $c \in E(x)$. Find a $g \in E(x)$ such that g is a one-one map of c into $TC(x)$.

4.12. Fix x. Show the predicate, γ is not x-reflecting, is E-recursive in x. Hint: make use of a Σ_1^{ZF} predicate $\mathscr{F}$, whose only free variable is x, such that

$$L(\kappa_r^x + 1,\ TC(\{x\})) \models \mathscr{F},$$

$$L(\kappa_r^x,\ TC(\{x\})) \models \sim\mathscr{F}.$$

Slaman has shown that $\mathscr{F}$ cannot be chosen effectively from x.

5. Moschovakis Witnesses

Suppose $\{e\}(x)$ diverges. A *Moschovakis witness* to the divergence of $\{e\}(x)$ is any infinite path in $>_U$, the universal computation tree, below $\langle e, x\rangle$. More precisely, a witness is a function $\lambda n\,|\,t(n)$ such that: $t(0) = \langle e, x\rangle$; and for each n, $t(n+1)$ is an immediate subcomputation instruction of $t(n)$. Note that "t is a witness to $\{e\}(x)\uparrow$" is E-recursively enumerable. This section is devoted to locating the least ordinal that suffices to enumerate a witness to the divergence of $\{e\}(x)$. In many cases that ordinal will belong to $E(x)$. Thus the process of forming an E-closure by closing with respect to convergent computations will often close with respect to divergence witnesses as well.

A structure $\mathscr{E}$ is said to admit divergence witnesses if for all $e < \omega$ and $x \in \mathscr{E}$: if $\{e\}(x)\uparrow$ then some witness to $\{e\}(x)\uparrow$ belongs to $\mathscr{E}$. It will be seen shortly that $L(\kappa)$ admits divergence witnesses if $L(\kappa)$ is E-closed but not Σ_1 admissible. The study of forcing and priority over E-closed structures in future chapters will depend heavily on the presence of divergence witnesses.

There is a connection between the current section and the Gandy basis theorem for Σ_1^1 predicates (cf. Part A). Suppose $x \in E(\omega)$ and $\{e\}(x)\uparrow$. Then there exists a witness t to $\{e\}(x)\uparrow$ first order definable over $L(\omega_1^{\mathrm{CK}})$. In addition t can be chosen so that $E(t) = L(\omega_1^{\mathrm{CK}}, t)$. Also $E(\omega)$ does not admit Moschovakis witnesses.

5.1 Theorem (Kechris). *Suppose $y \leq_E x$ and A is E-recursively enumerable in x. If $y - A$ is nonempty, then*

$$(\mathrm{E}b)[b \in y - A \quad \& \quad \kappa_r^{x,b} \leq \kappa_r^x].$$

Proof. First observe that

(1) $$\kappa_0^{x,b} \leq \kappa_r^x \rightarrow \kappa_r^{x,b} \leq \kappa_r^x.$$

Let $\mathscr{F}$ be a Σ_1 sentence of ZF with x as its only parameter. $\mathscr{F}$ reflects from $\kappa_r^{x,b}$ down to $\kappa_0^{x,b}$. Suppose $\kappa_0^{x,b} \leq \kappa_r^x$. Then $\mathscr{F}$ reflects from $\kappa_r^{x,b}$ down to κ_0^x. Hence $\kappa_r^{x,b} \leq \kappa_r^x$, since κ_r^x is the greatest x-reflecting ordinal.

By (1) it suffices to find a $b \in y - A$ such that $\kappa_0^{x,b} \leq \kappa_r^x$. Suppose there is no such b with the intent of showing $y \subseteq A$. Then

(2) $$y \subseteq A \cup \{b \mid \kappa_r^x < \kappa_0^{x,b}\}.$$

Note that the predicate, $\kappa_r^x < \kappa_0^{x,b}$, is E-recursively enumerable in x by Corollary 4.4 and Exercise 4.12 since it is equivalent to

$$(\mathrm{E}\delta)[\delta \leq_E x, b \quad \& \quad \delta \text{ is not } x\text{-reflecting}].$$

Thus y is contained in the union of two sets, as indicated in (2), each of which is E-recursively enumerable in x. For each $b \in y$, there is a $\delta_b \leq_E x, b$ such that either (i) or (ii) holds:

(i) δ_b is the length of a computation that puts y in A;
(ii) δ_b is not x-reflecting.

