Chapter II
Some Admissible Sets

Having gained some feeling for the theory KPU we turn to its intended models, admissible sets. Admissible sets come in many sizes and shapes. In this chapter the student is introduced to some of the more attractive ones in a cursory fashion. We will delve into their structure and properties later.

1. The Definition of Admissible Set and Admissible Ordinal

It facilitates matters if we fix a largest possible universe of sets over an arbitrary collection M of urelements once and for all. We define by recursion:

$$\mathrm{V}_M(0) = 0;$$
$$\mathrm{V}_M(\alpha+1) = \text{Power set of } (M \cup \mathrm{V}_M(\alpha));$$
$$\mathrm{V}_M(\lambda) = \bigcup_{\alpha<\lambda} \mathrm{V}_M(\alpha), \text{ if } \lambda \text{ is a limit; and}$$
$$\mathbb{V}_M = \bigcup_\alpha \mathrm{V}_M(\alpha),$$

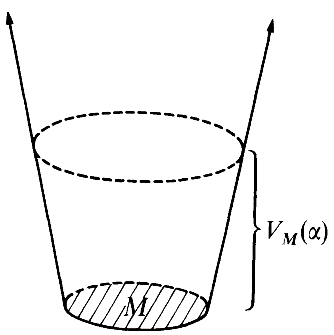

Fig. 1A. The universe $\mathbb{V}_M$ of sets on M

where the union in the last equation is taken over all ordinals α. (The reason for letting $V_M(0)=0$, rather than $V_M(0)=M$, is that $\mathbb{V}_M$ is to be a collection of *sets* on M.) We use $\in_M$ for the membership relation on $\mathbb{V}_M$, dropping the subscript if there is little room for confusion. If $\mathfrak{M}=\langle M,\text{---}\rangle$ we write $\mathbb{V}_{\mathfrak{M}}$ for $\mathbb{V}_M$. If M is the empty collection we write $V(\alpha)$ for $V_M(\alpha)$ and $\mathbb{V}$ for $\mathbb{V}_M$.

1.1 Definition. Let $\mathsf{L}^* = \mathsf{L}(\in,\ldots)$ and a structure $\mathfrak{M}$ for L be given. An *admissible set over* $\mathfrak{M}$ is a model $\mathfrak{A}_{\mathfrak{M}}$ of KPU of the form

$$\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A,\in,\ldots),$$

where $M\cup A$ is transitive in $\mathbb{V}_M$, and $\in$ is the restriction of $\in_M$ to $M\cup A$. The admissible set $\mathfrak{A}_{\mathfrak{M}}$ is *admissible above* $\mathfrak{M}$ if $M\in A$, i.e., if $\mathfrak{A}_{\mathfrak{N}}\models \mathrm{KPU}^+$. We use special Roman $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ to range over admissible sets. When we need to exhibit the underlying structure $\mathfrak{M}$ we write $\mathbb{A}_{\mathfrak{M}}$.

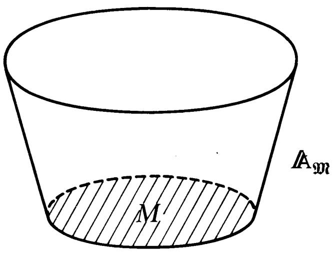

Fig. 1 B. A typical admissible set over $\mathfrak{M}$

In other words, admissible sets are models of KPU which are transitive hunks of $\mathbb{V}_M$ with the intended interpretation $\in_M$ of the membership symbol. *Warning:* While the interpretation of the membership symbol must be the natural one, Definition 1.1 makes no such demands on the interpretations of any other symbols in the list ... of $\mathsf{L}(\in,\ldots)$. They must fend for themselves. For example, if $\mathsf{L}^* = \mathsf{L}(\in, \mathsf{P})$ and the admissible set $\mathbb{A}_M = (\mathfrak{M}; A,\in,P)$ is a model of Power, then there is nothing to guarantee that $P(a)$ is the real power set of a; it may very well be only a small subset of the real power set of a.

