Part B

The Absolute Theory

"... the central notions of model theory are absolute, and absoluteness, unlike cardinality, is a logical concept."

G. Sacks, from
Saturated Model Theory

Chapter V

The Recursion Theory of Σ_1 Predicates on Admissible Sets

There are many equivalent definitions of the class of recursive functions on the natural numbers. Different definitions have different uses while the equivalence of all the notions provides evidence for Church's thesis, the thesis that the concept of recursive function is the most reasonable explication of our intuitive notion of effectively calculable function.

As the various definitions are lifted to domains other than the integers (e.g., admissible sets) some of the equivalences break down. This break-down provides us with a laboratory for the study of recursion theory. By studying the notions in the general setting one sees with a clearer eye the truths behind the results on the integers.

The most dramatic breakdown results in two competing notions of r.e. on admissible sets, notions which happen to coincide on countable admissible sets. We refer to these as the *syntactic* and *semantic* notions of r.e. and study the former in this chapter. The semantic notion is discussed in Chapter VIII.

1. Satisfaction and Parametrization

In view of Theorem II.2.3 (which shows that r.e. on ω is just Σ_1 on $\mathbb{HF}$) it is natural to ask oneself what properties of r.e. and recursive lift up to Σ_1 and Δ_1 on an arbitrary admissible set. Luckily, the more important results, results like Kleene's Enumeration and Second Recursion Theorem, lift to completely arbitrary admissible sets.

1.1 Definition. Let $\mathbb{A}$ be admissible and let R be a relation on $\mathbb{A}$.

(i) R is $\mathbb{A}$-*r.e.* if R is Σ_1 on $\mathbb{A}$.

(ii) R is $\mathbb{A}$-*recursive* if R is Δ_1 on $\mathbb{A}$.

(iii) R is $\mathbb{A}$-*finite* if $R \in \mathbb{A}$.

(iv) A function f with domain and range subsets of $\mathbb{A}$ is $\mathbb{A}$-*recursive* if its graph is $\mathbb{A}$-r.e.

If $\mathbb{A} = \mathrm{L}(\alpha)$ then we refer to these notions as α-*r.e.*, α-*recursive* and α-*finite*, respectively.

As in ordinary ω-recursion theory, a *total* $\mathbb{A}$-recursive function will have an $\mathbb{A}$-recursive graph.

The first result of ω-recursion theory we want to generalize is Kleene's Enumeration Theorem.

1.2 Definition. Let $\mathbf{S}$ be a collection of n-ary relations on some set X. Let $Y \subseteq X$. An $n+1$-ary relation T on X *parametrizes* $\mathbf{S}$ *(with indices from* Y*)* if $\mathbf{S}$ consists of all relations of the form

$$S_e = \{(x_1, \ldots, x_n) \mid T(e, x_1, \ldots, x_n)\}$$

as e ranges over Y.

1.3 Theorem. *Let* $\mathbb{A} = (\mathfrak{M}; A, \in, \ldots)$ *be an admissible set. There is an* $\mathbb{A}$*-r.e. relation* T_n *which parametrizes the class of n-ary* $\mathbb{A}$*-r.e. relations, with indices from* A.

To prove this theorem we make use of our earlier formalization in KPU of syntax and semantics. The proof is more important than the theorem itself.

There is a systematic ambiguity which has served us well until now. We have been using $\varphi, \psi, \ldots$ to range over formulas of our metalanguage L^* as well as over formulas of formalized languages. We must avoid this confusion in this section.

Let $\mathsf{L}^* = \mathsf{L}(\in, \ldots)$ be fixed and finite. For simplicity we assume L^* has only relation symbols. The extension to the general case is sketched in the exercises. We consider L^* here as a single sorted language with variables $x_1, x_2, \ldots$ and unary symbols U (for "urelement") and S (for "set"). Let l^* be some effective coding of L^* in $\mathbb{HF}$. For basic symbols like R we let $\ulcorner \mathsf{R} \urcorner$ be the set in $\mathbb{HF}$ which names R. For definiteness we take $v_n = \ulcorner x_n \urcorner = \langle 0, n \rangle$. To each formula φ of L^* there corresponds its formalized version $\ulcorner \varphi \urcorner$, an element of $l^*_{\omega\omega} \subseteq \mathbb{HF}$, defined by recursion equations

$$\ulcorner \varphi \wedge \psi \urcorner = \langle \bigwedge, \{\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner\} \rangle$$
$$\ulcorner \exists x_n \varphi \urcorner = \langle \exists, v_n, \ulcorner \varphi \urcorner \rangle$$

and so forth.

Define, in KPU, an operation $\mathfrak{N}_a$ on sets a by: $\mathfrak{N}_a$ is a structure for l^* with universe $\mathrm{TC}(a)$ which interprets the symbols of l^* as follows:

Symbol	*Interpretation*
$\ulcorner \mathsf{U} \urcorner$	$\{p \mid p \in \mathrm{TC}(a)\}$
$\ulcorner \mathsf{S} \urcorner$	$\{b \mid b \in \mathrm{TC}(a)\}$
$\ulcorner \in \urcorner$	$\{\langle x, y \rangle \mid x, y \in \mathrm{TC}(a), x \in y\}$
$\ulcorner \mathsf{R} \urcorner$	$\{\langle x_1, \ldots, x_n \rangle \in \mathrm{TC}(a)^n \mid R(x_1, \ldots, x_n)\}$.

Clearly $\mathfrak{N}_a$ is a Σ_1 operation of a. Recall the notation $\varphi^{(a)}$ from § I.4.

1.4 Lemma. *For each formula* $\varphi(x_1,\ldots,x_n)$ *of* L^* *the following is a theorem of* KPU: *for all sets* a *and all* $x_1,\ldots,x_n \in \mathrm{TC}(a)$, *if* $s=\{\langle v_i,x_i\rangle \mid i=1,\ldots,n\}$ *then*

$$\varphi^{(\mathrm{TC}(a))}(x_1,\ldots,x_n) \quad \textit{iff} \quad \mathfrak{N}_a \models \ulcorner\varphi\urcorner[s].$$

Proof. For φ atomic this follows from the definition of $\mathfrak{N}_a$. The result follows by induction on formulas. □

1.5 Definition. Let Σ-Sat_n be the following Σ_1 formula of L^* with variables $y, x_1,\ldots,x_n$:

"*y is a Σ formula of l^* with free variables among $v_1,\ldots,v_n$ and there is a transitive set a with $x_1,\ldots,x_n \in a$ such that*

$$\mathfrak{N}_a \models y[s]$$

where $s=\{\langle v_i,x_i\rangle \mid i=1,\ldots,n\}$".

That this can be expressed by a Σ_1 formula follows from the results in § III.1.

1.6 Proposition. *Let* $\varphi(x_1,\ldots,x_n)$ *be a* Σ *formula of* L^*. *The following is a theorem of* KPU: *for all* $x_1,\ldots,x_n$,

$$\varphi(x_1,\ldots,x_n) \quad \textit{iff} \quad \Sigma\text{-}\mathrm{Sat}_n(\ulcorner\varphi(x_1,\ldots,x_n)\urcorner, x_1,\ldots,x_n).$$

Proof. Assume the axioms of KPU. The following are equivalent:

$\varphi(x_1,\ldots,x_n)$
$\exists a\,[\mathrm{Tran}(a) \wedge x_1,\ldots,x_n \in a \wedge \varphi^{(a)}(x_1,\ldots,x_n)]$
$\exists a\,\exists s\,[\mathrm{Tran}(a) \wedge x_1,\ldots,x_n \in a \wedge \mathfrak{N}_a \models \ulcorner\varphi\urcorner[s]$, *where* $s=\{\langle v_i,x_i\rangle \mid i=1,\ldots,n\}]$
Σ-$\mathrm{Sat}_n(\ulcorner\varphi\urcorner, x_1,\ldots,x_n)$.

The first two lines are equivalent by Σ Reflection, the middle two by Lemma 1.4 and the last two by the definition of Σ-Sat_n. □

Define $T_n(e,x_1,\ldots,x_n)$ to be the Σ_1 formula:

"e is an ordered pair $\langle\psi,z\rangle$ and Σ-$\mathrm{Sat}_{n+1}(\psi,x_1,\ldots,x_n,z)$".

Proof of Theorem 1.3. Since T_n is Σ_1 any predicate defined by

$$R(x_1,\ldots,x_n) \quad \text{iff} \quad T_n(e,x_1,\ldots,x_n)$$

is $\mathbb{A}$-r.e. To prove the converse, let R be an n-ary $\mathbb{A}$-r.e. predicate. By using ordered pairs it has a Σ_1 definition on $\mathbb{A}$ with exactly one parameter z, say

$$R(x_1,\ldots,x_n) \quad \text{iff} \quad \mathbb{A} \models \psi(x_1,\ldots,x_n,z).$$

Then let $e = \langle \ulcorner \psi(x_1, \ldots, x_{n+1}) \urcorner, z \rangle$ and apply 1.6. $\square$

1.7 Corollary. *Let $\mathbb{A}$ be admissible. There is an $\mathbb{A}$-r.e. set which is not $\mathbb{A}$-recursive.*

Proof. Just as for ω-recursion theory define

$$K = \{e \in \mathbb{A} \mid \mathbb{A} \models T_1(e, e)\}.$$

If $\mathbb{A} - K$ were $\mathbb{A}$-r.e. there would be an e_0 such that for all $e \in \mathbb{A}$, $e \notin K$ iff $T_1(e_0, e)$, and hence $e_0 \notin K$ iff $e_0 \in K$. $\square$

Let $\mathfrak{M} = \langle M \rangle$ be an infinite set with no additional relations. Note that if $X \subseteq M$ is $\mathbb{HYP}_{\mathfrak{M}}$-r.e. then X is $\mathbb{HYP}_{\mathfrak{M}}$-finite since by II.9.3, X or $M - X$ is finite. Thus Corollary 1.7 cannot in general be improved to get a $\mathbb{A}$-r.e. subset of $\mathfrak{M}$ which is not $\mathbb{A}$-recursive.

1.8—1.10 Exercises

1.8. Suppose $\mathsf{L}^* = \mathsf{L}(\in, \mathsf{f}, \ldots)$ has a function symbol f. Show that under the standard treatment of function symbols as relation symbols, Δ_0 formulas transform into both Σ and Π formulas (but not necessarily into Δ_0 formulas). Hence Σ_1 formulas transform into Σ formulas.

1.9. Let L^* be a finite language with function symbols. Define Σ-Sat_n for L^* in such a way that 1.6 and hence 1.3 become provable.

1.10. Find an admissible set $\mathbb{A}_{\mathfrak{M}}$ such that the class of $\mathbb{A}_{\mathfrak{M}}$-r.e. subsets of $\mathfrak{M}$ cannot be parametrized by an $\mathbb{A}_{\mathfrak{M}}$-r.e. binary relation with indices from M.

2. *The Second Recursion Theorem for* KPU

The Second Recursion Theorem in ω-recursion theory is a mysterious device for implicitly defining recursive partial functions, or equivalently, r.e. predicates. The theorem is equally mysterious and equally useful in our setting.

Let $\mathsf{L}^* = \mathsf{L}(\in, \ldots)$ be a finite language (as in § 1) and let R be a new n-ary relation symbol, $n \geqslant 1$.

2.1 Definition. The collection of R-*positive* formulas of $\mathsf{L}^*(\mathsf{R})$ is the smallest class of formulas containing all formulas of L^*, all atomic formulas of $\mathsf{L}^*(\mathsf{R})$, and closed under

$$\wedge,\ \vee,\ \forall u \in v,\ \exists u \in v,\ \forall u,\ \exists u$$

for all variables u, v. We use the notation

$$\varphi(\mathsf{R}_+)$$

to indicate that φ is an R-positive formula.

Given a formula $\varphi(\mathsf{R})$ of $\mathsf{L}^*(\mathsf{R})$ and a formula $\psi(x_1,\ldots,x_n)$ of L^* we use the notations

$$\varphi(\psi/\mathsf{R})$$

$$\varphi(\lambda x_1,\ldots,x_n\,\psi(x_1,\ldots,x_n))$$

more or less interchangeably to denote the formula resulting by replacing each occurrence of an atomic formula of the form $\mathsf{R}(t_1,\ldots,t_n)$ in $\varphi(\mathsf{R})$ by $\psi(t_1/x_1,\ldots,t_n/x_n)$ (unless some t_i is not free for x_i in ψ in which case we must first rename bound variables in ψ, but then we agreed in Chapter I not to mention such details). Thus $x_1,\ldots,x_n$ do not occur free in $\varphi(\psi/\mathsf{R})$ (unless they are free in $\varphi(\mathsf{R})$), and R does not occur in $\varphi(\psi/\mathsf{R})$.

2.2 Lemma. *If $\varphi(\mathsf{R}_+)$ is a Σ formula of $\mathsf{L}^*(\mathsf{R})$ and if $\psi(x_1,\ldots,x_n)$ is a Σ formula of L^* then $\varphi(\psi/\mathsf{R})$ is a Σ formula of L^*.*

Proof. By induction on the class of R-positive formulas $\varphi(\mathsf{R}_+)$. □

2.3 The Second Recursion Theorem. *Let $\varphi(\vec{x},\vec{y},\mathsf{R}_+)$ be an R-positive Σ formula where R is n-ary, $\vec{x}=x_1,\ldots,x_n$ and $\vec{y}=y_1,\ldots,y_k$. There is a Σ formula $\psi(\vec{x},\vec{y})$ of L^* so that the following is a theorem of* KPU: *for all parameters $\vec{y}$ and all $x_1,\ldots,x_n$*

$$\psi(x_1,\ldots,x_n,\vec{y}) \quad \textit{iff} \quad \varphi(x_1,\ldots,x_n,\vec{y},\lambda x_1,\ldots,x_n\,\psi(x_1,\ldots,x_n,\vec{y})).$$

Proof. To simplify notation we assume $n=k=1$. Let $\theta(x,y,z)$ be the Σ formula

$$\varphi(x,y,\lambda x\,\Sigma\text{-}\mathrm{Sat}_3(z,x,y,z)).$$

Let $e=\ulcorner\theta(x,y,z)\urcorner\in\mathbb{HF}$ and let $\psi(x,y)$ be $\theta(x,y,e)$, or rather, the Σ formula equivalent to it obtained by replacing the constant e by a good Σ_1 definition of e. Then we have, in KPU, that the following are equivalent:

$$\psi(x,y)$$

$$\theta(x,y,e)$$

$$\varphi(x,y,\lambda x\,\Sigma\text{-}\mathrm{Sat}_3(e,x,y,e))$$

$$\varphi(x,y,\lambda x\,\theta(x,y,e))$$

$$\varphi(x,y,\lambda x\,\psi(x,y)).\quad □$$

Since any Σ formula is equivalent, in KPU, to a Σ_1 formula, we could have demanded that the ψ of 2.3 be Σ_1.

We give a simple application of the Second Recursion Theorem. In any admissible set $\mathbb{A}_{\mathfrak{M}}$, $\mathbb{HF}$ is an $\mathbb{A}_{\mathfrak{M}}$-recursive subset since

$$a \in \mathbb{HF} \text{ iff } \mathrm{sp}(a) = 0 \wedge (\mathrm{rk}(a) \text{ is a natural number}).$$

$\mathbb{HF}_{\mathfrak{M}}$, however, *is not always* Δ_1 *definable.* (The student can find an example of this in Exercise 2.6.)

2.4 Proposition. *There is a* Σ_1 *formula* $\psi(x)$ *such that in any admissible set* $\mathbb{A}_{\mathfrak{M}}$,

$$\mathbb{HF}_{\mathfrak{M}} = \{a \in \mathbb{A}_{\mathfrak{M}} \mid \mathbb{A}_{\mathfrak{M}} \models \psi[a]\}.$$

Proof. Let R be unary and let $\varphi(x, \mathsf{R}_+)$ be the Σ formula

$$(x \textit{ is a finite set}) \wedge \forall y \in x \ (\textit{if } y \textit{ is a set then } \mathsf{R}(y)).$$

Now apply the Second Recursion Theorem to get a formula ψ such that

$$\mathrm{KPU} \vdash \psi(x) \leftrightarrow (x \textit{ is a finite set} \wedge \forall y \in x \ (y \textit{ is a set} \rightarrow \psi(y))).$$

Now let $\mathbb{A}_{\mathfrak{M}}$ be admissible. A trivial proof by induction on $\in$ shows that

$$a \in \mathbb{HF}_{\mathfrak{M}} \quad \text{iff} \quad \mathbb{A}_{\mathfrak{M}} \models \psi[a]$$

for all $a \in \mathbb{A}_{\mathfrak{M}}$. $\square$

2.5—2.6 Exercises

2.5. Show that a formula $\varphi(\mathsf{R})$ is logically equivalent to an R-positive formula iff the result of pushing negations inside φ as far as possible (using de Morgan's laws) results in a formula in which $\neg\mathsf{R}$ does not occur.

2.6. Let $\mathfrak{M}$ be a recursively saturated model of Peano arithmetic, KP or ZF. Show that $\mathbb{HF}_{\mathfrak{M}}$ is not $\mathbb{HYP}_{\mathfrak{M}}$-recursive.

3. *Recursion Along Well-founded Relations*

In this section we use the Second Recursion Theorem to give a new principle of definition by recursion along well-founded relations. This serves as a useful warm-up exercise in the use of the Second Recursion Theorem.