By Corollary 4.4 δ_b can be construed as a partial E-recursive function of x, b defined for all $b \in y$. Let

$$\delta^s = \sup\{\delta_b \mid b \in y\}.$$

Then the bounding scheme yields $\delta^s \leq_E x$ and so $\delta^s < \kappa_r^x$. If (ii) holds for some b, then $\delta_b > \kappa_r^x > \delta^s$. Hence (ii) never holds, and $y \subseteq A$. $\square$

The Kechris basis theorem is analogous to the Gandy basis theorem for Σ_1^1 sets of reals (Part A). In both an element b of a "co-recursively enumerable" set is found so that some ordinal generated by b is minimized. This minimization is precisely what is needed to prove Lemma 5.3. The next result is a technicality required for the proof of 5.3.

5.2 Proposition. *Assume some wellordering of $TC(x)$ is E-recursive in x. Suppose $\langle c, y\rangle$ is an immediate subcomputation instruction of $\langle e, x\rangle$. Then some wellordering of $TC(y)$ is E-recursive in x, y. In addition if $\{c\}(y)\downarrow$, then some wellordering of $\{c\}(y)$ is E-recursive in x, y.*

Proof. Let e be $2^m \cdot 3^n$. Suppose $\{m\}(x)\downarrow$ and $z \in \{m\}(x)$. Let $v \leq_E x$ be a wellordering of $TC(x)$. By Proposition 2.8,

$$\{m\}(x),\ v \in L(\gamma, TC(\{x\}))$$

for some $\gamma \leq_E x$. A recursion on the ordinals less than γ extends v to a wellordering w of $L(\gamma, TC(\{x\}))$. $w \leq_E v, \gamma$, hence $w \leq_E x$. Since $L(\gamma, TC(\{x\}))$ is transitive, w induces a wellordering of $\{m\}(x)$ E-recursive in x, and a wellordering of $TC(z)$ E-recursive in x, z. If $\{n\}(z)\downarrow$, then another application of 2.8 yields a wellordering of $\{n\}(z)$ E-recursive in x, z. $\square$

Lemma 5.3 occupies a central place in E-recursion theory. It was inspired by an early theorem of Harrington [1973] in the setting of Kleene recursion in objects of finite type. His result, in the language of E-recursion, is: if $a \in 2^\omega$ and $\{e\}(a, 2^\omega)\uparrow$, then some Moschovakis witness to $\{e\}(a, 2^\omega)\uparrow$ is seen to be such via a computation of height $\kappa_r^{2^\omega, a}$.

5.3 Lemma. *Assume some wellordering of $TC(x)$ is E-recursive in x. If $\{e\}(x)\uparrow$, then some Moschovakis witness to $\{e\}(x)\uparrow$ is first order definable over $L(\kappa_r^x, TC(\{x\}))$.*

Proof. The witness, $\lambda t \mid \langle e_t, x_t \rangle$, is defined by recursion on $t < \omega$. $\langle e_0, x_0 \rangle$ is $\langle e, x \rangle$. Fix $t \geq 0$ and assume $\langle e_t, x_t \rangle$ has already been defined so that

$$\{e_t\}(x_t)\uparrow, \tag{1}$$

$$x_t \in L(\kappa_r^x, TC(\{x\})), \text{ and} \tag{2}$$

$$\kappa_r^x \geq \kappa_r^{x_0, \ldots, x_t}. \tag{3}$$

Let $e_t = 2^m \cdot 3^n$. (As in the proof of 2.5, only scheme (T) is considered.) (3) implies

$$\kappa_r^x \geq \kappa_0^{x_0, \ldots, x_t} \geq \kappa_0^{x_t}. \tag{4}$$

Hence anything E-recursive in x_t belongs to $L(\kappa_r^x, TC(\{x\}))$. In particular, if $\{m\}(x_t)\downarrow$, then its computation tree belongs to $L(\kappa_r^x, TC(\{x\}))$. Hence examination of $L(\kappa_r^x, TC(\{x\}))$ reveals whether or not $\{m\}(x_t)\downarrow$.

Case A: $\{m\}(x_t)\uparrow$.

Define $\langle e_{t+1}, x_{t+1} \rangle = \langle m, x_t \rangle$.

Case B: $\{m\}(x_t)\downarrow$.