1.2 Lemma. *Suppose* $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A,\in_M,\ldots)$ *and* $\mathfrak{B}_{\mathfrak{N}} = (\mathfrak{N}; B,\in_{\mathfrak{N}},\ldots)$ *and* $\mathfrak{A}_{\mathfrak{M}} \subseteq \mathfrak{B}_{\mathfrak{N}}$. *If* $M\cup A$ *is transitive in* $\mathbb{V}_M$, *then* $\mathfrak{A}_{\mathfrak{M}} \subseteq_{\mathrm{end}} \mathfrak{B}_{\mathfrak{N}}$.

Proof. Recall the definition given in I.8.1. If $a\in A$ then $a_{\in_M} = a = a_{\in_N}$ since $M\cup A$ is transitive in $\mathbb{V}_M$. ☐

This lemma also holds for $\mathfrak{B}_{\mathfrak{N}} = \mathbb{V}_{\mathfrak{N}}$, except that $\mathbb{V}_{\mathfrak{N}}$ is not a proper structure. This trivial lemma is of real importance. With the results of I.8 it insures that Δ predicates and Σ operations of KPU have the same meaning in all admissible sets that they have in $\mathbb{V}_{\mathfrak{M}}$.

1.3 A Comparison. Consider the two operations TC and P (Power set) and an admissible set $\mathbb{A}=(\mathfrak{M},A,\in,P)$ satisfying the Power set axiom. Given $a\in A$ the expressions

$$\mathrm{TC}(a),\quad \mathsf{P}(a)$$

each have *two* possible interpretations. For TC there are the sets b_0, b_1 such that

$$\mathbb{A}\models \mathrm{TC}(a)=b_0 \quad\text{and}\quad \mathbb{V}_{\mathfrak{M}}\models \mathrm{TC}(a)=b_1,$$

where $\mathbb{A}=(\mathfrak{M};A,\in)$. For P there are the sets c_0, c_1 such that

$$\mathbb{A}\models \mathsf{P}(a)=c_0 \quad\text{and}\quad \mathbb{V}_{\mathfrak{M}}\models \mathsf{P}(a)=c_1.$$

Since $\mathbb{A}_{\mathfrak{M}}\subseteq_{\mathrm{end}}\mathbb{V}_{\mathfrak{M}}$ and TC is a Σ operation, we have $\mathbb{V}_{\mathfrak{M}}\models \mathrm{TC}(a)=b_0$; and so $b_0=b_1$. Thus b_0 is the real transitive closure of a, so that

$$b_0=\bigcap\{b\,|\,b \text{ transitive, } a\subseteq b\}.$$

For P, however, this fails. Since $x\subseteq y$ is Δ_0 we get $c_0\subseteq c_1$ but that's all. Typically, c_0 will be a *proper* subset of the real power set c_1 of a.

1.4 Definitions. A *pure set* in $\mathbb{V}_M$ is a set a with empty support; i. e., one with $\mathrm{TC}(a)\cap M=0$. (For example, ordinals are pure sets.) A *pure admissible set* is an admissible set which is a model of KP; i. e., one without urelements. Pure admissibles can be written $\mathbb{A}=(A,\in,\ldots)$. If $\mathsf{L}^*=\{\in\}$ then we write A for $\mathbb{A}=\langle A,\in\rangle$.

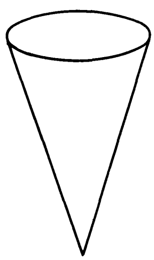

Fig. 1 C. A pure admissible set A

1.5 Theorem. *If* $\mathbb{A}_{\mathfrak{M}}=(\mathfrak{M};A,\in)$ *is admissible and* $A_0=\{a\in A\,|\,a \text{ is a pure set}\}$, *then* A_0 *is a pure admissible set, called the* pure part *of* $\mathbb{A}_{\mathfrak{M}}$. (See Fig. 1 D.)