3.1 Theorem. *Let* $\mathbb{A} = \mathbb{A}_{\mathfrak{M}}$ *be admissible, let* p *be an* $\mathbb{A}$*-recursive function and define a binary relation* $\prec$ *by*

$$x \prec y \quad \textit{iff} \quad x \in p(y)$$

for all $y \in \mathrm{dom}(p)$.

(i) *The well-founded part of* $\prec$, $\mathcal{W}\!f(\prec)$, *is* $\mathbb{A}$*-r.e.*

(ii) *If* G *is a total* $k+2$*-ary* $\mathbb{A}$*-recursive function then there is an* $\mathbb{A}$*-recursive* F *with*

$$\mathrm{dom}(F) = (M \cup A)^k \times \mathcal{W}\!f(\prec)$$

such that

$$F(\vec{z}, x) = G(\vec{z}, x, \{\langle y, F(\vec{z}, y)\rangle \mid y \prec x\})$$

for all $\vec{z} \in (M \cup A)^k$ *and all* $x \in \mathcal{W}\!f(\prec)$.

Proof. Recall that $\mathcal{W}\!f(\prec)$ is the largest subset B of Field$(\prec)$ such that:

$$x \prec y, \ y \in B \ \text{implies} \ x \in B, \ \text{and}$$

$$\prec \restriction B^2 \ \text{is well founded}.$$

There is such a largest set by II.8.2. Note that $\mathrm{pred}(x) \subseteq \mathcal{W}\!f(\prec)$ implies $x \in \mathcal{W}\!f(\prec)$. Part (i) of the theorem follows from part (ii) but we need (i) in the proof of (ii). Besides, (i) is an easy example of the use of the Second Recursion Theorem.

Define a Σ_1 formula $\psi(x, \alpha)$ such that

$$(1) \qquad \psi(x, \alpha) \ \text{iff} \ \exists z(z = p(x) \wedge \forall y \in z \, \exists \beta < \alpha \, \psi(y, \beta))$$

is a theorem of KPU and hence true in $\mathbb{A}$. Since this is only our second use of the Second Recursion Theorem, perhaps we should be a bit more explicit. Let $\eta(x, z)$ define the graph of p; η may have some other parameters which remain fixed throughout (the y's of the Second Recursion Theorem). Let R be a new binary relation symbol and let $\varphi(x, \alpha, \mathsf{R}_+)$ be the Σ formula

$$\exists z[\eta(x, z) \wedge \forall y \in z \, \exists \beta \in \alpha \, \mathsf{R}(y, \beta)]$$

of $\mathsf{L}^*(\mathsf{R})$. Note that R does indeed occur positively in this formula. Now apply the Second Recursion Theorem to get ψ satisfying (1). We will never again be this explicit; rather we'll just write an equation like (1) and leave it to the reader to see that the right-hand side is of the appropriate form. Now given ψ, one proves, for $\alpha \in \mathbb{A}$,

$$(2) \qquad \mathbb{A} \models \psi(x, \alpha) \ \text{implies} \ x \in \mathcal{W}\!f(\prec)$$

by a simple induction on α, using (1). A little less trivial is

(3) $$x \in \mathscr{W}\!f(\prec) \quad \text{implies} \quad \mathbb{A} \models \exists \alpha\, \psi(x,\alpha).$$

Assume $x \in \mathscr{W}\!f(\prec)$. Since $\mathscr{W}\!f(\prec) \subseteq \mathrm{Field}(\prec)$, $p(x)$ is defined. We may assume by induction ($\prec \restriction \mathscr{W}\!f(\prec)$ is well founded so induction over it is legitimate) that for each $y \in p(x)$

$$\mathbb{A} \models \exists \beta\, \psi(y,\beta)$$

and hence

$$\mathbb{A} \models \forall y \in p(x)\, \exists \beta\, \psi(y,\beta),$$

so by Σ Reflection there is an $\alpha \in \mathbb{A}$ with

$$\mathbb{A} \models \forall y \in p(x)\, \exists \beta < \alpha\, \psi(y,\beta).$$

By (1),

$$\mathbb{A} \models \psi(x,\alpha).$$

Combining (2), (3) we have

$$\mathscr{W}\!f(\prec) = \{x \in \mathbb{A} \mid \mathbb{A} \models \exists \alpha\, \psi(x,\alpha)\}$$

which makes $\mathscr{W}\!f(\prec)$ an $\mathbb{A}$-r.e. set.

To prove (ii) we use the Second Recursion Theorem again. We want to define the graph of F by a Σ_1 formula $\psi(\vec{z},x,w)$. Let us suppress the parameters since they are held fixed throughout. We want

$$\begin{aligned} \psi(x,w) \quad &\text{iff} \quad x \in \mathscr{W}\!f(\prec) \wedge F(x) = w \\ &\text{iff} \quad x \in \mathscr{W}\!f(\prec) \wedge \exists f[f = F \restriction p(x) \wedge w = G(x,f)]. \end{aligned}$$

The Second Recursion Theorem gives us a $\Sigma_1\, \psi$ so that

$$\begin{aligned} &\psi(x,w) \quad \text{iff} \quad x \in \mathscr{W}\!f(\prec) \wedge \exists f\, [f \textit{ is a function} \wedge \mathrm{dom}(f) = p(x) \\ &\wedge \forall y \in p(x)\, \psi(y, f(y)) \wedge w = G(x,f)] \end{aligned}$$

is true in $\mathbb{A}$ for all x, w. Using Σ Replacement one shows by induction on $\prec \restriction \mathscr{W}\!f(\prec)$ that

$$x \in \mathscr{W}\!f(\prec) \quad \text{implies} \quad \mathbb{A} \models \exists!\, w\, \psi(x,w)$$

so we may use $\psi(x,w)$ as a definition of an $\mathbb{A}$-recursive F. One then checks that F satisfies the desired equation, again by induction on $\prec \restriction \mathscr{W}\!f(\prec)$. $\square$

3.2 Definition. Let $\prec$ be a binary relation with nonempty wellfounded part. Define the *$\prec$-rank function* $\rho^{\prec}$, for $x \in \mathscr{W}\!\!f(\prec)$, by

$$\rho^{\prec}(x) = \sup\{\rho^{\prec}(y)+1 \mid y \prec x\}.$$

Define the *rank of* $\prec$, $\rho(\prec)$, by

$$\rho(\prec) = \sup\{\rho^{\prec}(x)+1 \mid x \in \mathscr{W}\!\!f(\prec)\}.$$

3.3 Corollary. *Let* $\mathbb{A}$ *be an admissible set with* $\prec$ *an element of* $\mathbb{A}$ *and* $\alpha = o(\mathbb{A})$.
 (i) $\rho(\prec) \leqslant \alpha$.
 (ii) *If* $\mathscr{W}\!\!f(\prec) \in \mathbb{A}$ *(for example, if* $\prec$ *is well founded) then* $\rho(\prec) < \alpha$.
 (iii) *If* $\mathscr{W}\!\!f(\prec) \notin \mathbb{A}$ *then* $\rho(\prec) = \alpha$.

Proof. To apply Theorem 3.1 define an $\mathbb{A}$-recursive function p by

$$p(x) = \{y \in \text{Field}(\prec) \mid y \prec x\}.$$

(This is the reason we assumed $\prec \in \mathbb{A}$.) Then the definition

$$\rho^{\prec}(x) = \sup\{\rho^{\prec}(y)+1 \mid y \prec x\}$$

falls under 3.1 (ii) so always gives values in $\mathbb{A}$. This proves (i).

If $\mathscr{W}\!\!f(\prec) \in \mathbb{A}$ then we may use Σ Replacement to form

$$\sup\{\rho^{\prec}(x)+1 \mid x \in \mathscr{W}\!\!f(\prec)\}$$

in $\mathbb{A}$. This gives (ii). To prove (iii), suppose $\rho(\prec) = \beta \in \mathbb{A}$, and let us prove $\mathscr{W}\!\!f(\prec) \in \mathbb{A}$. For $\gamma < \beta$ let

$$\begin{aligned} F(\gamma) &= \textstyle\bigcup_{\xi<\gamma} F(\xi) \cup \{x \in \text{Field}(\prec) \mid p(x) \subseteq \bigcup_{\xi<\gamma} F(\xi)\} \\ &= \{x \in \text{Field}(\prec) \mid \rho^{\prec}(x) \leqslant \gamma\} \end{aligned}$$

be defined by Σ Recursion for $\gamma < \beta$. But then

$$\mathscr{W}\!\!f(\prec) = \textstyle\bigcup_{\gamma<\beta} F(\gamma)$$

is in $\mathbb{A}$ by Σ Replacement. $\square$

While the most useful results of ω-recursion theory lift to an arbitrary admissible set, many of the more pleasing facts of recursion theoretic life on ω carry over only to special admissible sets. In particular, there are many results of recursion theory which use the effective well-ordering of the domain in an essential way.

3.4 Definition. Let $\mathbb{A} = (\mathfrak{M}; A, \in, \ldots)$ be an admissible set with $\alpha = o(\mathbb{A})$. $\mathbb{A}$ is *recursively listed* if there is an $\mathbb{A}$-recursive bijection of α onto $M \cup A$.

Lemma II.2.4 shows that $\mathbb{HF}$ is recursively listed. We will study the recursion theory of recursively listed admissible sets in the next section. They are related to this section by means of the following result.

3.5 Proposition. *$\mathbb{A}$ is recursively listed iff there is a total $\mathbb{A}$-recursive function p such that*

$$x \prec y \quad \textit{iff} \quad x \in p(y)$$

defines a well-ordering $\prec$ of $M \cup A$.

Proof. Suppose $e: \alpha \to \mathbb{A}$ is an $\mathbb{A}$-recursive enumeration of $M \cup A$. Note that e^{-1} is $\mathbb{A}$-recursive. Define $p(y) = \{e(\beta) \mid \beta < e^{-1}(y)\}$ and note that $x \in p(y)$ iff $e^{-1}(x) < e^{-1}(y)$.

Now suppose p is given as above. Note that, by 3.1, $\rho^{\prec}$ is an $\mathbb{A}$-recursive function. Since $\prec$ is a linear ordering, $\rho^{\prec}$ is one-one so we can let e be the inverse of $\rho^{\prec}$. By Σ Replacement, $\rho^{\prec}$ has range α so e has domain α. $\square$

Recall the definition of $\mathrm{L}(\alpha)$ given (in KPU) in II.5.

$$\mathrm{L}(0) = 0$$
$$\mathrm{L}(\alpha+1) = \mathscr{D}(\mathrm{L}(\alpha) \cup \{\mathrm{L}(\alpha)\})$$
$$\mathrm{L}(\lambda) = \textstyle\bigcup_{\alpha<\lambda} \mathrm{L}(\alpha) \quad \text{for } \lambda \text{ a limit ordinal}$$

where

$$\mathscr{D}(b) = b \cup \{\mathscr{F}_i(x,y) : x, y \in b,\ 1 \leqslant i \leqslant N\}.$$

We have shown that if α is admissible then $\mathrm{L}(\alpha)$ is the smallest admissible set $\mathbb{A}$ with $o(\mathbb{A}) = \alpha$. There is a natural well-ordering of $\mathrm{L}(\alpha)$ given by putting everything in $\mathrm{L}(\beta)$ before everything in $\mathrm{L}(\delta)$ for $\beta < \delta$ and ordering the elements a of $\mathrm{L}(\beta+1) - \mathrm{L}(\beta)$ according to which $\mathscr{F}_i(x,y) = a$. To make this precise define, in KPU, a Σ_1 formula $\psi(x,y)$, which we write as $x <_{\mathrm{L}} y$, as follows. First let

$$F(x) = \text{the least } \alpha (x \in \mathrm{L}(\alpha+1))$$

$$G(x) = \begin{cases} 0 \text{ if } x \in \mathscr{S}(\mathrm{L}(F(x))), \\ \text{the least } i,\ 1 \leqslant i \leqslant N, \quad \text{such that } x = \mathscr{F}_i(z_1, z_2) \text{ for some} \\ \qquad z_1, z_2 \in \mathscr{S}(\mathrm{L}(F(x))) \text{ otherwise}. \end{cases}$$

$\langle z_1, z_2 \rangle \lhd_\alpha \langle w_1, w_2 \rangle$ if

$z_1 <_{\mathrm{L}} w_1$ or

$z_1 \in \mathrm{L}(\alpha) \wedge w_1 = \mathrm{L}(\alpha)$ or

$z_1 = w_1$ and,

$z_2 <_{\mathrm{L}} w_2$ or

$z_2 \in \mathrm{L}(\alpha) \wedge w_2 = \mathrm{L}(\alpha)$.

Given that $<_{\mathrm{L}}$ already wellorders $\mathrm{L}(\alpha)$, $\lhd_\alpha$ wellorders the pairs z_1, z_2 from $\mathscr{S}(\mathrm{L}(\alpha)) = \mathrm{L}(\alpha) \cup \{\mathrm{L}(\alpha)\}$ "lexicographically after putting $\mathrm{L}(\alpha)$ itself at the end of the alphabet".

Now define $<_{\mathrm{L}}$ by

$$\begin{aligned} x <_{\mathrm{L}} y \quad &\text{if} \quad x \in \mathrm{L} \quad \text{and} \quad y \in \mathrm{L} \quad \text{and} \quad F(x) < F(y) \quad \text{or} \\ &F(x) = F(y) \wedge G(x) < G(y) \quad \text{or} \\ &F(x) = F(y) \wedge G(x) = G(y) \quad \text{and there is a pair} \\ &\qquad z_1, z_2 \in \mathscr{S}(L(F(x))) \quad \text{such that} \quad x = \mathscr{F}_{G(x)}(z_1, z_2) \\ &\qquad \text{but for all} \quad w_1, w_2 \in \mathscr{S}(L(F(x))), \quad \text{if} \quad y = \mathscr{F}_{G(x)}(w_1, w_2) \\ &\qquad \text{then} \quad \langle z_1, z_2 \rangle \lhd_{F(x)} \langle w_1, w_2 \rangle . \end{aligned}$$

We could define $<_{\mathrm{L}}$ explicitly, if we really had to, but for our purposes here we can be content to use any such formula given by the Second Recursion Theorem. To see that the Second Recursion Theorem applies we need only observe that, once $\lhd_\alpha$ is replaced by its · definition, the right-hand side is a Σ formula and that $<_{\mathrm{L}}$ occurs positively.

3.6 Lemma (of KPU). *For each* α, $<_{\mathrm{L}} \restriction \mathrm{L}(\alpha) \times \mathrm{L}(\alpha)$ *well orders* $\mathrm{L}(\alpha)$ *in such a way that for* $\beta < \gamma < \alpha$, *if* $x \in \mathrm{L}(\beta)$, $y \in \mathrm{L}(\gamma) - \mathrm{L}(\beta)$ *then* $x <_{\mathrm{L}} y$.

Proof. By induction on α. ☐

3.7 Theorem. *If* α *is an admissible ordinal then* $\mathrm{L}(\alpha)$ *is a recursively listed admissible set.*

Proof. Since, for $x, y \in \mathrm{L}$, $\neg(x <_{\mathrm{L}} y)$ iff $x = y \vee y <_{\mathrm{L}} x$, we see that $<_{\mathrm{L}}$ is Δ_1 when restricted to L. Also we can define

$$\begin{aligned} p(x) &= \{y \in \mathrm{L}(F(x)+1) \mid y <_{\mathrm{L}} x\} \\ &= \{y \in \mathrm{L} \mid y <_{\mathrm{L}} x\} \end{aligned}$$

for all $x \in \mathrm{L}(\alpha)$ so p is α-recursive, and

$$x <_{\mathrm{L}} y \quad \text{iff} \quad x \in p(y)$$

so we may apply Proposition 3.5. ☐

3.8—3.11 Exercises

3.8. An admissible set $\mathbb{A}_{\mathfrak{M}}$ is *resolvable* if there is an $\mathbb{A}_{\mathfrak{M}}$-recursive function f with $\operatorname{dom}(f) = o(\mathbb{A}_{\mathfrak{M}})$ such that $\mathbb{A}_{\mathfrak{M}} = \bigcup \operatorname{rng}(f)$.

i) Show that if $\mathbb{A}_{\mathfrak{M}}$ is resolvable then there is a function f with the above properties which also satisfies: $f(\beta)$ is always transitive and $\beta < \gamma$ implies $f(\beta) \in f(\gamma)$. Such an f is a *resolution* of $\mathbb{A}_{\mathfrak{M}}$.

ii) A well-founded relation $\prec$ is a *pre-wellordering* if for all $x, y \in \mathrm{Field}(\prec)$,

$$\rho^{\prec}(x) < \rho^{\prec}(y) \quad \text{implies} \quad x \prec y .$$

Show that an admissible set $\mathbb{A}_{\mathfrak{M}}$ is resolvable iff there is a total $\mathbb{A}_{\mathfrak{M}}$-recursive function p with $\mathbb{A}_{\mathfrak{M}} = \bigcup \mathrm{rng}(p)$ such that

$$x \prec y \quad \text{iff} \quad x \in p(y)$$

defines a pre-wellordering of $\mathbb{A}_{\mathfrak{M}}$.

3.9. Show that every admissible set of the form $\mathrm{L}(a, \alpha)$ is resolvable. In particular, $\mathbb{HYP}_{\mathfrak{M}}$ is resolvable.