By Proposition 5.2 there is a wellordering $\leq_w$ of $\{m\}(x_t)$ E-recursive in x_t. The choice of w can be made in a uniform way with the aid of Gandy selection (Theorem 4.1), as in the remarks following the proof of Corollary 4.5, because the predicate, $\{c\}(x_t)\downarrow$ & $\{c\}(x_t)$ is a wellordering of $\{m\}(x_t)$, with c as a free variable, is E-recursively enumerable. Define

$$e_{t+1} = n, \text{ and}$$

$$x_{t+1} = w\text{-least } u\,[u \in \{m\}(x_t)\ \&\ |\{n\}(u)| \geq \kappa_r^x].$$

Observe that $\{x_{t+1}\}$ is first order definable over $L(\kappa_r^x, TC(\{x\}))$. It remains to be seen that

(5) $$\{n\}(x_{t+1})\uparrow \;\&\; \kappa_r^{x_0,\ldots,x_t} \geq \kappa_r^{x_0,\ldots,x_{t+1}}.$$

The Kechris basis theorem (Lemma 5.1) provides a z such that

(6) $$z \in \{m\}(x_t) \;\&\; \{n\}(z)\uparrow \;\&\; \kappa_r^{x_0,\ldots,x_t} \geq \kappa_r^{x_0,\ldots,x_t,z}.$$

Let z_0 be the w-least z that satisfies (6). (5) is proved by showing $x_{t+1} = z_0$. Clearly $x_{t+1} \leq z_0$.

Every $z <_w z_0$ is such that either $\{n\}(z)\downarrow$ or

(7) $$\kappa_r^{x_0,\ldots,x_t} < \kappa_r^{x_0,\ldots,x_t,z}.$$

As in the proof of (1) of Theorem 5.1, (7) is equivalent to

(8) $$\kappa_r^{x_0,\ldots,x_t} < \kappa_0^{x_0,\ldots,x_t,z}.$$

Thus for all $z <_w z_0$, either $\{n\}(z)\downarrow$ or (8) holds. (8) is equivalent to

$$(Ee)[\{e\}(x_0 \ldots, x_t, z)\downarrow \;\&\; \kappa_r^{x_0,\ldots,x_t} < \{e\}(x_0, \ldots, x_t, z)].$$

It follows from Gandy selection that there is a partial E-recursive function $f(x_0, \ldots, x_t, z)$, defined for all $z <_w z_0$, such that either

(9) $$|\{n\}(z)| = f(x_0, \ldots, x_t, z) \quad \text{or} \quad \kappa_r^{x_0,\ldots,x_t} < f(x_0, \ldots, x_t, z).$$

Let $\gamma = \sup\{f(x_0, \ldots, x_t, z) \mid z <_w z_0\}$. The bounding scheme gives $\gamma \leq_E x_0, \ldots, x_t, z_0$. Consequently $\gamma < \kappa_r^{x_0,\ldots,x_t}$ by (6), and so

$$\{n\}(z)\downarrow \quad \text{for all} \quad z < z_0$$

by (9). If $x_{t+1} <_w z_0$, then

$$|\{n\}(x_{t+1}| < \kappa_r^{x_0,\ldots,x_t} \leq \kappa_r^x \leq |\{n\}(x_{t+1})|,$$

an absurdity. □

A reformulation of Lemma 5.3 will prove helpful. Let $>_U^\gamma$ be the universal computation tree restricted to computations of height less than γ. To be precise, clause (v)(b) of the definition of $>_U$ in subsection 2.1 is modified as follows. Replace "$\{d_i\}(x_1, \ldots, x^n)$ converges" by "$|\{d_i\}(x_1, \ldots, x_n)| < \gamma$". In the arguments to come only scheme (T) of the proof of Lemma 2.5 will be considered. The precise definition of $>_U^\gamma$ requires that "$\{m\}(x)\downarrow$" of clause (Ta) be replaced by

"$|\{m\}(x)| < \gamma$". Let $T^{\gamma}_{\langle e,x\rangle}$ be the tree-like object consisting of $\langle e,x\rangle$ and that part of $>^{\gamma}_{U}$ below $\langle e,x\rangle$.