Proof. By 1.2 we have $A_0\subseteq_{\mathrm{end}}\mathbb{A}_{\mathfrak{M}}\subseteq_{\mathrm{end}}\mathbb{V}_{\mathfrak{M}}$. By absoluteness, sp$(a)$ has the same meaning in $\mathbb{A}_{\mathfrak{M}}$ and $\mathbb{V}_{\mathfrak{M}}$. Let us check Δ_0 Collection leaving the easier axioms as exercises to help the student master absoluteness arguments. Suppose $a,b\in A_0$ and suppose A_0 satisfies $\forall x\in a\,\exists y\,\varphi(x,y,b)$, where φ is Δ_0. If $\varphi(x,y,b)$ is true

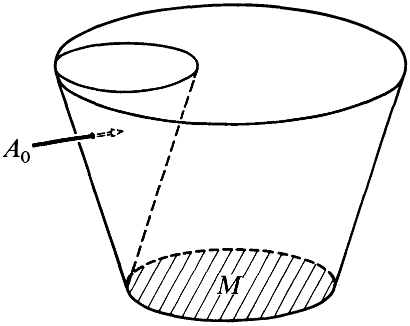

Fig. 1 D. The pure part A_0 of an admissible set $\mathbb{A}_{\mathfrak{M}}$

in A_0 it is also true in $\mathbb{A}_{\mathfrak{M}}$ by absoluteness so $\mathbb{A}_{\mathfrak{M}}$ satisfies:

$$\forall x \in a\, \exists y\, [\mathrm{sp}(y)=0 \wedge \varphi(x,y,b)].$$

Applying Σ collection in $\mathbb{A}_{\mathfrak{M}}$, we get a $c \in \mathbb{A}_{\mathfrak{M}}$ such that

(1) $\quad \forall x \in a\, \exists y \in c\, [\mathrm{sp}(y)=0 \wedge \varphi(x,y,b)]$, and

(2) $\quad \forall y \in c\, \exists x \in a\, [\mathrm{sp}(y)=0 \wedge \varphi(x,y,b)]$.

From (2) we get $\mathrm{sp}(c)=0$, since $\mathrm{sp}(c)=\bigcup\{\mathrm{sp}(y) \mid y \in c\}$; so $c \in A_0$. But then (1) is a Δ_0 formula with parameters from A_0, true in $\mathbb{A}_{\mathfrak{M}}$, hence true in A_0. □

1.6 Exercise. Verify Pair, Union and Δ_0 Separation for the proof of 1.5. Notice that Extensionality is trivial from the transitivity of A_0, and that Foundation is trivial by the well-foundedness of $\mathbb{A}_{\mathfrak{M}}$.

1.7 Definitions. The *ordinal of* an admissible set $\mathbb{A}_{\mathfrak{M}}$, denoted by $o(\mathbb{A}_{\mathfrak{M}})$, is the least ordinal not in $\mathbb{A}_{\mathfrak{M}}$; equivalently, it is the order type of the ordinals in $\mathbb{A}_{\mathfrak{M}}$. An ordinal α is *admissible* if $\alpha=o(\mathbb{A}_{\mathfrak{M}})$ for some $\mathfrak{M}$ and some admissible set $\mathbb{A}_{\mathfrak{M}}$. An ordinal α is *$\mathfrak{M}$-admissible* if $\alpha=o(\mathbb{A}_{\mathfrak{M}})$ for some $\mathbb{A}_{\mathfrak{M}}$ which is admissible above $\mathfrak{M}$ (in the sense of 1.1).