3.10. Let $\mathrm{L}(a, \alpha)$ be admissible and assume that there is a well-ordering $\prec$ of a, $\prec$ an element of $\mathrm{L}(a, \alpha)$. Modify the definition of $<_{\mathrm{L}}$ to show that $\mathrm{L}(a, \alpha)$ is recursively listed. In particular, if $\mathcal{N} = \langle \omega, +, \times \rangle$ and if $\mathrm{L}(\alpha)_{\mathcal{N}}$ is admissible then it is recursively listed. Hence $\mathbb{HYP}_{\mathcal{N}}$ is recursively listed.

3.11. Let $\mathbb{A}$ be admissible, $\prec \in \mathbb{A}$, $\prec$ not well-founded but

$$\mathbb{A} \models \text{“}\prec \text{is well founded”} .$$

(In other words, every subset X of $\mathrm{Field}(\prec)$ which happens to be an element of $\mathbb{A}$ has a $\prec$-minimal element.) Show that $\rho(\prec) = o(\mathbb{A})$.

4. *Recursively Listed Admissible Sets*

In this section we show how the elementary parts of the theory of r.e. sets generalize from ω-recursion theory to any recursively listed admissible set.

4.1 Theorem. *Let $\mathbb{A} = \mathbb{A}_{\mathfrak{M}}$ be a recursively listed admissible set, with $\alpha = o(\mathbb{A})$, and let B be a nonempty subset of $\mathbb{A}$. The following are equivalent:*

(i) *B is* $\mathbb{A}$-r.e.

(ii) *B is the range of a total $\mathbb{A}$-recursive function.*

(iii) *B is the range of an $\mathbb{A}$-recursive function with domain α.*

Proof. We have (iii)$\Rightarrow$(ii) since there is an $\mathbb{A}$-recursive bijection e of α onto $M \cup A$. Clearly (ii)$\Rightarrow$(i) so we prove (i)$\Rightarrow$(iii). Let

$$x \in B \quad \text{iff} \quad \mathbb{A} \models \exists y\, \varphi(x, y)$$

where φ is Δ_0. Fix $x_0 \in B$. Define an $\mathbb{A}$-recursive f by

$$\begin{aligned} f(\beta) &= 1^{\text{st}} e(\beta) \quad \text{if} \quad \varphi(1^{\text{st}} e(\beta), 2^{\text{nd}} e(\beta)) \\ &= x_0 \quad \text{otherwise}. \end{aligned}$$

Then $B = \operatorname{rng}(f)$ and $\alpha = \operatorname{dom}(f)$. $\square$

4.2 Reduction Theorem. *Let $\mathbb{A} = \mathbb{A}_{\mathfrak{M}}$ be a recursively listed admissible set. For any pair B, C of $\mathbb{A}$-r.e. sets there is a pair B_0, C_0 of disjoint $\mathbb{A}$-r.e. sets with $B_0 \subseteq B$, $C_0 \subseteq C$ and $B_0 \cup C_0 = B \cup C$.*

Proof. We may assume B and C are nonempty. Use 4.1 to choose $\mathbb{A}$-recursive functions F, G with domain $o(\mathbb{A})$ such that

$$B = \operatorname{rng}(F), \qquad C = \operatorname{rng}(G).$$

Define B_0 and C_0 by:

$$\begin{aligned} x \in B_0 \quad &\text{iff} \quad \exists \beta [F(\beta) = x \wedge \forall \gamma < \beta \, G(\gamma) \neq x] \\ x \in C_0 \quad &\text{iff} \quad \exists \gamma [G(\gamma) = x \wedge \forall \beta \leqslant \gamma \, F(\beta) \neq x] \end{aligned}$$

Then clearly B_0 and C_0 are disjoint $\mathbb{A}$-r.e. sets with $B_0 \subseteq B$, $C_0 \subseteq C$. If $x \in B - C$ then $x \in B_0$. If $x \in C - B$ then $x \in C_0$. If $x \in B \cap C$ then let β be the least ordinal with $F(\beta) = x$, γ the least with $G(\gamma) = x$. If $\beta \leqslant \gamma$ then $x \in B_0$ but if $\beta > \gamma$ then $x \in C_0$ so $B \cup C \subseteq B_0 \cup C_0$. $\square$

4.3 Corollary (Separation Theorem). *Let $\mathbb{A} = \mathbb{A}_{\mathfrak{M}}$ be a recursively listed admissible set. For any pair B, C of disjoint Π_1 sets on $\mathbb{A}$ there is an $\mathbb{A}$-recursive set containing B but disjoint from C.*

Proof. Apply 4.2 to $\mathbb{A} - B$, $\mathbb{A} - C$ to get disjoint sets B_0, C_0 with $B \subseteq B_0$, $C \subseteq C_0$, $B_0 \cup C_0 = A$. Then B_0 is $\mathbb{A}$-recursive. $\square$

4.4 Uniformization Theorem. *Let $\mathbb{A} = \mathbb{A}_{\mathfrak{M}}$ be a recursively listed admissible set and let R be an $\mathbb{A}$-r.e. binary relation. There is an $\mathbb{A}$-recursive function F with*

(i) $\operatorname{dom}(F) = \{x \mid \exists y R(x, y)\}$

(ii) *for* $x \in \operatorname{dom}(F)$,

$$R(x, F(x)).$$

Proof. Let e be an $\mathbb{A}$-recursive bijection of $o(\mathbb{A})$ onto $\mathbb{A}$. Let R be given by

$$R(x, y) \quad \text{iff} \quad \exists z \, S(x, y, z)$$

where S is $\mathbb{A}$-recursive. Define F by

$$F(x)=y \quad \text{iff} \quad \exists\beta[S(x,1^{\text{st}}e(\beta),2^{\text{nd}}e(\beta))\wedge\forall\gamma<\beta\,\neg S(x,1^{\text{st}}e(\gamma),2^{\text{nd}}e(\gamma))\wedge y=1^{\text{st}}e(\beta)]. \quad \square$$

The passage, in 4.4, from the Σ_1 definition of R to the Σ_1 definition of F was explicitly given, so we can get the following more complicated but stronger result. For $z\in\mathbb{A}$ we let

$$W_z^2=\{(x,y)\mid T_2(z,x,y)\}$$

where T_2 is the $\mathbb{A}$-r.e. relation which parametrizes the $\mathbb{A}$-r.e. binary relations, as it was defined in § 1.

4.5 Theorem. *Let $\mathbb{A}$ be a recursively listed admissible set. There is a total $\mathbb{A}$-recursive function G such that for all $z\in\mathbb{A}$:*
(i) *$W_{G(z)}^2$ is the graph of an $\mathbb{A}$-recursive function,*
(ii) *$W_{G(z)}^2\subseteq W_z^2$, and*
(iii) *$\operatorname{dom}(W_z^2)=\operatorname{dom}(W_{G(z)}^2)$.*

Proof. See 4.4 and remarks following it. $\square$

Using this we get the following analogue of Kleene's T-predicate for recursive partial functions.

4.6 Theorem. *Let $\mathbb{A}$ be a recursively listed admissible set. There is an $\mathbb{A}$-r.e. predicate T_2^* of three arguments which parametrizes the collection of all partial $\mathbb{A}$-recursive functions, with indices from the ordinals of $\mathbb{A}$.*

Proof. Let $e: o(\mathbb{A})\to\mathbb{A}$ be a recursive listing and let G be as given in 4.5. Define

$$T_2^*(\beta,x,y) \quad \text{iff} \quad T_2(G(e(\beta)),x,y).$$

Then for each β,

$$f_\beta=\{\langle x,y\rangle\mid T_2^*(\beta,x,y)\}$$

is a partial function with Σ_1 graph (by 4.4i). If $f=W_z^2$ then pick β so that $e(\beta)=z$. Then since

$$W_{G(z)}=W_z$$

by 4.4, $f_\beta=f$. $\square$

4.7 Corollary. *Let $\mathbb{A}$ be a recursively listed admissible set. There are disjoint $\mathbb{A}$-r.e. sets which cannot be separated by an $\mathbb{A}$-recursive set.*

Proof. Let B, C be the disjoint $\mathbb{A}$-r.e. sets defined by

$$B = \{\beta \mid T_2^*(\beta, \beta, 0)\}$$
$$C = \{\beta \mid T_2^*(\beta, \beta, 1)\}$$

where T_2^* is given in Theorem 4.6. Suppose D were an $\mathbb{A}$-recursive set with $B \subseteq D$, $C \cap D = 0$. Let

$$\begin{aligned} g(x) &= 1 \quad \text{if} \quad x \in D \\ &= 0 \quad \text{if} \quad x \notin D \end{aligned}$$

so that g is $\mathbb{A}$-recursive. Pick β so that

$$g(x) = y \quad \text{iff} \quad T_2^*(\beta, x, y).$$

If $\beta \in D$ then $g(\beta) = 1$ so $T_2^*(\beta, \beta, 1)$ which implies $\beta \in C$, but $C \cap D = 0$. If $\beta \notin D$ then $g(\beta) = 0$ so $T_2^*(\beta, \beta, 0)$ which implies $\beta \in B$, but $B \subseteq D$. But $\beta \in D$ or $\beta \notin D$ so we have a contradiction in either case. Thus there can be no such D. ▯

It is an open problem to determine whether the conclusion of 4.7 holds for arbitrary admissible sets.

4.8—4.10 Exercises

4.8. Let $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$ be countable and suppose there is a well-ordering of M which is Δ_1^1 on $\mathfrak{M}$. Prove the following:

(i) Let B be a Π_1^1 subset of $\mathfrak{M}$. There is a function F with domain $o(\mathbb{HYP}_{\mathfrak{M}})$ such that

$$B = \bigcup\nolimits_{\alpha < o(\mathbb{HYP}_{\mathfrak{M}})} F(\alpha)$$

and for each $\beta < o(\mathbb{HYP}_{\mathfrak{M}})$

$$\bigcup\nolimits_{\alpha < \beta} F(\alpha)$$

is Δ_1^1 on M. [Pick an F which is $\mathbb{HYP}_{\mathfrak{M}}$ recursive.]

(ii) (Reduction) If B, C are Π_1^1 subsets of $\mathfrak{M}$ then there are disjoint Π_1^1 subsets $B_0 \subseteq B$, $C_0 \subseteq C$ with $B_0 \cup C_0 = B \cup C$.

(iii) (Separation) If B, C are disjoint Σ_1^1 subsets of $\mathfrak{M}$ then there is a Δ_1^1 set D with

$$B \subseteq C, \qquad C \cap D = 0.$$

(iv) (Uniformization) If $R \subseteq M \times M$ is Π_1^1 on $\mathfrak{M}$ there is a Π_1^1 subrelation $R_0 \subseteq R$ such that

$$\mathrm{dom}(R_0) = \mathrm{dom}(R)$$
$$x \in \mathrm{dom}(R_0) \Rightarrow \exists! y R_0(x, y).$$

If $\mathrm{dom}(R)=M$ then R_0 is a Δ_1^1 relation.

4.9. Show that for any admissible $\mathbb{A}_{\mathfrak{M}}$ and any $B \subseteq \mathbb{A}_{\mathfrak{M}}$, B is $\mathbb{A}_{\mathfrak{M}}$-r.e. iff $B=\mathrm{dom}(f)$ for some $\mathbb{A}_{\mathfrak{M}}$-recursive function f.

4.10. Show that if $\mathbb{A}_{\mathfrak{M}}$ is resolvable then the Reduction and Separation Theorems, 4.2 and 4.3, still hold. In particular, show that 4.8(i), (ii), (iii) hold without the hypothesis that $\mathfrak{M}$ has a Δ_1^1 well-ordering.

5. Notation Systems and Projections of Recursion Theory

An important stimulus in the earlier development of admissible ordinals was the desire to understand the analogy between Π_1^1 and r.e. sets of natural numbers. The metarecursion theory of Kreisel-Sacks [1965] explained this by developing a recursion theory on ω_1^c, the first nonrecursive ordinal, with the property that a set of natural numbers in Π_1^1 on ω iff it is ω_1^c-r.e. The theory was developed by using a notation system for the recursive ordinals to define the notions of ω_1^c-recursive, ω_1^c-r.e. and ω_1^c-finite.

The development by means of admissible sets proceeds the other way around. Instead of using known facts about Π_1^1 sets to develop a recursion theory on ω_1^c by means of a notation system, we have a recursion theory given on ω_1^c (it is the first admissible ordinal $>\omega$; see 5.11) and then transfer the results to Π_1^1 subsets of ω via a notation system.

5.1 Definition. Let $\mathbb{A}=\mathbb{A}_{\mathfrak{M}}$ be admissible.

(i) A *notation system for* $\mathbb{A}$ is a total $\mathbb{A}$-recursive function π such that if $x \neq y$ then $\pi(x)$ and $\pi(y)$ are disjoint non-empty sets. (We think of $\pi(x)$ as a set of *notations for* x.)

(ii) The *domain* of a notation system π, D_π, is defined by (!)

$$D_\pi = \bigcup\nolimits_{x \in A} \pi(x). \quad \text{(Thus } D_\pi \text{ is the set of all notations.)}$$

(iii) Associated with a notation system π is a function $|\cdot|_\pi$ with domain D_π and range $A \cup M$ defined by

$$|y|_\pi = x \quad \text{iff} \quad y \in \pi(x).$$

(Thus, for any notation y, y is a notation for $|y|_\pi$.)

(iv) $\mathbb{A}$ is *projectible into* C if C is $\mathbb{A}$-r.e. and there is a notation system π with $D_\pi \subseteq C$.

It is best to think of the notation system as the triple D_π, $|\cdot|_\pi$, π even though the first two can be defined in terms of the third. We require C to be $\mathbb{A}$-r.e. in (iv) only because that is the only kind of C that interests us in this context.

5.2 Lemma. *Let π be a notation system for the admissible set $\mathbb{A}$.*

(i) *π is a one-one function.*
(ii) *D_π is $\mathbb{A}$-r.e. but not $\mathbb{A}$-finite.*
(iii) *The graph of $|\cdot|_\pi$ is an $\mathbb{A}$-*recursive *relation. In particular, $|\cdot|_\pi$ is an $\mathbb{A}$-recursive function.*

Proof. The only part which is not absolutely immediate is the fact that D_π is not $\mathbb{A}$-finite. But if $D_\pi \in \mathbb{A}$ then, by Σ Replacement, the range of $|\cdot|_\pi$ would be an element of $\mathbb{A}$ whereas this range is all of $M \cup A$. $\square$

Our plan for this section is to first exhibit some useful notation systems and then use them to transfer results.

5.3 Theorem.

(i) *For any structure $\mathfrak{M}$, $\mathbb{HYP}_\mathfrak{M}$ is projectible into $\mathbb{HF}_\mathfrak{M}$.*
(ii) *For any admissible set $\mathbb{A}$, $\mathbb{HYP}(\mathbb{A})$ is projectible into $\mathbb{A}$.*

The theorem is a simple consequence of the following lemma, an effective version of Theorem II.5.14.

5.4 Lemma. *Let L be a finite language, let $\mathfrak{M}$ be a structure for L, let $\mathsf{L}^* = \mathsf{L}(\in)$ and let $a \in \mathbb{V}_\mathfrak{M}$ be a transitive set with $M \subseteq a$. Let $\mathsf{L}' = \mathsf{L}^* \cup \{\overline{\mathsf{x}} \mid x \in a \cup \{a\}\}$ be the usual language with constant symbol $\overline{\mathsf{x}}$ for x. Let α be the least ordinal such that*

$$\mathbb{A}_\mathfrak{M} = (\mathfrak{M}; \mathrm{L}(a, \alpha), \in)$$

is admissible and assume L' is coded up on $\mathbb{A}_\mathfrak{M}$ in a way that makes the syntactic operations of $\mathsf{L}'_{\omega\omega}$ all $\mathbb{A}_\mathfrak{M}$-recursive. There is a total $\mathbb{A}_\mathfrak{M}$-recursive function π such that for each $x \in \mathbb{A}_\mathfrak{M}$, $\pi(x)$ is a set of good Σ_1 definitions of x with parameters from $a \cup \{a\}$.

Proof. We already know, from Theorem II.5.14, that each $x \in \mathbb{A}_\mathfrak{M}$ *has* a good Σ_1 definition with parameters from $a \cup \{a\}$. The object here is to use the Second Recursion Theorem to show how we can go $\mathbb{A}_\mathfrak{M}$-recursively from x to a set $\pi(x)$ of good Σ_1 definitions of x, by reexamining the proof of II.5.14. If we look back at that proof we see that this is really pretty obvious. We write out clauses in the definition of π. In each case it is assumed that none of the earlier cases hold. We also arrange things so that v is the only free variable in any formula considered.

Case 1ne. If $x \in a \cup \{a\}$ then $\pi(x)$ is the set whose only member is the $\mathsf{L}'_{\omega\omega}$ Δ_0 formula

$$v = \overline{\mathsf{x}}.$$

Case 2wo. If $x=\beta+1$ then $\pi(x)$ is the set of formulas

$$\exists w[v=\mathscr{S}(w) \wedge \varphi(w/v)]$$

where $\varphi(v)\in\pi(\beta)$ and w is the first variable not in $\varphi(v)$.