5.4 Reformulation of Lemma 5.3. Suppose some wellordering of $TC(x)$ is E-recursive in x. If $\{e\}(x)\uparrow$, then

$$T^{\kappa_r^x}_{\langle e,x\rangle} \text{ is not wellfounded,}$$

and both it and its left-most infinite path are first order definable over $L(\kappa_r^x, TC(\{x\}))$.

The term "left-most infinite path" in 5.4 refers to the witness $\lambda t \mid x_t$ defined in the proof of Lemma 5.3. "Left-most" is appropriate because x_{t+1} is the w-least z such that $z \in \{m\}(x_t)$ & $\{n\}(z)\uparrow$, when case B holds.

5.4 indicates the importance of another parameter. For all z define

$$\theta^z = \mu\gamma(e)\,[\{e\}(z)\uparrow \rightarrow T^{\gamma}_{\langle e,z\rangle} \text{ not wellfounded}].$$

Harrington's ground breaking results [1973] are:

$$\kappa_r^{2^\omega,a} = \theta^{2^\omega,a}$$

$$\kappa_r^{2^\omega,a,b} \geq \kappa_r^{2^\omega,a} \quad \text{for all} \quad a,b \in 2^\omega.$$

Warning: Slaman [1985a] has found an x and y such that

$$\kappa_r^x > \theta^x \quad \text{and} \quad \kappa_r^{x,y} < \kappa_r^x.$$

5.5 Lemma

(i) $\kappa_r^u \geq \theta^u$ *for all* u.
(ii) $\theta^{u,v} \geq \theta^u$ *for all* u *and* v.
(iii) $\kappa_r^x = \theta^x$ *if some wellordering of* x *is* E*-recursive in* x.
(iv) $\kappa_r^{x,u} \geq \kappa_r^x$ *for all* u, *if some wellordering of* x *is* E*-recursive in* x.

Proof. (i) Repeat the proof of Lemma 5.3 stripped of most definability considerations. Assume $\langle e_t, u_t\rangle$ has been defined so that

$$\{e_t\}(u_t)\uparrow \quad \text{and} \quad \kappa_r^u \geq \kappa_r^{u_0,\ldots,u_t}.$$

In case B choose u_{t+1} so that

$$u_{t+1} \in \{m\}(u_t),\ \{n\}(u_{t+1})\uparrow \quad \text{and} \quad \kappa_r^x \geq \kappa_r^{u_0,\ldots,u_{t+1}}.$$

The choice is made with the aid of Lemma 5.1. Hence $|\{m\}(u_t)| < \kappa_r^u$ when case B holds. Thus all the ordinals needed for checking convergence in the course of defining $\lambda t \mid u_t$ are less than κ_r^u. (Only scheme (T) is being considered.) So $\lambda t \mid u_t$ is a path through $>^{\kappa_r^x}_{U}$.

(ii) follows from the existence of an e^* such that the instructions for computing $\{e^*\}(u,v)$ are essentially the same as those for $\{e\}(u)$. Then $T_{\langle e^*, u, v\rangle}$ consists of finitely many trivial nodes above a copy of $T_{\langle e, u\rangle}$.

(iv) $\kappa_r^{x,u} \geq \theta^{x,u} \geq \theta^x = \kappa_r^x$ by (i), (ii) and (iii).

(iii) By (i) $\kappa_r^x \geq \theta^x$. In a moment it will be shown that there exists a recursive function h such that for all $e < \omega$:

(1) $$\{e\}(x)\downarrow \leftrightarrow \{h(e)\}(x)\downarrow;$$

(2) $$\text{if } \{h(e)\}(x)\uparrow, \text{ then } T_{\langle h(e), x\rangle} \text{ has a unique infinite path.}$$

Intuitively, $T_{\langle h(e), x\rangle}$ consists of the leftmost, infinite path of $T_{\langle e, x\rangle}$ and everything to the left of that path.

Suppose $\{e\}(x)\uparrow$. By (1) and (2) $T_{\langle h(e), x\rangle}$ has a unique infinite path t. t is first order definable over $L(\theta^x, TC(\{x\}))$. Suppose for a contradiction that $\theta^x < \kappa_r^x$. Then by reflection there is a divergence witness for $\{h(e)\}(x)\uparrow$ E-recursive in x.