1.8 Corollary. *The ordinal α is admissible iff $\alpha=o(A)$ for some pure admissible set.*

Proof. If $\alpha=o(\mathbb{A}_{\mathfrak{M}})$ and $\mathbb{A}_{\mathfrak{M}}$ is admissible, then $\alpha=o(A_0)$, where A_0 is the pure part of $\mathbb{A}_{\mathfrak{M}}$. □

What kinds of ordinals are admissible? In the next section we will see that ω is admissible. From our development of ordinal arithmetic in Chapter I we see that if α is admissible then α is closed under ordinal successor, addition, multiplication, exponentiation and similar functions of ordinal arithmetic. Thus the least admissible $\alpha>\omega$ is bigger that

$$\omega+\omega,\ \omega\cdot\omega,\ \omega^{\omega},\ \omega^{\omega^{\omega}},\ldots,\varepsilon_0,\ldots,$$

where the operations are from ordinal (*not* cardinal) arithmetic. In § 3 we will prove that every infinite cardinal κ is admissible and that for any $\beta<\kappa$, there are κ admissible ordinals α between β and κ. (Thus, κ is a limit of admissibles.)

1.9 Definitions. Let $\mathbb{A}=\mathbb{A}_{\mathfrak{M}}=(\mathfrak{M}; A,\in,\ldots)$. We often use the following notation and terminology. An object x is *in* $\mathbb{A}$ if $x\in M\cup A$, and we write $x\in\mathbb{A}$. A *relation on* $\mathbb{A}$ is a relation on $M\cup A$. An n-ary relation S on $\mathbb{A}$ is Σ_1 *on* $\mathbb{A}$ if there is a Σ_1 formula φ, possibly having constants $y_1,\ldots,y_k$ from $\mathbb{A}$, such that

(1) $S(x_1,\ldots,x_n)$ iff $\mathbb{A}\models\varphi[x_1,\ldots,x_n]$

for all $x_1,\ldots,x_n\in\mathbb{A}$. The relation S is Π_1 *on* $\mathbb{A}$ if (1) holds for some Π_1 formula φ, and S is Δ_1 *on* $\mathbb{A}$ if S is both Σ_1 and Π_1 on $\mathbb{A}$. A function F *on* $\mathbb{A}$ is a function with domain a subset of $(M\cup A)^n$ for some n and range a subset of $M\cup A$. We say F is Σ_1 *on* $\mathbb{A}$ if its graph is Σ_1 on $\mathbb{A}$.

1.10 Proposition. *Let* $\mathbb{A}$ *be admissible.*

(i) *If* $a\in\mathbb{A}$ *then* a *is* Δ_1 *on* $\mathbb{A}$.

(ii) *If* $x\in\mathbb{A}$ *then* $\{x\}$ *is* Δ_1 *on* $\mathbb{A}$.

(iii) *The* Σ_1 *relations of* $\mathbb{A}$ *are closed under* $\wedge$, $\vee$, $\exists x\in a$, $\forall x\in a$, $\exists x$.

Proof. (i) $x\in a$ iff $\mathbb{A}\models x\in a$; so a is Δ_1 as a subset of $\mathbb{A}$. Part (ii) follows from (i). Part (iii) is immediate from the fact that every Σ formula is equivalent, over $\mathbb{A}$, to a Σ_1 formula and the Σ formulas are closed under the operations mentioned. □

1.11 Exercise. Let $\mathbb{A}=\mathbb{A}_{\mathfrak{M}}$ be admissible and let G be an operation defined on all triples in $\mathbb{A}_{\mathfrak{M}}$ whose restriction to $\mathbb{A}_{\mathfrak{M}}$ is Σ_1 definable on $\mathbb{A}_{\mathfrak{M}}$. Define, in $\mathbb{V}_{\mathfrak{M}}$,

$$F(x,y)=G(x,y,\{F(x,z)\,|\,z\in\mathrm{TC}(y)\})$$

by Σ recursion. Show that $x\in\mathbb{A}_{\mathfrak{M}}$ implies $F(x)\in\mathbb{A}_{\mathfrak{M}}$, and that $F\restriction\mathbb{A}_{\mathfrak{M}}\times\mathbb{A}_{\mathfrak{M}}$ is Σ_1 on $\mathbb{A}_{\mathfrak{M}}$. (This should be easy if the student has understood what has come before.)