Case 3hree. If $\pi(\beta)$ is defined then $\pi(\mathrm{L}(a,\beta))$ is the set of formulas of the form

$$\exists w[v=\mathrm{L}(\bar{\mathsf{a}},w) \wedge \varphi(w/v)]$$

where $\varphi(v)\in\pi(\beta)$. We may use "$v=\mathrm{L}(\bar{\mathsf{a}},w)$" since $\mathrm{L}(\cdot,\cdot)$ is a Σ_1 operation symbol.

Case 4our. If $x\in\mathrm{L}(a,\beta+1)-\mathscr{S}(L(a,\beta))$ then $\pi(x)$ is defined as follows. Find the least i, $1\leqslant i\leqslant N$, such that for some $y,z\in\mathrm{L}(a,\beta)\cup\{\mathrm{L}(a,\beta)\}$,

$$x=\mathscr{F}_i(y,z)\,.$$

Then $\pi(x)$ is the set of all formulas of the form

$$\exists w_1\,\exists w_2[v=\mathscr{F}_i(w_1,w_2)\wedge\varphi(w_1/v)\wedge\psi(w_2/v)]$$

where, for some $y,z\in\mathscr{S}(L(a,\beta))$, $x=\mathscr{F}_i(y,z)$ and $\varphi(v)\in\pi(y)$ and $\psi(v)\in\pi(z)$ and w_1,w_2 are the first two distinct variables not appearing anywhere in φ or ψ. The set of all such formulas exists by Σ Replacement. This clause in the definition of $\pi(x)=y$ is Σ, as can be seen by writing it out.

Case 5ive. If $\beta<\alpha$ is a limit ordinal then $\pi(\beta)$ is defined $\mathbb{A}_{\mathfrak{M}}$-effectively as follows. Find the first Δ_0 formula $\varphi(x,y,z_1,\ldots,z_n)$ of L^* (first in some effective well-ordering of $\mathbb{HF}$, say that given by II.2.4 or 3.7 of this chapter) such that for some $d,z_1,\ldots,z_n\in\mathrm{L}(a,\beta)$

$$\mathrm{L}(a,\beta)\models\forall x\in d\,\exists y\,\varphi(x,y,z_1,\ldots,z_n) \tag{1}$$

but

$$\mathrm{L}(a,\beta)\models\neg\exists b\,\forall x\in d\,\exists y\in b\,\varphi(x,y,z_1,\ldots,z_n)\,. \tag{2}$$

Now given φ let $\theta(\beta)(=\theta(\beta,d,z_1,\ldots,z_n))$ be formed from φ just as in the proof of II.5.14. Let $\pi(\beta)$ be the set of all formulas of the form

$$\exists w,w_1,\ldots,w_n[\theta(v,w,w_1,\ldots,w_n)\wedge\psi(w/v)\wedge\textstyle\bigwedge_{j=1}^n\sigma_j(w_j/v)]$$

such that for some $d,z_1,\ldots,z_n\in\mathrm{L}(a,\beta)$, (1) and (2) hold and $\psi\in\pi(d)$ and, for $1\leqslant j\leqslant n$, $\sigma_j(v)\in\pi(z_j)$. Again, this clause in the definition of $\pi(x)=y$ can be seen to be Σ and so, by the Second Recursion Theorem, π is an $\mathbb{A}_{\mathfrak{M}}$-recursive function. $\square$

Proof of 5.3. For (i) simply note that $\mathsf{L}'_{\omega\omega}$ can be coded up on $\mathbb{HF}_{\mathfrak{M}}$ in this case. For (ii) we can code $\mathsf{L}'_{\omega\omega}$ on $\mathbb{A}_{\mathfrak{M}}$ itself. The admissibility of $\mathbb{A}_{\mathfrak{M}}$ comes in only in that this coding can be done on $\mathbb{A}_{\mathfrak{M}}$ and is far stronger than we need. □

We will see in §VI.4 that if $\mathfrak{M}$ has a "built in pairing function" then $\mathbb{HYP}_{\mathfrak{M}}$ is projectible into $\mathfrak{M}$.

5.5 Corollary. *Let $\mathcal{N}=\langle\omega,+,\cdot\rangle$ be the structure of the natural numbers. $\mathbb{HYP}_{\mathcal{N}}$ is projectible into $\mathcal{N}$.*

Proof. The simplest proof is just to observe that in this case the coding used in the proof of 5.3 (i) can be done on $\mathcal{N}$ itself. An alternate explicit proof will appear in §VI.4. □

We now give some examples of the use of notation systems. Combined with 5.5 and the results of §IV.3, the next two results show that, over $\mathcal{N}$, the Π_1^1 relations are parameterized by a Π_1^1 relation, that there are Π_1^1 sets which are not Δ_1^1, and that there are Δ_1^1 sets which are not first order definable over $\mathcal{N}$.

5.6 Theorem. *Let $\mathbb{A}$ be an admissible set which is projectible into C.*

(i) *For $n\geq 1$ there is an $(n+1)$-ary $\mathbb{A}$-r.e. relation S on C which parametrizes the class of all n-ary relations on C which are $\mathbb{A}$-r.e.*

(ii) *There is subset of C which is $\mathbb{A}$-r.e. but not $\mathbb{A}$-recursive.*

Proof. (ii) follows from (i) just as in the proof of 1.7. To prove (i) let π be a notation system for $\mathbb{A}$ with $D_\pi\subseteq C$. Let T_n be the $(n+1)$-ary relation on $\mathbb{A}$ which parametrizes the n-ary $\mathbb{A}$-r.e. relations. Define

$$S(y,x_1,\ldots,x_n) \quad\text{iff}\quad x_1,\ldots,x_n\in C\,,\quad y\in D_\pi \quad\text{and}\quad T_n(|y|_\pi,x_1,\ldots,x_n)\,.$$

S is $\mathbb{A}$-r.e. since C and D_π are $\mathbb{A}$-r.e. and $|\cdot|_\pi$ is $\mathbb{A}$-recursive. Now let $R\subseteq C^n$ be $\mathbb{A}$-r.e. Pick a z such that

$$R(x_1,\ldots,x_n) \quad\text{iff}\quad T_n(z,x_1,\ldots,x_n)\,.$$

Then for any $y\in\pi(z)$,

$$R(x_1,\ldots,x_n) \quad\text{iff}\quad S(y,x_1,\ldots,x_n)\,. \quad \square$$

5.7 Theorem. *Let $\mathbb{A}$ be an admissible set with $o(\mathbb{A})>\omega$. Let $\mathfrak{N}=\langle N,\ldots\rangle$ be a structure (for a language K) which is an element of $\mathbb{A}$ and suppose that $\mathbb{A}$ is projectible into N.*

(i) *There is an $\mathbb{A}$-recursive $(n+1)$-ary relation S on N which parametrizes the n-ary relations on $\mathfrak{N}$ which are first order definable over $\mathfrak{N}$ (using parameters).*

(ii) *There is a subset of $\mathfrak{N}$ which is $\mathbb{A}$-recursive but not first order definable over $\mathfrak{N}$.*

Proof. As usual (ii) follows from (i) by diagonalization. To prove (i) define

$S_0(y, x_1, \ldots, x_n)$ iff $y = \langle \varphi, s \rangle$ where $\varphi(v_1, \ldots, v_n, w_1, \ldots, w_m)$ is a formula of $\mathsf{K}_{\omega\omega}$ and s is an assignment with values in $\mathfrak{N}$, $s(v_i) = x_i$ all $i \leqslant n$, and $\mathfrak{N} \models \varphi[s]$.

S_0 is clearly Δ_1 on $\mathbb{A}$. Since $o(\mathbb{A}) > \omega$ the set X of all relevant pairs $\langle \varphi, s \rangle$ is an element of $\mathbb{A}$. Let π be the notation system for $\mathbb{A}$ with $D_\pi \subseteq N$. Define

$$S(z, x_1, \ldots, x_n) \quad \text{iff} \quad \exists y \in X [\pi(y) = z \wedge S_0(y, x_1, \ldots, x_n)].$$

Since $X \in \mathbb{A}$, the quantifier on y is bounded so S is indeed $\mathbb{A}$-recursive. It clearly parametrizes the relations definable over $\mathfrak{N}$. $\square$

We now turn to a result, Theorem 5.9, which will allow us to identify $O(\mathcal{N})$. A notation system π is *univalent* if each $\pi(x)$ is a singleton, that is, if it assigns a unique notation to each $x \in \mathbb{A}$.

5.8 Proposition. (i) *Let $\mathbb{A}$ be a recursively listed admissible set projectible into C. There is a univalent notation system which projects $\mathbb{A}$ into C.*

(ii) $\mathbb{HYP}_{\mathcal{N}}$ *has a univalent notation system which projects into $\mathcal{N}$.*

Proof. (i) If π projects $\mathbb{A}$ into C then define π_1, the univalent notation system, by

$$\pi_1(x) = \{y\} \quad \text{where } y \text{ is the first member of } \pi(x).$$

Part (ii) follows from (i) and 3.10. $\square$

5.9 Theorem. *Let $\mathbb{A}$ be an admissible set which is projectible into C.*

$$\begin{aligned} o(\mathbb{A}) &= \{\rho(\prec) \mid \prec \text{ is a well-founded relation, } \prec \subseteq C^2, \prec \in \mathbb{A}\} \\ &= \{\rho(\prec) \mid \prec \text{ is a pre-wellordering, } \prec \subseteq C^2, \prec \in \mathbb{A}\}. \end{aligned}$$

If there is a univalent notation system projecting $\mathbb{A}$ into C then

$$o(\mathbb{A}) = \{\rho(\prec) \mid \prec \text{ a well-ordering, } \prec \subseteq C^2, \prec \in \mathbb{A}\}.$$

Proof. Every well-founded relation $\prec \in \mathbb{A}$ has $\rho(\prec) < o(\mathbb{A})$ by 3.3(ii) so we need only show that each $\beta \in \mathbb{A}$ is of the form $\rho(\prec)$ for some pre-wellordering $\prec \in \mathbb{A}$, $\prec \subseteq C^2$. Let π be a notation system projecting $\mathbb{A}$ into C. Let $b = \bigcup \mathrm{rng}(\pi \restriction \beta) \in \mathbb{A}$. Now $b \subseteq D_\pi \subseteq C$ and b is the set of all notations for ordinals $\gamma < \beta$. Define $\prec \subseteq b \times b$ by

$$x \prec y \quad \text{iff} \quad |x|_\pi < |y|_\pi.$$

Then $\prec$ is a pre-wellordering of b of length β and it is a well-ordering if π happens to be univalent. $\square$

5.10 Corollary. *$O(\mathcal{N}) = \{\rho(\prec) \mid \prec$ is a Δ_1^1 well-ordering, $\prec \subseteq \mathcal{N} \times \mathcal{N}\}$*
$= \omega_1^c$.

Proof. The first equality is immediate 5.9, 5.10 and §IV.3. The second follows from the first and the result from ordinary recursion theory that every Δ_1^1 well-ordering of $\mathcal{N}$ has order type some $\alpha < \omega_1^c$. □

The reader unfamiliar with the result used in the above proof can take

$$\omega_1^c = \{\rho(\prec) \mid \prec \text{ is a } \Delta_1^1 \text{ well-ordering, } \prec \subseteq \mathcal{N} \times \mathcal{N}\}$$

as the definition of ω_1^c.

5.11 Corollary. *ω_1^c is the first admissible ordinal greater than ω.*

Proof. ω_1^c is admissible by 5.10. Let α be the least admissible $> \omega$ so that $L(\alpha)$ is admissible and $\omega_1^c \geqslant \alpha$. But if $\omega_1^c > \alpha$ then α is the order type of some Δ_1^1 well-ordering $\prec$ of $\mathcal{N}$ and hence of some Δ_1^1 well-ordering $\prec$ of ω. But then $\prec \in L(\alpha)$ by §IV.3 which contradicts 3.3(ii). □

5.12—5.13 Exercises

5.12. *For any $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$ show that*

$$O(\mathfrak{M}) = \{\rho(\prec) \mid \prec \textit{ is a pre-wellordering, } \prec \in \mathbb{HYP}_{\mathfrak{M}},\ \prec \subseteq \mathbb{HF}_{\mathfrak{M}}^2\}.$$

5.13. Let $\mathbb{A}$ be a recursively listed admissible set. Show that there is a single-valued notation system with domain $o(\mathbb{A})$. Hence the recursion theory of $\mathbb{A}$ can be transfered to $o(\mathbb{A})$.

5.14 Notes. Notation systems are standard tools in ordinal recursion theory but don't seem to have been treated systematically before over arbitrary admissible sets. The definitions used above are stronger than those of Moschovakis [1974]. In the case where $\mathbb{A}$ is projectible into some $C \in \mathbb{A}$ (the only case of interest to Moschovakis) they are equivalent.

Corollary 5.11 is due to Kripke and Platek, but with more complicated proofs.

6. Ordinal Recursion Theory: Projectible and Recursively Inaccessible Ordinals

In the final sections of this chapter we return to the origins of the theory of admissible sets, recursion theory on admissible ordinals. We are thus in the domain of admissible sets without urelements.

Let τ_β be the β^{th} admissible ordinal; that is, let

$$\tau_0 = \omega,$$

$$\tau_\beta = \text{least } \alpha[\alpha \text{ is admissible } \wedge \alpha > \tau_\gamma \text{ for all } \gamma < \beta].$$

In this section we begin looking at the sequence of admissible ordinals and the relationships between various members of it.

6.1 Definition. An admissible ordinal α is *projectible into* β (where $\beta \leqslant \alpha$) if there is a total α-recursive function mapping α one-one into β. The least β such that α is projectible into β is called the *projectum* of α and is denoted by α^*. If $\alpha^* < \alpha$ then α is said to be *projectible*; otherwise α is *nonprojectible*.

If α is admissible then $\mathrm{L}(\alpha)$ is recursively listed so we see that α is projectible into β in the sense of 6.1 iff $\mathrm{L}(\alpha)$ is projectible into β in the sense of 5.1 (iv). Similarly, if β is also admissible then α is projectible into β (in the sense of 6.1) iff $\mathrm{L}(\alpha)$ is projectible into $\mathrm{L}(\beta)$ (in the sense of 5.1(iv)).

6.2 Proposition

(i) *If $\kappa \geqslant \omega$ is a cardinal then κ is nonprojectible.*
(ii) *For any $\beta, \tau_{\beta+1}$ is projectible into τ_β.*
(iii) *If τ_β is projectible into τ_γ und τ_γ is projectible into δ then τ_β is projectible into δ.*

Proof. (i) is obvious by cardinality considerations since otherwise κ would have the same cardinality as some $\beta < \kappa$. For (ii), note that $\mathrm{L}(\tau_{\beta+1}) = \mathbb{HYP}(\mathrm{L}(\tau_\beta))$ so $\mathrm{L}(\tau_{\beta+1})$ is projectible into $\mathrm{L}(\tau_\beta)$ by 5.3. Part (iii) is obvious. We simply compose projections. □

From this proposition we see that there are many projectible ordinals. We also see that $\tau_n^* = \omega$ for all $n = 0, 1, 2, \ldots$.

As we mentioned at the beginning of this chapter, one use of generalized recursion theory is as a laboratory for understanding ordinary recursion theory. One important aspect of ordinary recursion theory is the number of different versions of the notion of finite that arise. For examples, a set $B \subseteq \omega$ is finite iff any one of the following hold: B is recursive and bounded, B is R.E. and bounded, or B is bounded. By defining a set $B \subseteq \mathrm{L}(\alpha)$ to be α-finite if $B \in \mathrm{L}(\alpha)$ we have chosen to use the first. This means that when we meet some use of a different version of "finite" in ordinary recursion theory we may have trouble lifting this to α-recursion theory. The following theorem shows us that if α is projectible then there are going to be α-r.e. subsets of ordinals $\beta < \alpha$ which are *not* α-finite. Thus, for projectible ordinals we may expect some aspects of ordinary recursion theory to become more subtle. This is particularly true in the study of α-degrees, a subject not treated in this book.

6.3 Theorem. *Let α be admissible. The following are equivalent:*

(i) *α is nonprojectible.*

(ii) $L(\alpha) \models \Sigma_1$ Separation.

(iii) *If $\beta < \alpha$ and B is an α-r.e. subset of β then B is α-finite.*

Proof. We first prove (i)$\Rightarrow$(ii). Suppose $B \subseteq a \in L(\alpha)$ and B is Σ_1 definable on $L(\alpha)$. We wish to prove $B \in L(\alpha)$. Pick $\beta_0 < \alpha$ such that $a \in L(\beta_0)$; hence $B \subseteq L(\beta_0)$. The recursive listing f of $L(\alpha)$ given by $<_L$ puts everything in $L(\beta_0)$ before everything in $L(\alpha) - L(\beta_0)$. If we show that the set

$$C = \{\gamma \mid f(\gamma) \in B\}$$

is an element of $L(\alpha)$, then $B \in L(\alpha)$ by Σ Replacement. But $C \subseteq \beta_1$ for some $\beta_1 < \alpha$. Use 4.1 to pick an α-recursive function G mapping α onto C and define H by Σ Recursion as follows:

$$H(\beta) = G(\text{least } \gamma [G(\gamma) \notin \{H(\delta) \mid \delta < \beta\}]) .$$

Now H is α-recursive, one-one, and is defined on some initial segment of α. It cannot be defined for all $\beta < \alpha$, however, for this would give a projection of α into $\beta_1 < \alpha$ and α is nonprojectible. Let β_2 be the least ordinal for which H is not defined. The only reason $H(\beta_2)$ can be undefined is that

$$C = \{H(\beta) \mid \beta < \beta_2\}$$

so that $C \in L(\alpha)$ by Σ Replacement.