But then for any e the question, does $\{e\}(x)\downarrow$?, can be effectively decided by searching through the sets E-recursive in x until either a wellfounded $T_{\langle e, x\rangle}$ turns up or a divergence witness to $\{h(e)\}(x)\uparrow$ is found. The search is effective by Gandy selection, as in the remarks following the proof of Corollary 4.5.

h is defined by abridging the Normann schemes so as not to allow computations to continue past the leftmost path. Suppose $e = 2^m \cdot 3^n$. (Only scheme (T) is being considered.) The immediate subcomputation instructions of $\langle h(e), x\rangle$ will be as follows:

(a) $\langle h(m), x\rangle$ is an immed. subcomp. instruc. of $\langle h(e), x\rangle$;
(b) suppose $\{h(m)\}(x)\downarrow$; let $<_w$ be the wellordering of $\{h(m)\}(x)$ E-recursive in x supplied by Proposition 5.2. If $\{h(n)\}(z)\downarrow$ for all $z <_w y$, then $\langle h(n), y\rangle$ is an immed. subcomp. instruc. of $\langle h(e), x\rangle$.

Let $h \simeq \{d\}$, where $\{d\}$ is a fixed point of $\psi(c)$; that is $\{d\} \simeq \{\psi(d)\}$.

$\psi(c)$ is a partial recursive function such that the instructions for computing $\{\{\psi(c)\}(e)\}(x)$ are as follows.

First compute $\{\{c\}(m)\}(x)$. If it converges, then compute $<_w$, a wellordering of $\{\{c\}(m)\}(x)$, as in Proposition 5.2. Then compute $\{\{c\}(n)\}(y)$ for all $y \in \{\{c\}(m)\}(x)$ *in the order* imposed by $<_w$. If this procedure terminates, then the final value is

$$\{\{\{c\}(n)\}(y) \mid y \in \{\{c\}(m)\}(x)\}.$$

Otherwise the procedure is stopped by the failure of $\{\{c\}(m)\}(x)$ to converge, or by the failure of $\{\{c\}(n)\}(y_0)$ to converge, where y_0 is the w-least y such that $\{\{c\}(n)\}(y_0)\uparrow$. In the latter case the procedure does compute $\{\{c\}(n)\}(y)$ for all $y <_w y_0$, but not for any $y >_w y_0$. $\square$

5.6 Lemma. *Let x be a set of ordinals. Suppose $L(\kappa, x)$ is E-closed but not Σ_1 admissible. Then $\kappa_r^{x,y} < \kappa$ for all $y \in L(\kappa, x)$.*

Proof. Suppose $\kappa_r^{x,y} \geq \kappa$ for some $y \in L(\kappa, x)$ with the intent of showing $L(\kappa, x)$ Σ_1 admissible. Assume

$$L(\kappa, x) \models (u)_{u \in d}\,(\mathrm{E}v)\,\mathscr{F}(u, v, p).$$

$d, p \in L(\kappa, x)$, and $\mathscr{F}$ is Δ_0^{ZF}. For the moment assume

$$\text{(1)} \qquad \kappa_r^{x,y,p,b} \geq \kappa_r^{x,y} \quad \text{for all } b \in d.$$

Then by reflection,

$$(b)_{b \in d}(\mathrm{E}c)\,[c \in L(\kappa_0^{x,y,p,b}, TC(\{x, y, p, b\})) \quad \& \quad \mathscr{F}(b, c, p)].$$

In fact

$$(b)_{b \in d}\,(\mathrm{E}c)\,[c \leq_E x, y, p, b \quad \& \quad \mathscr{F}(b, c, p)].$$

It follows from Corollary 4.5 that c can be construed as a partial recursive function of x, y, p, b defined for all $b \in d$. Let

$$e = \{c(x, y, p, b) \mid b \in d\}.$$

Then $e \in L(\kappa)$, since $e \leq_E x, y, p, d$ and $L(\kappa)$ is E-closed. Consequently

$$L(\kappa, x) \models (u)_{u \in d}\,(\mathrm{E}v)_{v \in e}\,\mathscr{F}(u, v, p).$$

To prove (1) apply Lemma 5.5 (iv). All that is needed is a wellordering w of $TC(\{x, y\})$ E-recursive in x, y. Since $\kappa_r^{x,y} \geq \kappa$, there is a wellordering of $TC(\{x, y\})$ in $L(\kappa_r^{x,y}, TC(\{x, y\}))$. Then w exists by reflection.