2. *Hereditarily Finite Sets*

A set $a\in\mathbb{V}_{\mathfrak{M}}$ is *hereditarily finite* if $\mathrm{TC}(a)$ is finite. $\mathbb{HF}_M$ is the set of hereditarily finite sets of $\mathbb{V}_{\mathfrak{M}}$. It can also be defined by:

$$\mathrm{HF}_M(0)=0;$$
$$\mathrm{HF}_M(n+1)=\text{set of all } \textit{finite} \text{ subsets of } (M\cup\mathrm{HF}_M(n));$$
$$\mathbb{HF}_M=\textstyle\bigcup_{n<\omega}\mathrm{HF}_M(n).$$

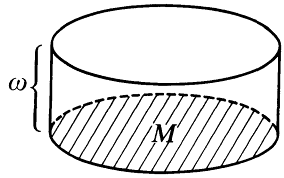

Fig. 2A. $\mathbb{HF}_{\mathfrak{M}}$

2.1 Theorem. *$\mathbb{HF}_{\mathfrak{M}}$ is the smallest admissible set over $\mathfrak{M}$. More precisely, let $\mathsf{L}^* = \mathsf{L}(\in, \ldots)$ and let $\mathbb{HF}_{\mathfrak{M}} = (\mathfrak{M}; \mathrm{HF}_M, \in, \ldots)$ be an L^*-structure.*

(i) *$\mathbb{HF}_{\mathfrak{M}}$ is admissible.*

(ii) *If $\mathbb{A}_{\mathfrak{M}} = (\mathfrak{M}; A, \in, \ldots)$ is admissible, then $\mathbb{HF}_{\mathfrak{M}} \subseteq \mathbb{A}_{\mathfrak{M}}$.*

There is a difference between $\mathbb{HF}_{\mathfrak{M}}$ as a set and as an L^*-structure, but it is usually clear which we have in mind.

Proof of 2.1. (ii) is trivial since A must be closed under pair and union so that $\mathrm{HF}_M(n) \subseteq A$ for all n, by induction on n. Let us prove that $\mathbb{HF}_{\mathfrak{M}}$ is admissible. Since $\mathbb{HF}_{\mathfrak{M}}$ is transitive in $\mathbb{V}_{\mathfrak{M}}$ we get extensionality and foundation for free. Note that each $\mathrm{HF}_{\mathfrak{M}}(n)$ is also transitive. If $x, y \in \mathrm{HF}_{\mathfrak{M}}(n)$ then $\{x, y\} \in \mathrm{HF}_{\mathfrak{M}}(n+1)$ so we have Pair. If $a \in \mathrm{HF}_{\mathfrak{M}}(n)$ then $\bigcup a$ is a finite subset of $\mathrm{HF}_{\mathfrak{M}}(n)$ so is an element of $\mathrm{HF}_{\mathfrak{M}}(n+1)$, and we have Union. If $a \subseteq b \in \mathrm{HF}_{\mathfrak{M}}(n)$ then $a \in \mathrm{HF}_{\mathfrak{M}}(n)$ since a subset of a finite set is finite, so we have full separation, hence Δ_0 Separation. Similarly, we have full collection for if $a \in \mathbb{HF}_{\mathfrak{M}}$ has say k elements $y_1, \ldots, y_k$ and for each of these y_i there is an x_i such that $\varphi(x_i, y_i)$ holds, then all $x_1, \ldots, x_k$ occur in some $\mathrm{HF}_{\mathfrak{M}}(n)$, hence $\{x_1, \ldots, x_k\} \in \mathrm{HF}_{\mathfrak{M}}(n+1)$. ☐

2.2 Corollary. *The smallest admissible set is*

$$\mathbb{HF} = \{a \in \mathbb{V} \mid a \text{ is a pure hereditarily finite set}).$$

The smallst admissible ordinal is ω.

Proof. $\mathbb{HF}$ is the pure part of any $\mathbb{HF}_{\mathfrak{M}}$, *and* $o(\mathbb{HF}) = \omega$. ☐

$\mathbb{HF}$ is really where the study of admissible sets began. It was in attempting to generalize recursion theory on the integers that admissible sets developed (by a rather tortuous route) and, as we now show, recursion theory on the integers amounts to the study of Σ_1 and Δ_1 on $\mathbb{HF}$.