The implication (ii)$\Rightarrow$(iii) is trivial. We prove (iii)$\Rightarrow$(i) by contraposition. Thus, let $p: \alpha \to \beta$ be an α-recursive one-one mapping of α into β, $\beta < \alpha$, and let $B = \text{rng}(p)$. Then B is α-r.e., $B \subseteq \beta$ but B cannot be α-finite, since

$$\alpha = \{p^{-1}(x) \mid x \in B\}$$

and p^{-1} is α-recursive. □

6.4 Corollary. *If α is projectible into β then there is an α-r.e. subset of β which is not α-finite.* □

6.5 Corollary. *If α is nonprojectible then* $L(\alpha) \models$ *Beta.*

Proof. $L(\alpha) \models \Sigma_1$ Separation, and Σ_1 Separation implies Beta. □

6.6 Corollary. *Let κ be an uncountable cardinal. For every $\beta < \kappa$ there is a nonprojectible α between β and κ.*

Proof. $L(\kappa) \models \Sigma_1$ Separation, so apply Theorem II.3.3 with $\mathbb{A}_{\mathfrak{M}} = L(\kappa)$, $A_0 = \beta$. The resulting admissible set satisfies the axiom $V = L$ (i.e. $\forall x\, L(x)$) and so is $L(\alpha)$ for some $\alpha < \kappa$. Since $L(\alpha) \equiv L(\kappa)$, $L(\alpha) \models \Sigma_1$ Separation and hence α is nonprojectible. □

Now that we know there are lots of nonprojectible ordinals we can ask how big the first one is. So far, all we know is that it is bigger than τ_n for each $n<\omega$. Is it τ_ω? To shed some light on the size of the first nonprojectible we introduce the recursively inaccessible ordinals.

6.7 Definition. An admissible ordinal α is *recursively inaccessible* if α is the least upper bound of all admissibles less than α.

6.8 Theorem. *If α is nonprojectible and greater than ω then α is recursively inaccessible.*

Proof. Assume that α is admissible, $\alpha>\omega$ but that the ordinal

$$\beta=\sup\{\gamma<\alpha \mid \gamma \text{ is admissible}\}$$

is less than α. We will prove that α is projectible into β. Let $e\colon \alpha\to L(\alpha)$ be the recursive listing of $L(\alpha)$ given by $<_L$. Since β is a sup of admissible ordinals, $e\restriction\beta$ is the canonical listing of $L(\beta)$ by ordinals $<\beta$. Thus, if $L(\alpha)$ were projectible into $L(\beta)$, then it would be projectible into β and so α would be projectible with $\alpha^*\leqslant\beta$. But $L(\alpha)$ is the smallest admissible set with $L(\beta)$ as an element, i.e. $L(\alpha)=\mathbb{HYP}(L(\beta))$ so $L(\alpha)$ is projectible into $L(\beta)$ by Lemma 5.4. ▯

If we combine Theorem 6.8 with the next result we see that the first nonprojectible is fairly large, much larger than τ_ω.

6.9 Theorem. *If τ_α is recursively inaccessible then $\tau_\alpha=\alpha$, and conversely.*

We isolate part of the proof of 6.9 which will be used again.

6.10 Lemma. *Define $G(\beta)=\tau_\beta$ for $\beta<\alpha$. Then G is a τ_α-recursive function.*

Proof. The result is literally trivial if $\alpha=0$. For $\alpha>0$ we can define G by

$$G(0)=\omega,$$
$$G(\beta)=\text{least}\,\gamma\,[L(\gamma)\models KP \wedge \gamma\notin\{G(\delta)\mid \delta<\beta\}]$$

for $\beta<\alpha$. Since KP is an ω-recursive set of axioms, it is in $L(\tau_\alpha)$ so this is a Σ Recursive definition of G. ▯

Proof of 6.9. Note first that $\tau_\alpha\geqslant\alpha$ for all α, by induction. Suppose $\tau_\alpha=\alpha$. Then for each $\beta<\tau_\alpha$, $\beta\leqslant\tau_\beta<\tau_\alpha$ so τ_α is the sup of all smaller admissibles. Now suppose τ_α is recursively inaccessible, but that $\tau_\alpha>\alpha$. Note that α is a limit ordinal, since $\tau_{\beta+1}$ can never be recursively inaccessible. Let G be as in Lemma 6.10 and observe that

$$\tau_\alpha=\sup\{G(\beta)\mid\beta<\alpha\}.$$

But this is a contradiction, for G is τ_α-recursive and hence the right-hand side of this equality is in $L(\tau_\alpha)$ by Σ Replacement. □

We see, by 6.9, that none of the following are recursively inaccesible and, hence, all are projectible:

$$\tau_1, \tau_2, \ldots, \tau_\omega, \tau_{\omega+1}, \ldots, \tau_{\tau_1}, \tau_{\tau_1+1}, \ldots, \tau_{\tau_2}, \ldots, \tau_{\tau_{\tau_2}}, \quad \text{etc.}$$

What are their projectums? We will show in the next section that all are projectible into ω by showing that projectums are always admissible.

The interest in projectums stems largely from the following property which is quite useful in priority arguments involving α-degrees.

6.11 Theorem. *Let α be admissible and let α^* be its projectum. If B is α-r.e., $B \subseteq \beta$ for some $\beta < \alpha^*$, then B is α-finite.*

Proof. The proof is like the proof of (i) $\Rightarrow$ (ii) in Theorem 6.3. Define an α-recursive function F by

$$F(\gamma) = \gamma^{\text{th}} \quad \text{member of } B$$

i.e.

$$F(\gamma) = G(\text{least}\, \delta(G(\delta) \notin \{F(\xi) \mid \xi < \gamma\}))$$

where G maps α onto B. Now, since $\beta < \alpha^*$, F cannot be a one-one mapping of α into β. Thus $F(\gamma)$ is undefined for some $\gamma < \alpha$. If γ_0 is the least such then $B = \{F(\gamma) : \gamma < \gamma_0\}$ so $B \in L(\alpha)$ by Σ Replacement. □

To prove stronger facts about nonprojectible ordinals we need to use the notion of *stable* ordinal introduced in the next section.

6.12 Exercise. Let α be a limit of admissibles. Prove that $L(\alpha) \models \text{Beta}$ even if α is not admissible. This is an improvement of 6.5.

6.13 Notes. The concepts and results of this section are all due to Kripke and Platek. The student interested in the uses of the projectum in the study of α-degrees should consult Simpson's excellent survey article, Simpson [1974].

7. Ordinal Recursion Theory: Stability

Given structures $\mathfrak{A}_{\mathfrak{M}} \subseteq \mathfrak{B}_{\mathfrak{N}}$, we write $\mathfrak{A}_{\mathfrak{M}} \prec_1 \mathfrak{B}_{\mathfrak{N}}$ if for every Σ_1 formula $\varphi(v_1, \ldots, v_n)$ and every $x_1, \ldots, x_n \in \mathfrak{A}_{\mathfrak{M}}$,

$$\mathfrak{B}_{\mathfrak{N}} \models \varphi[x_1, \ldots, x_n] \quad \text{iff} \quad \mathfrak{A}_{\mathfrak{M}} \models \varphi[x_1, \ldots, x_n].$$

7.1 Definition. An ordinal α is *stable* if $L(\alpha) \prec_1 L$. The sequence of stable ordinals is defined by

$$\sigma_0 = \text{the least stable ordinal},$$
$$\sigma_\gamma = \text{the least stable ordinal greater than each } \sigma_\beta \text{ for } \beta < \gamma.$$

The first theorem shows that there are lots of stable ordinals and that they are better behaved under sups than the admissible ordinals.

7.2 Theorem. (i) *If $\lambda > 0$ is a limit ordinal then $\sigma_\lambda = \sup\{\sigma_\beta \mid \beta < \lambda\}$.*
(ii) *Every uncountable cardinal is stable.*
(iii) *If $\omega \leqslant \beta < \kappa$, where κ is a cardinal, then there is a stable ordinal $\alpha, \beta < \alpha < \kappa$.*
(iv) *If κ is a cardinal then $\kappa = \sigma_\kappa$.*

Proof. To prove (i) let $\lambda > 0$ be a limit. Since $\sigma_\lambda \geqslant \sup\{\sigma_\beta \mid \beta < \lambda\}$ by definition, it suffices to prove that the ordinal

$$\gamma = \sup\{\sigma_\beta \mid \beta < \lambda\}$$

is stable. Let φ be a Σ_1 formula, let $x_1, \ldots, x_n \in L(\gamma)$ and suppose

$$L \models \varphi[x_1, \ldots, x_n].$$

Pick $\beta < \lambda$ such that $x_1, \ldots, x_n \in L(\sigma_\beta)$. Then $L(\sigma_\beta) \models \varphi[x_1, \ldots, x_n]$ by stability, and then $L(\gamma) \models \varphi[x_1, \ldots, x_n]$ by persistence of Σ_1 formulas. (What we are really proving here is that the union $\mathfrak{A} = \bigcup_{\beta<\lambda} \mathfrak{A}_\beta$ of a chain of $\prec_1$-extensions $\mathfrak{A}_\beta$ is a $\prec_1$-extension of each $\mathfrak{A}_\beta$.)

Now let $\kappa > \omega$ be a cardinal and suppose that $x_1, \ldots, x_n \in L(\kappa)$ and that

$$L \models \exists y\, \psi[x_1, \ldots, x_n]$$

where ψ is Δ_0. We need to see that $L(\kappa)$ satisfies the same formula. But, for large enough cardinal λ,

$$H(\lambda) \models \exists \alpha\, \exists y \in L(\alpha)\, \psi[x_1, \ldots, x_n]$$

so, by II.3.5,

$$H(\kappa) \models \exists \alpha\, \exists y \in L(\alpha)\, \psi[x_1, \ldots, x_n]$$

and so there is an $\alpha < \kappa$ such that $L(\alpha) \models \exists y\, \psi[x_1, \ldots, x_n]$ and hence $L(\kappa) \models \exists y\, \psi[x_1, \ldots, x_n]$, as desired.

To prove (iii) we apply Theorem II.3.3. Note that we need only prove the result for κ regular since every singular κ is a limit of regular cardinals. Let $\alpha_0 = \beta + 1$. Given α_n apply II.3.3 to get an admissible set $\mathbb{B}$ such that

$$L(\alpha_n) \subseteq \mathbb{B},$$
$$\operatorname{card}(\alpha_n) = \operatorname{card}(\mathbb{B}) < \kappa,$$
$$L(\kappa) \models \varphi[\vec{x}] \quad \text{iff} \quad \mathbb{B} \models \varphi[\vec{x}]$$

for every formula φ and every $x_1, \ldots, x_n \in L(\alpha_n)$. Now since $\mathbb{B} \equiv L(\kappa)$, $B = L(\gamma)$ for some admissible $\gamma < \kappa$ and we let α_{n+1} be this γ. Let $\alpha = \sup_{n<\omega} \alpha_n < \kappa$. We claim that α is stable; i.e., that $L(\alpha) \prec_1 L$. It suffices to prove that $L(\alpha) \prec_1 L(\kappa)$ since κ is stable. Let φ be Σ_1 and $L(\kappa) \models \varphi[x_1, \ldots, x_n]$, where $x_1, \ldots, x_n \in L(\alpha)$. Pick $k < \omega$ so that $x_1, \ldots, x_n \in L(\alpha_k)$. Then $L(\alpha_{k+1}) \models \varphi[x_1, \ldots, x_n]$ by choice of α_{k+1} and then $L(\alpha) \models \varphi[x_1, \ldots, x_n]$ by persistence of Σ_1 formulas.

Part (iv) follows from (i) and (iii). In fact, if f is any continuous increasing function on the ordinals such that for all cardinals $\kappa > \omega$, $f(\alpha) < \kappa$ implies $f(\alpha+1) < \kappa$, one always has for all $\kappa > \omega$, $f(\kappa) = \kappa$. First assume κ is regular and consider the set B of β such that $f(\beta) < \kappa$. B is an initial segment of the ordinals and has no largest element so B is a limit ordinal λ. But then, by continuity,

$$f(\lambda) = \sup\{f(\beta) \mid \beta < \lambda\} \leqslant \kappa$$

but $f(\lambda) \not< \kappa$ since $\lambda \notin B$ so $f(\lambda) = \kappa$. Since κ is regular, $\lambda = \kappa$. Now for singular κ the result follows by continuity since every singular κ is the sup of regular cardinals. For if $\kappa = \sup_{\beta<\gamma} \lambda_\beta$, where the λ_β are regular, then $f(\kappa) = \sup_{\beta<\gamma} f(\lambda_\beta) = \sup_{\beta<\gamma} \lambda_\beta = \kappa$. □

There is a useful relative notion of stability.

7.3 Definition. An ordinal α is *β-stable* if $\alpha \leqslant \beta$ and

$$L(\alpha) \prec_1 L(\beta).$$

Since we have allowed $\alpha = \beta$ there is always at least one β-stable ordinal.

7.4 Proposition. (i) *If $\alpha \leqslant \beta \leqslant \gamma$ and α is γ-stable then α is β-stable.*
(ii) *If α is β-stable and β is γ-stable then α is γ-stable.*
(iii) *If β is stable and $\alpha < \beta$ then α is stable iff α is β-stable.*
(iv) *If B is a nonempty set of β-stable ordinals and $\alpha = \sup B$ then α is β-stable.*

Proof. These are all simple consequences of the definition and the persistence of Σ_1 formulas. □

7.5 Theorem. *If $\alpha < \beta$ and α is β-stable then α is admissible. In particular, every stable ordinal is admissible.*

Proof. Suppose $\alpha < \beta$ and $L(\alpha) \prec_1 L(\beta)$. Note that since the operations $\mathscr{F}_1, \ldots, \mathscr{F}_N$ all have Δ_0 graphs, and for $x, y \in L(\alpha)$,

$$L(\beta) \models \exists z(\mathscr{F}_i(x, y) = z),$$

we have

$$L(\alpha) \models \exists z(\mathscr{F}_i(x, y) = z),$$

so $L(\alpha)$ is closed under the operations $\mathscr{F}_1, \ldots, \mathscr{F}_N$. Thus $L(\alpha)$, in addition to being transitive, is closed under pair and union and satisfies Δ_0 separation. It remains to check Δ_0 Collection. Suppose

$$L(\alpha) \models \forall x \in a\, \exists y\, \varphi(x, y, z)$$

where φ is Δ_0 and $a, z \in L(\alpha)$. Then, letting $b = L(\alpha) \in L(\beta)$, we have

$$L(\beta) \models \forall x \in a\, \exists y \in b\, \varphi(x, y, z)$$

so

$$L(\beta) \models \exists b\, \forall x \in a\, \exists y \in b\, \varphi(x, y, z)$$

and so, by $L(\alpha) \prec_1 L(\beta)$,

$$L(\alpha) \models \exists b\, \forall x \in a\, \exists y \in b\, \varphi(x, y, z)\,. \quad \square$$

7.6 Corollary. (i) *If β is admissible, $\alpha < \beta$ and α is β-stable then α is recursively inaccessible.*

(ii) *Every stable ordinal is recursively inaccessible.*

Proof. (i) Let $\beta = \tau_\gamma$ and $\alpha = \tau_\delta$ where $\delta \leqslant \alpha$ and $\delta < \gamma$. We need to see that $\delta = \alpha$. Suppose $\delta < \alpha$. Then $L(\beta) \models \exists x[x = \tau_\delta]$, so, by Lemma 6.10, and $L(\alpha) \prec_1 L(\beta)$, $L(\alpha) \models \exists x[x = \tau_\delta]$ (one needs to observe that no parameters occur in the definition of G in 6.10) from which we have $\tau_\delta \in L(\alpha)$, which is ridiculous since $\alpha = \tau_\delta$ and $\alpha \notin L(\alpha)$. Part (ii) follows from (i). $\square$

The definition of α is β-stable appears to be model theoretic until one reformulates it as follows: If f is a β-recursive function then whenever $x_1, \ldots, x_n \in L(\alpha)$, if $f(x_1, \ldots, x_n)$ is defined then $f(x_1, \ldots, x_n) \in L(\alpha)$. This reformulation suggests a way of generating the β-stable and the stable ordinals. First, however, a lemma.

Notice that we did not assume $\mathfrak{A}_{\mathfrak{M}} \subseteq_{\text{end}} \mathfrak{B}_{\mathfrak{N}}$ in the definition of $\mathfrak{A}_{\mathfrak{M}} \prec_1 \mathfrak{B}_{\mathfrak{N}}$. We are going to apply the notion to a case where we do not know, ahead of time, that this holds.