5.7 Theorem. *Suppose x is a set of ordinals and $L(\kappa, x)$ is E-closed. If $L(\kappa, x)$ is not Σ_1 admissible, then $L(\kappa, x)$ admits Moschovakis witnesses.* $\square$

Proof. Suppose $y \in L(\kappa, x)$ and $\{e\}(y)\uparrow$. Let $w \in L(\kappa, x)$ be a wellordering of $TC(\{x, y\})$. By Lemma 5.6, $\kappa_r^{x,y,w} < \kappa$. According to Lemma 5.3 there is a witness t to $\{e\}(y)\uparrow$ first order definable over

$$L(\kappa_r^{x,y,w},\ TC(\{x, y, w\})).$$

$t \leq_E \kappa_r^{x,y,w}, x, y, w$; hence $t \in L(\kappa, x)$, since $L(\kappa, x)$ is E-closed. $\square$

5.8 Theorem. *Let x be a set of ordinals. The following are equivalent.*

(i) $E(x)$ is not Σ_1 admissible.
(ii) $\kappa_r^{x,y} \in E(x)$ for all $y \in E(x)$.
(iii) $E(x)$ admits Moschovakis witnesses.

Proof. (i) → (ii) by Lemma 5.6. (ii) → (iii) as in the proof of Theorem 5.7. Assume (iii) to prove (i). For each $a \in TC(x)$, define $f(e, a, x)$ to be the least γ such that $|\{e\}(a, x)| < \gamma$ or some witness to $\{e\}(a, x)\uparrow$ is first order definable over $L(\gamma, TC(\{x\}))$.
By (iii) $f(e, a, x)$ is defined on $\omega \times TC(x) \times \{x\}$. f is Σ_1 over $E(x)$, and

$$E(x) \subseteq L(\sup \text{range} f, TC(\{x\})).$$

Consequently sup range $f = \kappa^x$, and $L(\kappa^x, x)$ is not Σ_1 admissible. □

It is not known if Lemma 5.6 can be strengthened so that its conclusion reads: Then $\kappa_r^y < \kappa$ for all $y \in L(\kappa, x)$. Slaman [1985] has found a Σ_1 inadmissible E-closed set that does not admit Moschovakis witnesses. Thus the supposition, x is a set of ordinals, of Theorem 5.7 was not lightly made. Existing proofs of the existence of Moschovakis witnesses rely on some form of dependent choice (DC_ω) obtained by either effectively wellordering the underlying set x or closing it under the formation of ω-sequences. As an example of the latter, let $x = 2^\omega$. Then $(2^\omega)^\omega = 2^\omega$ and so $E(2^\omega)$ satisfies DC_ω, if it is agreed that V, the class of all sets, satisfies DC_ω. Otherwise DC_ω can be justified in $E(2^\omega)$ by assuming either (a) $V = L$, or (b) $V = L(2^\omega)$ and the axiom of determinateness holds in $L(2^\omega)$ (Kechris).

5.9 Recursive Enumerability on $\mathscr{E}$. Suppose $\mathscr{E}$ is E-closed and $A \subseteq \mathscr{E}$. A is said to be E-recursively enumerable on $\mathscr{E}$ if for some $p \in \mathscr{E}$ and $e < \omega$,

(1) $$A = \{z \mid z \in \mathscr{E} \;\&\; \{e\}(p, z)\downarrow\}.$$

Equivalently, $A = \mathscr{E} \cap B$ for some B E-recursively enumerable in some $p \in \mathscr{E}$.

Warning: A can be E-recursively enumerable on $\mathscr{E}$ without being E-recursively enumerable in any $p \in \mathscr{E}$.

In the next chapter it will be shown by a forcing argument that there exists an E-closed $L(\kappa)$ such that $L(\kappa)$ is not E-recursively enumerable in any $p \in L(\kappa)$.

5.10 Theorem. *Let x be a set of ordinals. Then (i) ↔ (ii).*

(i) $E(x)$ is Σ_1 admissible.

(ii) For all $A \subseteq E(x)$: A is Σ_1 definable over $E(x)$ iff A is E-recursively enumerable on $E(x)$.