2.3 Theorem. *Let S be a relation on natural numbers.*

(i) *S is* r.e. *iff S is Σ_1 on* $\mathbb{HF}$.

(ii) *S is recursive iff S is Δ_1 on* $\mathbb{HF}$.

There are relativized versions of 2.3 that are just as easy to prove. For example, S is recursive *in f* iff S is Δ_1 on $\langle \mathbb{HF}, \in, f \rangle$, which by 2.1 is admissible.

For the proof of 2.3 we assume familiarity with the elements of ordinary recursion theory.

Proof of 2.3 ($\Rightarrow$). Note that (i) implies (ii) since S is recursive iff S and $\neg S$ are r.e. Nevertheless, first we prove the ($\Rightarrow$) part of (ii) to help us prove the corresponding half of (i). It clearly suffices to show that every recursive total function on the integers f can be extended to a Σ_1 function $\hat{f}$ on $\mathbb{HF}$ by the definition:

$$\begin{aligned}\hat{f}(x) &= f(x), \quad \text{for} \quad x \in \omega \\ &= 0, \qquad \text{for} \quad x \notin \omega.\end{aligned}$$

To prove this we take a definition of recursive function where one starts with basic total functions and closes under some operations which take one from total functions to total functions. We choose the one given in Shoenfield [1967], though any other will go through just as easily. Thus, the (total) recursive functions are the smallest class containing $+$, $\cdot$, $K_<$ (the characteristic function of $<$), $F(x_1,\ldots,x_n)=x_i$ (the projection functions), closed under composition and closed under the μ-operation (if G is a recursive function such that $\forall \vec{n}\, \exists m\, [G(\vec{n},m)=0]$ and for all $\vec{n}$,

$$F(\vec{n}) = \mu m[G(\vec{n},m)=0], \quad \textit{the least } m \textit{ such that} \quad G(\vec{n},m)=0,$$

then F is recursive).

We have already defined Σ_1 operations $+$ and $\cdot$ in § I.6 and the Δ_0 relation $\alpha<\beta$ in Table 2. The composition of total Σ_1 functions is total and Σ_1 so we need only verify that the class of f with Σ_1 $\hat{f}$ are closed unter the μ-operator. Suppose $\forall \vec{n}\, \exists m\, (G(\vec{n},m)=0)$, that G is recursive, that $\hat{G}$ is Σ_1 on $\mathbb{HF}$ by the inductive hypothesis and that $F(\vec{n})=\mu m[G(\vec{n},m)=0]$.

Then $\hat{F}(\vec{x})=y$ iff

> Some x_i is not a natural number $\wedge\, y=0$; or all x_i and y are natural numbers and $G(\vec{x},y)=0$ and $\forall z<y\, \exists n\, [n\neq 0 \wedge G(\vec{x},z)=n]$.

This is Σ (since G is Σ_1), and hence it is Σ_1 by I.4.3. Thus every recursive function and predicate on ω is Δ_1 on $\mathbb{HF}$. But every r.e. predicate $S(\vec{x})$ can be written in the form $\exists n\, R(\vec{x},n)$, where R is recursive by a standard result of recursion theory; so every r.e. predicate is Σ_1 on $\mathbb{HF}$. □

To prove the other half of 2.3 we need the following lemma.