7.7 Lemma (Tarski Criterion for $\prec_1$). *If $\mathfrak{A}_{\mathfrak{M}} \subseteq \mathfrak{B}_{\mathfrak{N}}$ then $\mathfrak{A}_{\mathfrak{M}} \prec_1 \mathfrak{B}_{\mathfrak{N}}$ iff the following condition holds for every Δ_0 formula $\varphi(v_1, \ldots, v_n)$ and every $x_1, \ldots, x_{n-1} \in \mathfrak{A}_{\mathfrak{M}}$: if*

$$\mathfrak{B}_{\mathfrak{N}} \models \exists v_n \varphi[x_1, \ldots, x_{n-1}]$$

then there is an $x_n \in \mathfrak{A}_{\mathfrak{M}}$ such that

$$\mathfrak{B}_{\mathfrak{N}} \models \varphi[x_1, \ldots, x_{n-1}, x_n]\,.$$

Proof. $\mathfrak{A}_{\mathfrak{M}} \prec_1 \mathfrak{B}_{\mathfrak{N}}$ clearly implies the condition. To prove the converse, one first uses the criterion to prove

$$\mathfrak{A}_{\mathfrak{M}} \models \psi[x_1, \ldots, x_n] \quad \text{iff} \quad \mathfrak{B}_{\mathfrak{N}} \models \psi[x_1, \ldots, x_n]$$

for all Δ_0 formulas ψ and all $x_1, \ldots, x_n \in \mathfrak{A}_{\mathfrak{M}}$, by induction on ψ. The atomic cases hold by $\mathfrak{A}_{\mathfrak{M}} \subseteq \mathfrak{B}_{\mathfrak{N}}$, the propositional connectives take care of themsevles and the criterion gets us past bounded quantifiers. Now suppose $\exists v_n \varphi(v_1, \ldots, v_n)$ is a Σ_1 formula, $x_1, \ldots, x_{n-1} \in \mathfrak{A}_{\mathfrak{M}}$. If $\mathfrak{A}_{\mathfrak{M}} \models \exists v_n \varphi[x_1, \ldots, x_{n-1}]$ then there is an $x_n \in \mathfrak{A}_{\mathfrak{M}}$ such that $\mathfrak{A}_{\mathfrak{M}} \models \varphi[x_1, \ldots, x_{n-1}, x_n]$ so $\mathfrak{B}_{\mathfrak{N}} \models \varphi[x_1, \ldots, x_{n-1}, x_n]$, since φ is Δ_0, and hence

$$\mathfrak{B}_{\mathfrak{N}} \models \exists v_n \varphi[x_1, \ldots, x_{n-1}].$$

The proof of the converse first uses the criterion to pick $x_n \in \mathfrak{A}_{\mathfrak{M}}$ and then applies the result for Δ_0 formulas. □

We now come to the main theorem on the generation of stable ordinals. The proof is rather amusing since we use the Collapsing Lemma to collapse a set that is already transitive.

7.8 Theorem. *Let β be an admissible ordinal and let $0 \leqslant \gamma < \beta$. Let A be the set of those $a \in \mathrm{L}(\beta)$ for which there is a Σ_1 definition of a in $\mathrm{L}(\beta)$ using parameters $< \gamma$ (Σ_1 definable as elements in the sense of* II.5.13). *Let α be the least ordinal not in A. Then*

(i) *$A = \mathrm{L}(\alpha)$, and*

(ii) *α is the least β-stable ordinal $\geqslant \gamma$.*

Proof. It is not transparent that A is even transitive, let alone admissible. The first step in the proof is to show

(1) $\langle A, \in \cap A^2 \rangle \prec_1 \langle L(\beta), \in \rangle$.

We use the Tarski Criterion. Suppose $\mathrm{L}(\beta) \models \exists y \varphi[a_1, \ldots, a_n]$, where $a_1, \ldots, a_n \in A$. We need to find a $b \in A$ such that $\mathrm{L}(\beta) \models \varphi[a_1, \ldots, a_n, b]$.

Since each $a_i \in A$ is Σ_1 definable by a formula with parameters $< \gamma$, we may replace each a_i by its definition and assume all the parameters are ordinals $< \gamma$, except that φ now becomes Σ_1 instead of Δ_0. Write φ as $\exists z \psi(v_1, \ldots, v_m, y, z)$, so that $\mathrm{L}(\beta) \models \exists y \exists z \psi(\lambda_1, \ldots, \lambda_m, y, z)$, where $\lambda_1, \ldots, \lambda_m < \gamma$. Let $b = 1^{\mathrm{st}}(c)$ where c is the least pair $\langle y, z \rangle$ in $\mathrm{L}(\beta)$ (least under the ordering $<_{\mathrm{L}}$) such that $\mathrm{L}(\beta) \models \psi(\lambda_1, \ldots, \lambda_m, y, z)$. Then b is Σ_1 definable in $\mathrm{L}(\beta)$ with parameters $\lambda_1, \ldots, \lambda_m$ so $b \in A$ and $\mathrm{L}(\beta) \models \varphi(a_1, \ldots, a_n, b)$. Which proves (1).

Let $B = \mathrm{clpse}(A)$ so that B is transitive, and

$$c_A : \langle A, \in \cap A^2 \rangle \cong \langle B, \in \rangle.$$

Let τ be the least ordinal not in B and note that $\tau \leqslant \beta$ since there is an embedding of τ into β. We claim that

(2) $B \subseteq \mathrm{L}(\tau)$.

The predicate (of x and δ)

$$x \in \mathrm{L}(\delta)$$

is Δ_1 in KP so we can find a Σ_1 formula equivalent to it in KP:

$$x \in \mathrm{L}(\delta) \quad \text{iff} \quad \exists y\, \theta(x, \delta, y)$$

where θ is Δ_0. Now pick any $x \in B$. We will show that $x \in \mathrm{L}(\delta)$ for some $\delta \leqslant \tau$. Write x as $c_A(a)$ for some $a \in A$. Since $A \subseteq \mathrm{L}(\beta)$ there is an ordinal $\lambda < \beta$ such that $a \in \mathrm{L}(\lambda)$. By (1), A is a model of

$$\exists \lambda, y\, \theta(a, \lambda, y)\,.$$

Hence B is a model of

$$\exists \lambda, y\, \theta(x, \lambda, y)\,,$$

but then x really is in $L(\delta)$ for some $\delta \in B$, proving (2).

Next we prove that

(3) $A = B$.

To prove this it suffices to prove that $c_A(a) = a$ for all $a \in A$. Since $\gamma \subseteq A$, $c_A(\lambda) = \lambda$ for all $\lambda < \gamma$. Let $a \in A$ be Σ_1 definable in $\mathrm{L}(\beta)$ by the Σ_1 formula $\varphi(x, \lambda_1, \ldots, \lambda_n)$ where $\lambda_1, \ldots, \lambda_n < \gamma$,

$$\mathrm{L}(\beta) \models \exists !\, x\, \varphi(x, \lambda_1, \ldots, \lambda_n),$$

$$\mathrm{L}(\beta) \models \varphi[a, \lambda_1, \ldots, \lambda_n]\,.$$

If we can prove that $\mathrm{L}(\beta) \models \varphi[c_A(a), \lambda_1, \ldots, \lambda_n]$ then we will have $a = c_A(a)$. But from (1) it follows that $\langle A, \in \cap A^2 \rangle \models \varphi[a, \lambda_1, \ldots, \lambda_n]$, so $B \models \varphi[c_A(a), c_A(\lambda_1), \ldots, c_A(\lambda_n)]$. As we mentioned, $c_A(\lambda_i) = \lambda_i$ so $B \models \varphi[c_A(a), \lambda_1, \ldots, \lambda_n]$. By (2), $B \subseteq \mathrm{L}(\beta)$ so $\mathrm{L}(\beta) \models \varphi[c_A(a), \lambda_1, \ldots, \lambda_n]$ by persistence of φ. This proves $A = B$.

Since B is transitive and $B \prec_1 \mathrm{L}(\beta)$, it follows that B is admissible and that $B = \mathrm{L}(\tau)$. But of course $\tau = \alpha$ so $A = B = \mathrm{L}(\alpha)$. Thus α is β-stable. Since $\gamma \subseteq A$, $\gamma \leqslant \alpha$. If $\gamma \leqslant \alpha' \leqslant \beta$ and α' is also β-stable then every element of A must be in $\mathrm{L}(\alpha')$ so $\alpha \leqslant \alpha'$. Hence α is the least β-stable ordinal $\geqslant \gamma$. □

7.9 Corollary. *The stable ordinals are generated as follows.*

(i) $\sigma_0 = \{\alpha \mid \alpha$ *is* Σ_1 *definable in* L *without parameters*$\}$,
$\mathrm{L}(\sigma_0) = \{x \in L \mid x$ *is* Σ_1 *definable in* L *without parameters*$\}$;

(ii) $\sigma_{\gamma+1} = \{\alpha \mid \alpha$ *is* Σ_1 *definable in* L *with parameters* $\leqslant \sigma_\gamma\}$,
$\mathrm{L}(\sigma_{\gamma+1}) = \{x \in L \mid x$ *is* Σ_1 *definable in* L *with parameters* $\leqslant \sigma_\gamma\}$;

(iii) *If* λ *is a limit ordinal then*

$$\sigma_\lambda = \sup\{\sigma_\gamma \mid \gamma < \lambda\}\,,$$

$$\mathrm{L}(\sigma_\lambda) = \textstyle\bigcup_{\gamma < \lambda} \mathrm{L}(\sigma_\gamma)\,.$$

Proof. For (i) apply 7.8 with $\beta=\omega_1$, $\gamma=0$. For (ii) apply 7.8 with β equal to some cardinal $>\sigma_\gamma$ and the γ of 7.8 equal to $\sigma_\gamma+1$. Part (iii) is just a restatement of part of 7.2 included for completeness. ▯

We will study σ_0 in some depth in the next section and give a classical description of it. Part (i) of the next theorem will play a crucial role.

7.10 Theorem. (i) *σ_0 is projectible into ω.*
(ii) *$\sigma_{\gamma+1}$ is projectible into σ_γ.*
(iii) *If λ is a limit ordinal then σ_λ is nonprojectible.*

Proof. Let's dispose of (iii) first since it's fairly trivial. We prove that

$$\mathrm{L}(\sigma_\lambda)\models\Sigma_1\ \textit{Separation}$$

and then apply Theorem 6.3 to see that σ_λ is nonprojectible. Let $a\in\mathrm{L}(\sigma_\lambda)$, let φ be Σ_1 and form the set

$$b=\{x\in a\,|\,\mathrm{L}(\sigma_\lambda)\models\varphi[x]\}\,.$$

Pick $\gamma<\lambda$ large enough that a and the parameters in φ are members of $\mathrm{L}(\sigma_\gamma)$. Then, by $\mathrm{L}(\sigma_\gamma)\prec_1\mathrm{L}(\sigma_\lambda)$, we have

$$b=\{x\in a\,|\,\mathrm{L}(\sigma_\gamma)\models\varphi[x]\}$$

so $b\in\mathrm{L}(\sigma_\lambda)$ by Δ Separation.

Now for (i). The idea is that we want to assign to each $\alpha<\sigma_0$ some Σ_1 definition of α, thus projecting σ_0 into $\mathbb{HF}$. The trouble is that

$$\mathrm{L}(\sigma_0)\models\exists!x\,\varphi(x)$$

is not a σ_0-r.e. predicate of the formula φ. To get around this we use the Uniformization Theorem, Theorem 4.4.

Recall the Σ_1 formula $\Sigma\text{-Sat}_1(z,y)$ from § 1. Let F be given by 4.4 so that F is σ_0-recursive,

$$\begin{aligned}\mathrm{dom}(F)&=\{\psi(x)\,|\,\psi(x)\text{ is a }\Sigma_1\text{ formula }\wedge\mathrm{L}(\sigma_0)\models\exists y\,\psi(y)\}\\&=\{z\,|\,\mathrm{L}(\sigma_0)\models\text{“}z\text{ is }\Sigma_1\wedge\exists y\,\Sigma\text{-Sat}_1(z,y)\text{”}\}\end{aligned}$$

and for each $\psi(x)\in\mathrm{dom}(F)$, $\mathrm{L}(\sigma_0)\models\psi(F(\psi))$. Now whenever a is Σ_1 definable there is a ψ such that

$$\mathrm{L}(\sigma_0)\models\exists!y\,\psi(y)\wedge\psi[a]$$

so $F(\psi)=a$. We may project $\mathrm{L}(\sigma_0)$ into $\mathbb{HF}$ by

$$\pi(a)=\text{least }\Sigma_1\text{ formula }\psi\text{ such that }F(\psi)=a\,,$$

where by least we mean in some well-ordering of $\mathbb{HF}$ as given in II.2.4. The proof for (ii) is similar, using Σ-Sat_3 instead of Σ-Sat_1, once we observe that every $a \in \mathrm{L}(\sigma_{\gamma+1})$ is definable by a formula

$$\exists! x\, \varphi[x, b, \sigma_\gamma]$$

with $b \in \mathrm{L}(\sigma)$ and no other parameters, by just using b to code a finite sequence of ordinals. Now apply Uniformization to get a $\sigma_{\gamma+1}$-recursive F such that

$$\mathrm{dom}(F) = \{\langle \psi(x,y,z), b\rangle \mid \psi \text{ is } \Sigma \text{ and } \mathrm{L}(\sigma_{\gamma+1}) \models \exists x\, \psi(x, b, \sigma_\gamma)\}$$

and, if $F(\psi(x,y,z), b)$ is defined then

$$\mathrm{L}(\sigma_{\gamma+1}) \models \psi(F(\psi, b), b, \sigma_\gamma).$$

Then define

$$\pi(a) = \text{least pair } \langle \psi, b\rangle \text{ such that } F(\psi, b) = a.$$

This π projects $\mathrm{L}(\sigma_{\gamma+1})$ into $\mathrm{L}(\sigma_\gamma)$. Since $\mathrm{L}(\sigma_\gamma)$ is recursively listed, this amounts to projecting $\mathrm{L}(\sigma_{\gamma+1})$ into σ_γ. □

The use of Uniformization in 7.10 is very typical of more advanced work in L. We also use it to prove the next result.

7.11 Theorem. *Let β be an admissible ordinal whose projectum β^* is not ω. Then β^* is the limit of smaller β-stable ordinals. Hence β^* is β-stable and admissible.*

Before proving 7.11 we state some of its consequences.

7.12 Corollary. *If $\alpha > \omega$ is nonprojectible then α is the limit of smaller α-stable ordinals.* □

Next we present the result promised at the end of the last section.

7.13 Corollary. *For any admissible ordinal α, α^* is admissible and nonprojectible.*

Proof. By 7.11, α^* is admissible if $\alpha^* > \omega$. But if $\alpha^* = \omega$ it is also admissible. Nonprojectibility is obvious. □

Thus, if α is an admissible ordinal less than the first nonprojectible then $\alpha^* = \omega$. We saw that the first nonprojectible ordinal was recursively inaccessible. We can iterate this result using 7.11. We give only a sample result which shows that the first nonprojectible is much larger than the first recursively inaccessible.

7.14 Corollary. *Let ρ_β be the β^{th} recursively inaccessible ordinal. If α is nonprojectible and $\alpha > \omega$ then $\alpha = \rho_\alpha$.*

Proof. Assume that $\alpha^*=\alpha$ but that $\alpha=\rho_\gamma$ for some $\gamma<\alpha$. Apply 7.12 to find an α-stable ordinal λ, $\gamma<\lambda<\alpha$. The predicate

$$\gamma<\alpha \quad \text{and } \gamma \text{ is recursively inaccessible}$$

is α-recursive since it holds iff

$$\gamma \text{ is admissible } \wedge \forall x<\gamma\, \exists\tau<\lambda\, (x<\tau \wedge \tau \text{ is admissible}).$$

Define $H(x)=\rho_x$ for $x<\gamma$. Then H is α-recursive and

$$\mathrm{L}(\alpha)\models \forall x<\gamma\, \exists y\, (H(x)=y)$$

so

$$\mathrm{L}(\lambda)\models \forall x<\gamma\, \exists y\, (H(x)=y)$$

since λ is α-stable. But this is ridiculous for λ itself is recursively inaccessible by 7.6, so $\lambda=H(x)$ for some $x<\gamma$. $\square$

Some authors refer to ordinals α such that $\alpha=\rho_\alpha$ as being *recursively hyper-inaccessible.*

We now return to prove Theorem 7.11.

7.15 Lemma (Π_2-reflection). *Let* $\alpha>\omega$ *be admissible and let* $\forall x\, \exists y\, \varphi(x,y)$ *be a sentence which holds in* $\mathrm{L}(\alpha)$, *where* φ *is* Δ_0. *Then for every* $\gamma<\alpha$ *there is a* λ, $\gamma\leqslant\lambda<\alpha$ *such that*

$$\mathrm{L}(\lambda)\models \forall x\, \exists y\, \varphi(x,y).$$

Proof. Let $\gamma\leqslant\lambda_0<\alpha$ where all parameters in φ are members of $\mathrm{L}(\lambda_0)$. Let λ_{n+1} be the least ordinal such that for all $x\in\mathrm{L}(\lambda_n)$ there is a $y\in\mathrm{L}(\lambda_{n+1})$ such that $\varphi(x,y)$. There is such a λ_{n+1} by Σ Reflection. The sequence $\langle\lambda_n : n<\omega\rangle$ is α-recursive so

$$\lambda=\sup_{n<\omega}\lambda_n$$

is less than α. $\square$

Proof of Theorem 7.11. For several years all that was known about the projectum β^* of an admissible ordinal was that

(4) β^* is admissible or the limit of admissibles.

For suppose $\beta^*<\beta$ but that there is an admissible ordinal τ_γ such that

$$\beta\geqslant\tau_\gamma>\beta^*>\sup_{\delta<\gamma}\tau_\delta.$$

But by Theorem 6.8 (and its proof) τ_γ is projectible into $\sup_{\delta<\gamma}\tau_\delta$ and hence so is β, contradicting the definition of β^*.