Proof. (i) → (ii). Let $E(x) = L(\kappa, x)$. Suppose $A \subseteq E(x)$ and

$$z \in A \leftrightarrow L(\kappa, x) \models (\mathrm{E}v)\, \mathscr{F}(z, v, p).$$

for some Δ_0^{ZF} $\mathscr{F}$ and $p \in L(\kappa, x)$. By Theorem 5.8 there is a $y \in L(\kappa, x)$ such that $\kappa_r^{y,x} \geq \kappa$. As in the proof of Lemma 5.6, v can be construed as a partial E-recursive function of x, y, p and z defined for all $z \in A$. Then

$$z \in A \leftrightarrow f(x, y, p, z)\downarrow \quad \& \quad L(\kappa, x) \models \mathscr{F}(z, f(x, y, p, z), z),$$

and so A is E-recursively enumerable on $E(x)$ via y, p.

Now assume (ii) holds and (i) fails in hope of a contradiction. By Theorem 5.8, $E(x)$ admits Moschovakis witnesses, and so the predicate,

$$z \in E(x) \;\; \& \;\; \{e\}(e, z)\uparrow$$

with z and e as free variables, is Σ_1 definable over $E(x)$. Hence it is E-recursively enumerable on $E(x)$ via some parameter $q \in E(x)$. Then the set

$$\kappa_q = \{e \mid \{e\}(e, q)\uparrow\}$$

is E-recursively enumerable on $E(x)$ via q, hence E-recursively enumerable in q, since $x \in \omega$ is an E-recursive predicate. Thus for some $c \in \omega$,

$$\kappa_q = \{e \mid \{c\}(e, q)\downarrow\}.$$

But then $\{c\}(c, q)\uparrow$ iff $\{c\}(c, q)\downarrow$. $\square$

5.11 The Divergence-Admissibility Split. Theorems 5.8 and 5.10 yield a useful split of all E-closed $L(\kappa)$'s into two *disjoint* classes.

Class I: $L(\kappa)$ admits Moschovakis witnesses.

Class II: $L(\kappa)$ is Σ_1 admissible; and for all $A \subseteq L(\kappa)$, A is Σ_1 definable over $L(\kappa)$ iff A is E-recursively enumerable on $L(\kappa)$.

In brief, either $L(\kappa)$ admits divergence witnesses or Σ_1 equals E-recursively enumerable on $L(\kappa)$, but not both. A typical application of the split, made in Chapter XIII occurs in the solution to Post's problem in the sense of E-recursion theory for every E-closed $L(\kappa)$. If $L(\kappa) \in$ Class II, then the admissibility methods of Part B suffice. If $L(\kappa) \in$ Class I, then Moschovakis witnesses shoulder most of the burden of proof.

For proof of the split first assume $L(\kappa) \neq E(x)$ for any x. Every $x \in L(\kappa)$ is wellordered in $L(\kappa)$, so Lemma 5.3 implies $L(\kappa) \in I$ and $\{\langle e, x\rangle \mid \{e\}(x)\uparrow\}$ is Σ_1 over $L(\kappa)$. If $L(\kappa)$ were in II, then $\{\langle e, x\rangle \mid \{e\}(x)\uparrow\}$ would be E-r.e.; hence E-recursive, an absurdity.

Now suppose $L(\kappa) = E(x)$ for some x. It is safe to assume x is a wellordering. Then by Theorems 5.8 and 5.10, $L(\kappa) \notin I$ iff $L(\kappa)$ is Σ_1 admissible iff "Σ_1 definability over" agrees with "E-recursively enumerable on" for $L(\kappa)$.

5.12–5.17 Exercises

5.12. Make precise and prove the reformulation of Lemma 5.3 given in subsection 5.4.

5.13. Supply the details of the proof of Lemma 5.5 (i).

5.14. Suppose $A \subseteq q \in \mathscr{E}$, $\mathscr{E}$ is E-closed, and A is E-recursively enumerable on $\mathscr{E}$ via p. Show A is E-recursively enumerable in p, q.

5.15. Show $E(\omega_1)$ is not Σ_1 admissible.

5.16. (Kleene) Show $2^\omega \cap E(\omega)$ is E-recursively enumerable in ω.

5.17. Suppose $P(\gamma)$ is E-co-recursively enumerable in a, and $(E\gamma)_{\gamma < c}\, P(\gamma)$. Let γ_∞ be $\mu\gamma_{\gamma < c}\, P(\gamma)$. Show

$$\kappa_r^{\gamma_\infty, a, c} \leq \kappa_r^{a, c}.$$