2.4 Lemma. *There is a function* $e\colon \omega \to \mathbb{HF}$ *with the following properties:*
- (i) *e is a bijection (e is one-one and onto);*
- (ii) *e is Σ_1 on $\langle \mathbb{HF}, \in \rangle$;*
- (iii) *$n=e(m)$ is a recursive relation of m, n; and*
- (iv) *for any Δ_0 formula $\varphi(x_1,\ldots,x_k)$ the relation $\langle \mathbb{HF}, \in \rangle \models \varphi(e(n_1),\ldots,e(n_k))$ of $n_1,\ldots,n_k$ is recursive.*

Proof. Let us define:

$$\begin{array}{ll}
e(0)=0 & \\
e(1)=\{e(0)\}=\{0\} & (1=2^0) \\
e(2)=\{e(1)\} & (2=2^1) \\
\vdots & \vdots \\
e(5)=\{e(2), e(0)\} & (5=2^2+2^0) \\
\vdots & \vdots \\
e(2^{n_1}+2^{n_2}+\cdots+2^{n_k})=\{e(n_1),\ldots,e(n_k)\} & (n_1>n_2>\cdots>n_k).
\end{array}$$

We are using the binary expansion of integers, so $e(n)$ is defined for all n by Σ recursion. Hence e is Σ_1 by I.6.4. and 2.16. An easy induction shows that e is one-one and onto. To prove (iii), note that if $e(k)$ is an integer n, then $e(k+2^k)=n+1$. To prove (iv), note that $e(n)\in e(m)$ iff n is an exponent in the binary expansion $2^{k_1}+\cdots+2^{k_l}$ of m. Other Δ_0 formulas follow by induction on Δ_0 formulas using familiar closure properties of the recursive predicates. □

Proof of 2.3 ($\Leftarrow$). Now suppose S in Σ_1 on $\mathbb{HF}$, say $S(n)$ iff $\mathbb{HF}\models\exists y\,\varphi(n,y)$, where φ is Δ_0, the case where S has more than one argument being similar. Then $S(n)$ iff $\exists k\exists m\,[e(k)=n\wedge\varphi(e(k),e(m))]$. The part within brackets is recursive by 2.4 (iii) and 2.4 (iv), so S is r.e. □

There is another way one might want to consider ordinary recursion theory. Suppose we think of the natural numbers not as finite ordinals but as primitive objects (urelements) given to us with some structure, say

$$\mathfrak{N}=\langle N,\otimes,\oplus\rangle$$

where we use $\mathbf{0},\mathbf{1},\mathbf{2},\ldots$ for these natural numbers, $N=\{\mathbf{0},\mathbf{1},\mathbf{2},\ldots\}$, and $\otimes$, $\oplus$ for addition and multiplication in $\mathfrak{N}$.

2.5 Theorem. *Let S be a relation on $\mathfrak{N}=\langle N,\otimes,\oplus\rangle$. Then*
(i) *S is* r.e. *iff S is Σ_1 on $\mathbb{HF}_{\mathfrak{N}}$;*
(ii) *S is recursive iff S is Δ_1 on $\mathbb{HF}_{\mathfrak{N}}$.*

The proof is similar to 2.3. For a different proof one can use Theorem VI.4.12. We include 2.5 because it suggests that one might consider Δ_1 and Σ_1 on $\mathbb{HF}_{\mathfrak{M}}$ as definitions of recursive and r.e. on $\mathfrak{M}$, for an arbitrary structure $\mathfrak{M}$. This is, in effect, what Montague suggested in Montague [1968] for the case of what he calls $\aleph_0$-recursion theory.

Another definition of a recursion theory over an arbitrary structure $\mathfrak{M}$ was presented in Moschovakis [1969a], the generalizations of recursive and r.e. being called *search computable* and *semi-search computable.* What Moschovakis did was this. He started with $\mathfrak{M}=\langle M,R_1,\ldots,R_k\rangle$, chose a new object $0\notin M$ and closed $M\cup\{0\}$ under an ordered pair function, calling the result M^*. Then in M^* he introduced, via an inductive definition similar to Kleene's for higher type recursion theory, the class of search computable functions. Theorem 2.6

below, due to Gordon [1970] shows that these two approaches coincide. This result will not be used in this book. The reader unfamiliar with search computability should consider 2.6 as a *definition.* A proof is sketched in the notes for those familiar with the notions involved.