For the purposes of this proof we call an ordinal γ *nice* if $<_L\restriction L(\gamma)\times L(\gamma)$ has order type γ so that the function enumerating L, definable in KP, maps γ onto $L(\gamma)$. We know that every admissible ordinal is nice. The only point of proving (4) was to prove that

(5) *if* $\omega\leqslant\xi<\beta^*$ *then there is a nice limit ordinal* γ, $\xi\leqslant\gamma<\beta^*$.

If β^* is admissible, this follows by Π_2 Reflection. If β^* is the limit of admissibles we pick γ to be an admissible.

We are now ready to prove that if $\omega\leqslant\gamma<\beta^*$ then there is a β-stable α, $\gamma\leqslant\alpha<\beta^*$. By (5) it suffices to prove this for nice γ. Let α be the least β-stable ordinal $\geqslant\gamma$. Now $L(\alpha)$ is the set of $a\in L(\beta)$ definable by Σ_1 formulas with parameters $<\gamma$. Since γ is nice, though, we can code all these parameters into one so that

$$L(\alpha)=\{a\in L(\beta)\mid \text{for some } \Sigma_1 \text{ formula } \varphi(v_1,v_2) \text{ and some } \xi<\gamma,\ L(\beta) \models(\exists! v_1\varphi(v_1,\xi)\wedge\varphi(a,\xi))\}.$$

As in Theorem 7.10, let F be a β-recursive function uniformizing Σ-Sat_2. Note that Σ-Sat_2 and hence the graph of F are β-r.e. definable by Σ_1 formulas without parameters. Thus

$$L(\alpha)=\mathrm{rng}(F\restriction(\mathbb{HF}\times\gamma)).$$

Since γ is nice we can identify $\mathbb{HF}\times\gamma$ with γ and apply Theorem 6.11 to see that $\mathrm{dom}(F\restriction(\mathbb{HF}\times\gamma))\in L(\beta)$ since it is, essentially, a β-r.e. subset of $\gamma<\beta^*$. But if the domain of a β-recursive function is in $L(\beta)$, so is its range, so $L(\alpha)\in L(\beta)$. That is, $\alpha<\beta$. We need to see that $\alpha<\beta^*$. Suppose $\beta^*\leqslant\alpha$. The inverse of $F\restriction(\mathbb{HF}\times\gamma)$ maps $L(\alpha)$ into γ so we could then project $L(\beta)$ into γ, contradicting the definition of β^*. Thus $\alpha<\beta^*$ so we have proven that β^* is the sup of smaller β-stable ordinals. Thus β^* is itself β-stable and admissible. ☐

7.16—7.25 Exercises

7.16. Prove that every stable ordinal is the limit of smaller nonprojectible ordinals. In particular, the first nonprojectible ordinal is less than the first stable ordinal, even though the first stable ordinal is projectible into ω.

7.17. Compute σ_γ^*. [Hint: $\sigma_{\omega+1}^*=\omega$.]

7.18 (Jensen). Show that α is admissible iff α is a limit ordinal and $L(\alpha)$ satisfies Δ_1 Separation.

7.19. Prove the converse of Lemma 7.15. That is show that a limit ordinal α is admissible and $>\omega$ iff every Π_2 sentence $\forall x\,\exists y\,\varphi(x,y,z)$ true in $L(\alpha)$ is true in $L(\beta)$ for arbitrarily large $\beta<\alpha$.

7.20. Let α be admissible, $\alpha > \omega$. An ordinal $\beta < \alpha$ is an *α-cardinal* if there is no $f \in \mathrm{L}(\alpha)$ mapping β one-one into an ordinal $\gamma < \beta$.

(i) Show that if α^* (the projectum of α) is $< \alpha$ then α^* is an α-cardinal.

(ii) Prove that every α-cardinal $\kappa > \omega$ (if there are any) is α-stable. [Show that the proof of 7.2(ii) can be effectivized so as to hold inside $\mathrm{L}(\alpha)$.]

(iii) Prove that if κ is an α-cardinal $> \omega$ and $\gamma < \kappa$ then there is an α-stable ordinal β, $\gamma < \beta < \kappa$. [Modify the proof of 7.11.]

7.21. Suppose α is admissible and $\omega < \alpha^* < \alpha$. Show that any α-r.e. subset of some $\mathrm{L}(\beta)$, for $\beta < \alpha^*$, is α^*-finite, not just α-finite as stated in 6.11. [Use 7.11.]

7.22. Assume that there is an α such that $\mathrm{L}(\alpha)$ is a model of ZF. Show that the least such α is less than σ_0.

7.23. Let $\mathbb{A}_{\mathfrak{M}}, \mathbb{B}_{\mathfrak{N}}$ be countable, admissible sets and suppose that $\mathbb{A}_{\mathfrak{M}}$ is $\mathbb{B}_{\mathfrak{N}}$-stable; i.e., that

$$\mathbb{A}_{\mathfrak{M}} \prec_1 \mathbb{B}_{\mathfrak{N}}.$$

Let T be a theory of $\mathsf{L}_{\mathbb{B}}$ which is definable over by a Σ_1 formula with parameters from $\mathbb{A}_{\mathfrak{M}}$. Show that if every $T_0 \subseteq T$ with $T_0 \in \mathbb{A}_{\mathfrak{M}}$ has a model then T has a model. [Use the Extended Completeness Theorem.]

7.24. Let $\mathbb{A}_{\mathfrak{M}}$ be countable, admissible. Show that the following are equivalent:

(i) $\mathbb{A}_{\mathfrak{M}} \prec_1 \mathbb{HYP}(\mathbb{A}_{\mathfrak{M}})$;

(ii) $\mathbb{A}_{\mathfrak{M}}$ is Π_1^1 reflecting; i.e. if $\Phi(v)$ is a Π_1^1 formula and $\mathbb{A}_{\mathfrak{M}} \models \Phi[x]$, then there is an admissible set $\mathbb{A}'_{\mathfrak{M}} \in \mathbb{A}_{\mathfrak{M}}$ such that $\mathbb{A}'_{\mathfrak{M}} \models \Phi[x]$. In particular, if $\alpha < \omega_1$ then τ_α is $\tau_{\alpha+1}$-stable iff $\mathrm{L}(\tau_\alpha)$ is Π_1^1 reflecting. [Use the Completeness Theorem.]

7.25. An admissible ordinal α is *recursively Mahlo* if every α-recursive closed unbounded subset of α contains an admissible ordinal. (This is the "effective version" of the definition of Mahlo cardinal. See Chapter VIII.)

(i) Prove that if α is recursively Mahlo then it is recursively inaccessible, recursively hyperinaccessible, etc.

(ii) Prove that if α is the limit of smaller α-stable ordinals then α is recursively Mahlo.

(iii) Prove that if α is nonprojectible then it is recursively Mahlo.

7.26 Notes. The stability of uncountable cardinals is due to Takeuti [1960]. The concepts and other results in 7.1—7.10 are due to Kripke and Platek, independently. Theorem 7.11 (and hence 7.12, 7.13, 7.20, 7.21) are due to Kripke. The student interested in further similar results should study Jensen's theory of the fine structure of L as presented, for example, in Devlin [1973]. Exercise 7.23 appears in Barwise [1969]. Exercise 7.24 is due to Aczel-Richter [1973] and, in an absolute form, to Moschovakis [1974]. Exercise 7.25 goes back to Kripke and Platek.

Putting admissible ordinals $\alpha > \omega$ in their place.

$$
\begin{tikzcd}
 & \text{Cardinal } > \omega \arrow[dl] \arrow[ddr] & \\
\text{stable (i.e., } L(\alpha) \prec_1 L) \arrow[d] & & \\
\text{limit of smaller nonprojectibles} \arrow[dr] & & \text{nonprojectible (iff } L_\alpha \models \Sigma_1 \textit{ Separation}) \arrow[dl] \\
 & \text{recursively Mahlo} \arrow[d, dotted] & \\
 & \text{recursively hyper-hyper-inaccessible} \arrow[d] & \\
 & \text{recursively hyper-inaccessible (i.e. } \alpha = \rho_\alpha) \arrow[d] & \\
 & \text{recursively inaccessible (i.e. } \alpha = \tau_\alpha) \arrow[d] & \\
 & \text{admissible} &
\end{tikzcd}
$$

Notes:

1. No arrows are missing.
2. No arrows reverse.
3. The first stable ordinal σ_0 is projectible into ω; the $\beta+1^{\text{st}}$ stable ordinal $\sigma_{\beta+1}$ is projectible into σ_β.
4. For λ a limit, σ_λ is nonprojectible.
5. If α is projectible then its projectum α^* is admissible and nonprojectible.

8. Shoenfield's Absoluteness Lemma and the First Stable Ordinal

In § 5 we saw that the first admissible ordinal $\tau_1 > \omega$ is the least ordinal not the order type of a Δ_1^1 well-ordering of $\mathbb{HF}$ and that a set $X \subseteq \mathbb{HF}$ is τ_1-r.e. iff X is Π_1^1 on $\mathbb{HF}$. In this section we prove an analogous result for the first stable ordinal σ_0.

A relation R on $\mathbb{HF}$ is Σ_2^1 if it can be defined by a second order formula of the form $\exists \mathrm{S}_1 \forall \mathrm{S}_2 \varphi$, where φ is first order:

$$R(\vec{x}) \quad \text{iff} \quad \langle \mathbb{HF}, \in \rangle \models \exists \mathrm{S}_1 \forall \mathrm{S}_2 \varphi(\vec{x}, \mathrm{S}_1, \mathrm{S}_2).$$

If the complement $\mathbb{HF}^n - R$ of R is Σ_2^1, then R is said to be Π_2^1. If R is both Σ_2^1 and Π_2^1 then R is Δ_2^1.

At first glance the step from Δ_1^1 to Δ_2^1 seems a small one. We will show, however, that it is an enormous jump, taking us from τ_1 past the first recursively inaccessible, past the first nonprojectible all the way to σ_0, the first stable ordinal. The precise statement is contained in Corollary 8.3 below. The main step in the proof is the following theorem, known as the Shoenfield-Lévy Absoluteness Lemma.

8.1 Theorem. *Any Σ_1 sentence without parameters true in $\mathbb{V}$ is true in* L.

Warning: this does not say that $\mathrm{L} \prec_1 \mathbb{V}$ because parameters are not permitted. Some extensions with parameters are discussed in the exercises.

We defer the proof of 8.1 to the end of the section (Corollary 8.11) since it leads away from our chief concern.

8.2 Theorem. *Let σ_0 be the first stable ordinal and let R be a relation on* $\mathbb{HF}$.

(i) *R is Σ_2^1 on $\langle \mathbb{HF}, \in \rangle$ iff R is σ_0-r.e.*

(ii) *R is Δ_2^1 on $\langle \mathbb{HF}, \in \rangle$ iff $R \in \mathrm{L}(\sigma_0)$.*

Proof. As usual, (ii) follows from (i). We first prove the ($\Leftarrow$) half of (i). Let R be Σ_1 on $\mathrm{L}(\sigma_0)$. We know that $\mathrm{L}(\sigma_0) \prec_1 \mathrm{L}$ by the definition of σ_0 and that every $x \in \mathrm{L}(\sigma_0)$ is Σ_1 definable (as an element) in L by a formula without parameters (by 7.9). It follows that every $x \in \mathrm{L}(\sigma_0)$ is Σ_1 definable in $\mathrm{L}(\sigma_0)$ by a Σ_1 formula without parameters. Thus any parameters in a Σ_1 definition of the relation R can be eliminated so we may assume that

$$R(x) \quad \text{iff} \quad \mathrm{L}(\sigma_0) \models \exists y \, \varphi(x, y)$$

where φ is Δ_0 and contains no parameters. But then we claim that

(1) $R(x)$ iff $\exists \alpha [\alpha \text{ admissible} \wedge \mathrm{L}(\alpha) \models \exists y \, \varphi(x, y)]$.

The proof of ($\Rightarrow$) in (1) is trivial since we can let $\alpha = \sigma_0$. The other half ($\Leftarrow$) of (1) follows from $\mathrm{L}(\sigma_0) \prec_1 \mathrm{L}$ for $\mathrm{L}(\alpha) \models \exists y \, \varphi(x, y)$ implies $\mathrm{L} \models \exists y \, \varphi(x, y)$ and hence

$L(\sigma_0) \models \exists y\, \varphi(x,y)$ by stability. Using (1), it is not too difficult to rewrite R to be Σ_2^1 over $\langle \mathbb{HF}, \in \rangle$. Namely, $R(x)$ holds iff $\langle \mathbb{HF}, \in \rangle$ satisfies

(2) $\exists E, F[\langle \mathbb{HF}, E\rangle \models \mathrm{KP} + \mathrm{V} = \mathrm{L} \wedge E$ *is well founded* $\wedge F$ *is an isomorphism of* $\langle \mathbb{HF}, \in \rangle$ onto an *initial submodel of* $\langle \mathbb{HF}, E\rangle \wedge \langle \mathbb{HF}, E\rangle \models \exists y\, \varphi(F(x), y)]$

since any such $\langle \mathbb{HF}, E\rangle$ is isomorphic to $L(\alpha)$ for some admissible α. By the techniques of IV.2, everything inside the brackets is Δ_1^1 in $\in, E, F$ except the condition

E *is well founded*.

But this is Π_1^1 by the very definition:

$$\forall X[\exists z(z \in X) \to \exists z(z \in X \wedge \forall w(w \in X \to \neg wEz))].$$

Hence the whole of (2) has the form

$$\exists E, F[\text{---}]$$

where $[\text{---}]$ is Π_1^1 so (2) is Σ_2^1. (If you insist, you can always collapse the two existential second order quantifiers to one.)

To prove the other half of (i), let $R \subseteq \mathbb{HF}$ be Σ_2^1 over $\langle \mathbb{HF}, \in \rangle$, say

$$R(x) \text{ iff } \langle \mathbb{HF}, \in \rangle \models \exists \mathsf{S}_1 \forall \mathsf{S}_2\, \varphi(x, \mathsf{S}_1, \mathsf{S}_2).$$

For each relation S_1 on $\langle \mathbb{HF}, \in \rangle$, let $\sigma(S_1)$ be the infinitary sentence

$$\mathrm{Diagram}(\langle \mathbb{HF}, \in, S_1\rangle) \wedge \forall v \bigvee_{x \in \mathbb{HF}} (v = \overline{\mathsf{x}}).$$

Thus S_1 occurs in $\sigma(S_1)$.

We claim that $R(x)$ holds iff $L(\sigma_0)$ is a model of

(3) $\exists S_1 \exists P[S_1 \subseteq \mathbb{HF} \wedge P$ *is a proof of* $(\sigma(S_1) \to \varphi(\overline{\mathsf{x}}, \mathsf{S}_1, \mathsf{S}_2))]$,

which will show that R is Σ_1 over $L(\sigma_0)$. To show this, first suppose $R(x)$ holds. Then there is an S_1 such that

$$(\mathbb{HF}, \in, S_1) \models \forall \mathsf{S}_2\, \varphi(x, \mathsf{S}_1, \mathsf{S}_2)$$

and hence

$$\sigma(S_1) \to \varphi(\overline{\mathsf{x}}, \mathsf{S}_1, \mathsf{S}_2)$$

is logically valid. It is a countable infinitary sentence, so it is provable. Hence (3) holds in $\mathbb{V}$. The only parameter in (3) is x and it is Σ_1 definable, being in $\mathbb{HF}$. Thus (3) holds in L by 8.1 and hence in $L(\sigma_0)$. Thus $R(x)$ implies $L(\sigma_0) \models (3)$. To prove the converse, suppose (3) holds in $L(\sigma_0)$. Then there is an $S_1 \in L(\sigma_0)$ such that

$$\sigma(S_1) \to \varphi(\overline{\mathsf{x}}, \mathsf{S}_1, \mathsf{S}_2)$$

is provable, and hence, is logically valid. Thus,

$$(\mathbb{H}\mathrm{F}, \in, S_1) \models \forall \mathrm{S}_2\, \varphi(x, \mathrm{S}_1, \mathrm{S}_2)$$

so $R(x)$ holds. □

8.3 Corollary. *The first stable ordinal is the least ordinal not the order type of some well-ordering which is Δ_2^1 on $\langle \mathbb{H}\mathrm{F}, \in \rangle$.*

Proof. Every Δ_2^1 well-ordering R is in $\mathrm{L}(\sigma_0)$ so its order type is less than σ_0, by 3.3. To prove the converse, recall that σ_0 is projectible into ω by Corollary 7.10.

Let p be some one-one σ_0-recursive map of σ_0 into ω. For $\beta < \sigma_0$ let

$$R_\beta = \{\langle p(x), p(y)\rangle \mid x < y < \beta\}$$

which is in $\mathrm{L}(\sigma_0)$ by Σ Replacement. But then R_β is a well-ordering of order type β and R_β is Δ_2^1 by Theorem 8.2(ii). □

We can now project the recursion theory from σ_0-r.e. sets of ordinals to Σ_2^1 sets of integers using Section 5. We state some of the simplest results.

8.4 Corollary. (i) *For any Σ_2^1 subsets B, C of* $\mathbb{H}\mathrm{F}$ *there are disjoint Σ_2^1 sets B_0, C_0 with $B_0 \subseteq B$, $C_0 \subseteq C$ and $B \cup C = B_0 \cup C_0$.*

(ii) *Any two disjoint Π_2^1 subsets of* $\mathbb{H}\mathrm{F}$ *can be separated by a Δ_2^1 set.*

(iii) *There are disjoint Σ_2^1 subsets of* $\mathbb{H}\mathrm{F}$ *which cannot be separated by a Δ_2^1 set.*

Proof. These are translations and projections of results we know about σ_0. □

8.5 Corollary. *Every Σ_2^1 subset of* $\mathbb{H}\mathrm{F}$ *is constructible.*

Proof. If R is Σ_2^1 on $\mathbb{H}\mathrm{F}$ then it is Σ_1 on $\mathrm{L}(\sigma_0)$ and hence an element of $\mathrm{L}(\sigma_0 + \omega)$. □

It follows, of course, that every Π_2^1 subset of $\mathbb{H}\mathrm{F}$ is constructible, but this is as far as one can go. It is consistent with ZFC to assume there is a nonconstructible Δ_3^1 subset of $\mathbb{H}\mathrm{F}$, where Δ_3^1 means expressible in both the forms

$$\exists \mathrm{S}_1 \forall \mathrm{S}_2 \exists \mathrm{S}_3(---),$$
$$\forall \mathrm{S}_1 \exists \mathrm{S}_2 \forall \mathrm{S}_3(---).$$

We now turn to the proof of Theorem 8.1. We need several preliminary lemmas. A finitary formula $\varphi(x_1, \dots, x_n)$ is an $\forall\exists$-*formula* if it has the form

$$\forall y_1, \dots, y_k \exists z_1, \dots, z_l\, \psi(\vec{x}, \vec{y}, \vec{z})$$

where ψ is quantifier-free.

8.6 Lemma (Skolem $\forall\exists$ normal form). *Let* K *be a language and let* Ψ *be a finite set of formulas of* $\mathsf{K}_{\omega\omega}$. *There is an expansion*

$$\mathsf{L} = \mathsf{K} \cup \{\mathsf{S}_1, \ldots, \mathsf{S}_n\}$$

by a finite number of new relation symbols and an $\forall\exists$*-sentence* φ *of* $\mathsf{L}_{\omega\omega}$ *with the following properties:*

(i) *Every* K*-structure* $\mathfrak{M}$ *has a unique expansion* $\mathfrak{M}' = (\mathfrak{M}, S_1, \ldots, S_n)$ *with* $\mathfrak{M}' \models \varphi$.

(ii) *For each formula* $\psi(y_1, \ldots, y_n)$ *in* Ψ *there is a quantifier free formula* ψ_0 *of* $\mathsf{L}_{\omega\omega}$ *such that*

$$\models \varphi \to \forall \vec{y}\,[\psi(\vec{y}) \leftrightarrow \psi_0(\vec{y})]\,.$$

Proof. We may assume Ψ is closed under subformulas. Introduce, for each $\psi(y_1, \ldots, y_n) \in \Psi$ a new relation symbol $\mathsf{S}_{\psi(y_1, \ldots, y_n)}$. Let φ be the conjunction of the universal closures of the following:

$\mathsf{S}_\psi(y_1, \ldots, y_n) \leftrightarrow \psi(y_1, \ldots, y_n)$ if $\psi \in \Psi$ is atomic,
$\mathsf{S}_{\neg\psi}(y_1, \ldots, y_n) \leftrightarrow \neg \mathsf{S}_\psi(y_1, \ldots, y_n)$ if $\neg\psi \in \Psi$,
$\mathsf{S}_{\theta\wedge\psi}(y_1, \ldots, y_n) \leftrightarrow \mathsf{S}_\theta(y_1, \ldots, y_n) \wedge \mathsf{S}_\psi(y_1, \ldots, y_n)$ if $(\theta \wedge \psi) \in \Psi$,
$\mathsf{S}_{\theta\vee\psi}(y_1, \ldots, y_n) \leftrightarrow \mathsf{S}_\theta(y_1, \ldots, y_n) \vee \mathsf{S}_\psi(y_1, \ldots, y_n)$ if $(\theta \vee \psi) \in \Psi$,
$\mathsf{S}_{\exists y_m\psi}(y_1, \ldots, \hat{y}_m, \ldots, y_n) \leftrightarrow \exists y_m\, \mathsf{S}_\psi(y_1, \ldots, y_n)$ if $(\exists y_m \psi) \in \Psi$,
$\mathsf{S}_{\forall y_m\psi}(y_1, \ldots, \hat{y}_m, \ldots, y_n) \leftrightarrow \forall y_m\, \mathsf{S}_\psi(y_1, \ldots, y_m)$ if $(\forall y_m \psi) \in \Psi$.

Here we use $y_1, \ldots, \hat{y}_m, \ldots, y_n$ to denote $y_1, \ldots, y_{m-1} y_{m+1}, \ldots, y_m$ if $m \leqslant n$, to denote $y_1, \ldots, y_n$ if $m > n$. Now φ clearly has the desired properties. □

8.7 Corollary. *Let* ψ *be any sentence of* $\mathsf{K}_{\omega\omega}$. *There is an expansion* L *of* K *by a finite number of new relation symbols and an* $\forall\exists$*-sentence* ψ' *of* $\mathsf{L}_{\omega\omega}$ *such that*

(i) *Every model* $\mathfrak{M}$ *of* ψ *has a unique expansion to a model* $\mathfrak{M}'$ *of* ψ'.

(ii) *If* $\mathfrak{M}' \models \psi'$ *and* $\mathfrak{M}$ *is the reduct of* $\mathfrak{M}'$ *to a* K*-structure then* $\mathfrak{M} \models \psi$.

Proof. Let $\Psi = \{\psi\}$ and apply 8.6. Let φ be as given there and let ψ_0 be quantifier free such that

$$\models \varphi \to (\psi \leftrightarrow \psi_0)\,.$$

The desired ψ' is $(\varphi \wedge \psi_0)$, or rather, the $\forall\exists$-sentence equivalent to it after one moves the quantifiers in φ out front. □

The next lemma gives us an easy way to construct models of $\forall\exists$-sentences and accounts for our sudden preoccupation with them.

8.8 Lemma. *Let* φ *be an* $\forall\exists$*-sentence of* $\mathsf{L}_{\omega\omega}$, *say*

$$\forall x_1, \ldots, x_n \exists y_1, \ldots, y_k \psi(\vec{x}, \vec{y})$$

where ψ *is quantifer-free. Let*

$$\mathfrak{M}_0 \subseteq \mathfrak{M}_1 \subseteq \cdots \subseteq \mathfrak{M}_l \subseteq \cdots$$

be a chain of L*-structures. Suppose that for each* $l<\omega$ *and each* $x_1,\ldots,x_n\in\mathfrak{M}_l$,

$$\mathfrak{M}_{l+1}\models\exists y_1,\ldots,y_k\psi(x_1,\ldots,x_n,y_1,\ldots,y_k).$$

If $\mathfrak{M}=\bigcup_{l<\omega}\mathfrak{M}_l$, *then* $\mathfrak{M}\models\varphi$.

Proof. Trivial, since $\mathfrak{M}_{l+1}\models\exists\vec{y}\,\psi(\vec{x},\vec{y})$ implies $\mathfrak{M}\models\exists\vec{y}\psi(\vec{x},\vec{y})$. ▯

The next lemma contains the secret to proving a number of important results, including Theorem 8.1.

8.9 Lemma. *Let* $\langle X,\prec\rangle$ *be a non-wellfounded partially ordered structure which is constructible (i.e. is an element of* L*). There is a sequence* $\langle x_n\rangle_{n<\omega}$ *in* L *such that*

$$x_{n+1}\prec x_n$$

for all $n<\omega$.

Proof. The hypothesis is that $\langle X,\prec\rangle\in\mathrm{L}$ and that

$$\mathbb{V}\models\langle X,\prec\rangle \quad \text{is not well founded}.$$

We claim that

(4) $\mathrm{L}\models\langle X,\prec\rangle$ is not well founded.

For otherwise, since $\mathrm{L}\models Beta$, there would be a function $f\in\mathrm{L}$ such that $f(x)=\{f(y)\mid y\prec x\}$ for all $x\in X$. But then $\langle X,\prec\rangle$ really would be well founded (see Exercise I.9.9). Now since (4) holds, there is a nonempty $X_0\in\mathrm{L}$, $X_0\subseteq X$ such that

$$\forall y\in X_0\,\exists z\in X_0(z\prec y).$$

But then, using the axiom of (dependent) choice in L, there is a sequence of the desired kind. ▯

8.10 Theorem (of ZF). *Let* φ *be a finitary sentence in a language* L *containing* $\in$ *and some other relation symbols* $\mathsf{R}_1,\ldots,\mathsf{R}_l$. *If* φ *is true in some structure* $\mathbb{A}=\langle A,\in,R_1,\ldots,R_l\rangle$ *where* A *is transitive, then there is a transitive structure* $\mathbb{B}=\langle B,\in,R_1',\ldots,R_l'\rangle$ *which is constructible and a model of* φ.

Proof. We may assume that extensionality is a consequence of φ since it holds in $\mathbb{A}$. By 8.7 we may also assume that φ is $\forall\exists$, say

$$\forall\vec{x}\psi(\vec{x})$$

where $\psi(\vec{x})$ is

$$\exists\vec{y}\theta(\vec{x},\vec{y})$$

and θ is quantifier free. Let $\beta = \mathrm{rk}(A)$. We define a non-wellfounded structure $\langle X, \prec \rangle \in \mathrm{L}$. The set X consits of all pairs $\langle \mathfrak{B}, f \rangle$ such that $\mathfrak{B} = \langle B, E, R_1, \ldots, R_l \rangle$ is a finite structure with $B \subseteq \omega$, $f: B \to \beta$ and xEy implies $f(x) < f(y)$. We define

$$\langle \mathfrak{B}_1, f_1 \rangle \prec \langle \mathfrak{B}_0, f_0 \rangle$$

to mean that $\mathfrak{B}_0 \subseteq \mathfrak{B}_1$, $f_0 \subseteq f_1$ and for every $\vec{x} \in B_0$,

$$\mathfrak{B}_1 \models \psi[\vec{x}].$$

Now the definitions of X and $\prec$ are absolute so $\langle X, \prec \rangle \in \mathrm{L}$. We claim that

(5) $\langle X, \prec \rangle$ is not well founded.

Assuming (5) for a moment, let us finish the proof of the theorem. By Lemma 8.9 there is a sequence

$$\langle \mathfrak{B}_n, f_n \rangle_{n < \omega}$$

in L such that

$$\langle \mathfrak{B}_{n+1}, f_{n+1} \rangle \prec \langle \mathfrak{B}_n, f_n \rangle$$

for each n. Let $\mathfrak{B} = \bigcup \mathfrak{B}_n$. By Lemma 8.8, $\mathfrak{B} \models \varphi$. Let $f = \bigcup_n f_n$. Then, if $\mathfrak{B} = \langle B, E, R'_1, \ldots, R'_l \rangle$, then $f: B \to \beta$ and xEy implies $f(x) < f(y)$ so E is well founded. Now since $\mathrm{L} \models \mathrm{Beta}$, there is a transitive structure $\mathbb{B} \in \mathrm{L}$ isomorphic to $\mathfrak{B}$. This $\mathbb{B}$ satisfies the conclusion of the theorem.

Now let's go back and prove (5). Let X_0 be the set of those $(\mathfrak{B}, f) \in X$ such that there is an embedding i of $\mathfrak{B}$ into the original $\mathbb{A}$ such that

$$f(x) = \mathrm{rk}(i(x))$$

for all $x \in B$. The set X_0 is nonempty since $\langle \mathfrak{A}_0, \{\langle 0, 0 \rangle\} \rangle \in X_0$ where $\mathfrak{A}_0$ is the substructure of $\mathbb{A}$ with universe $\{0\}$. It remains to show that X_0 has no $\prec$ minimal member. Let $\langle \mathfrak{B}_0, f_0 \rangle \in X_0$ with $i_0: \mathfrak{B}_0 \to \mathbb{A}$ the associated embedding. Let $\mathfrak{A}_0 \subseteq \mathbb{A}$ be isomorphic to $\mathfrak{B}_0$ via i_0. Since

$$\mathbb{A} \models \forall \vec{x}\, \exists \vec{y}\, \theta(\vec{x}, \vec{y})$$

there is a finite structure $\mathfrak{A}_1$, $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathbb{A}$, such that for all $\vec{x} \in \mathfrak{A}_0$,

$$\mathfrak{A}_1 \models \psi[\vec{x}].$$

Now choose $\mathfrak{B}_1 = \langle B_1, E_1, \ldots \rangle$ extending $\mathfrak{B}_0$ with $B_1 \subseteq \omega$ so that for some i_1 extending i_0,

$$i_1: \mathfrak{B}_1 \cong \mathfrak{A}_1.$$

Let $f_1(x)=\mathrm{rk}(i_1(x))$ for $x\in B_1$. Then $\langle\mathfrak{B}_1,f_1\rangle\prec\langle\mathfrak{B}_0,f_0\rangle$ and $\langle\mathfrak{B}_1,f_1\rangle\in X_0$. □

Theorem 8.1 is the informal version of the next result.

8.11 Corollary. *For any* Σ *sentence* φ *of set theory*

$$(\varphi\rightarrow\varphi^{(\mathrm{L})})$$

is a theorem of ZF.

Proof. We work in ZF. Assume φ. Then there is a transitive $\langle A,\in\rangle\models\varphi$. But then by Theorem 8.10, there is a transitive $\langle B,\in\rangle\in\mathrm{L}$ such that $\langle B,\in\rangle\models\varphi$. And $\langle B,\in\rangle\subseteq_{\mathrm{end}}\langle \mathrm{L},\in\rangle$ so $\langle \mathrm{L},\in\rangle\models\varphi$; i.e., $\varphi^{(\mathrm{L})}$. □

Some extensions of these results are sketched in the exercises.

8.12—8.20 Exercises

8.12. Show that if

$$\langle\mathbb{HF},\in\rangle\models\exists \mathsf{S}_1\,\exists \mathsf{S}_2\,\forall \mathsf{R}\,\varphi(\mathsf{S}_1,\mathsf{S}_2,\mathsf{R},x)$$

where φ is first order, then there is an $S_1\in\mathrm{L}(\sigma_0)$ such that

$$\langle\mathbb{HF},\in,S_1\rangle\models\exists \mathsf{S}_2\,\forall \mathsf{R}\,\varphi(\mathsf{S}_1,\mathsf{S}_2,\mathsf{R},x)\,.$$

This is the original version of Shoenfield's Absoluteness Lemma. A proof of it can be discovered inside the proof of Theorem 8.2.

8.13. Show that there is a Σ^1_2 well-ordering of a subset of ω of order type σ_0.

8.14. Improve 8.11 by replacing ZF by $\mathrm{KP}+\mathrm{Beta}$.

8.15. Improve Theorem 8.10 as follows. Let φ, $\mathbb{A}$ be as in 8.10, let $\beta=\mathrm{rk}(A)$. Let α be the least admissible $\tau_\gamma>\beta$ if $\mathrm{L}(\tau_\gamma)\models\mathrm{Beta}$, otherwise let $\alpha=\tau_{\gamma+1}$. Show that there is a transitive model $\mathbb{B}$ of φ which is an element of $\mathrm{L}(\alpha)$.

8.16. Let α be a limit of admissibles. Show that any Σ_1 sentence (without parameters) true in $\mathrm{V}(\alpha)$ is true in $\mathrm{L}(\alpha)$. [Use 8.15.]

8.17. Let T be a countable set of finitary sentences true in some transitive structure $\mathbb{A}=\langle A,\in,R_1,R_2,\ldots\rangle$. Show that there is a transitive model $\mathbb{B}$ of T which is an element of $\mathrm{L}(T)$ and is countable in $\mathrm{L}(T)$. [Hint: Modify the definition of $\langle X,\prec\rangle$ in the proof of 8.10 so that bigger structures take care of more of the sentences in T.]

8.18. Prove that the following is a theorem of ZF (by using 8.17): for each Σ formula $\varphi(v)$

$$\forall x \subseteq \mathbb{HF}\,[\varphi(x) \rightarrow \varphi(x)^{L(x)}]\,.$$

8.19. Let α be the constructible $\aleph_1$, i.e. the ordinal which, in L, is the first uncountable cardinal. Prove that

$$L(\alpha) \prec_1 \mathbb{V}\,.$$

It is consistent with ZFC to assume α is countable. Prove that if α *is* countable and if $\beta > \alpha$ then $L(\beta) \not\prec_1 \mathbb{V}$.

8.20. Let Φ be a Σ_2^1 sentence true in some countable structure $\mathfrak{M}$. Prove that there is an $\mathfrak{M} \in L(\sigma_0)$ which satisfies Φ. If Φ is Π_1^1 you can improve this bound. How?

8.21 Notes. The original Shoenfield Absoluteness Lemma (Exercise 8.12) was proved in Shoenfield [1961]. Theorem 8.1 appears as Theorem 43 in Lévy [1965]. The proof given in this section and some of the generalizations found in the Exercises appeared in Barwise-Fisher [1970]. Exercise 8.16 is due to Jensen-Karp [1972].

Theorem 8.2 and its Corollary 8.3 are due to Kripke [1963] and Platek [1965] in the form stated here. Their content, however, goes back to Takeuti-Kino [1962].