2.6 Theorem. *Let $\mathfrak{M}=\langle M,R_1,\ldots,R_k\rangle$, and let S be a relation on $\mathfrak{M}$.*
(i) *S is semi-search computable on $\mathfrak{M}$ iff S is Σ_1 on $\mathbb{HF}_{\mathfrak{M}}$.*
(ii) *S is search computable on $\mathfrak{M}$ iff S is Δ_1 on $\mathbb{HF}_{\mathfrak{M}}$.*

In the context of recursion theory one often works with $\mathbb{HF}_{\mathfrak{M}}$ as opposed to $\mathfrak{M}$ itself since the relations on $\mathfrak{M}$ which are semi-search computable are not always definable at all over $\mathfrak{M}$ itself. The trouble with your average structure $\mathfrak{M}$ is that it lacks coding ability. This lack is what rests behind the need for the following class of formulas. We will not use them until Chapters IV and VI.

2.7 Definition. The *extended first order formulas* of $\mathsf{L}^*=\mathsf{L}(\in,\ldots)$ form the smallest collection containing:
(i) all formulas of L,
(ii) all Δ_0 formulas of L^*,
and closed under:
(iii) $\wedge$, $\vee$, $\forall u\in v$, $\exists u\in v$ (u, v any kind of variables), $\forall p$, $\exists p$,
(iv) $\exists a$.
The *coextended first order formulas* of L^* form the smallest collection containing (i) and (ii) and closed under (iii) and under:
(v) $\forall a$.

The extended first order formulas do not allow unbounded universal quantifiers over sets. The coextended formulas form the dual collection. That these collections are more natural than they seem at first is shown by the next result and the fact that its converse also holds. The converse is a theorem of Feferman [1968] and will not be needed here.

2.8 Proposition. *Let $\varphi(v_1,\ldots,v_n)$ be extended first order. For any structures $\mathfrak{A}_{\mathfrak{M}}\subseteq_{\mathsf{end}}\mathfrak{B}_{\mathfrak{M}}$, and any $x_1,\ldots,x_n\in\mathfrak{A}_{\mathfrak{M}}$:*
(i) *$\mathfrak{A}_{\mathfrak{M}}\models\varphi[x_1,\ldots,x_n]$ implies $\mathfrak{B}_{\mathfrak{M}}\models\varphi[x_1,\ldots,x_n]$.*

Proof. The difference between this and Lemma I.8.4 rests in the fact that these structures have the same urelement base $\mathfrak{M}$. The proof is a trivial proof by induction. □

2.9 Example. Let L be the language of number theory with $\mathbf{0}, \mathbf{1}, \otimes, \oplus$. In a model $\mathfrak{N}$ of arithmetic the set of standard finite integers is defined in $\mathbb{HF}_{\mathfrak{N}}$ by the extended first-order formula $\psi(x)$ shown here:

$$\exists a\,[x\in a\wedge\forall z\in a\,[z\neq\mathbf{0}\rightarrow\exists y\in a\,((y\oplus\mathbf{1})=z)]].$$

This formula is Σ_1, in fact, so that *the set of finite integers is semi-search computable over* $\mathfrak{N}$. The sentence $\forall p\, \psi(p)$ is extended first order, and $\mathbb{HF}_{\mathfrak{N}} \models \forall p\, \psi(p)$ iff $\mathfrak{N}$ is the standard model of arithmetic.

The extended and coextended first order formulas of $\mathsf{L}(\in)$, when interpreted over $\mathbb{HF}_{\mathfrak{M}}$, form a very small fragment of so called weak second-order logic. *Weak second-order logic over* $\mathfrak{M}$ just consists of the language $\mathsf{L}(\in)$ interpreted in $\mathbb{HF}_{\mathfrak{M}}$. At least that is one way of describing it.

2.10—2.16 Exercises

2.10. Prove that $\mathbb{HF}_{\mathfrak{M}} \subseteq \mathrm{V}_{\mathfrak{M}}(\omega)$, and that $\mathbb{HF}_{\mathfrak{M}} = \mathrm{V}_{\mathfrak{M}}(\omega)$ iff $\mathfrak{M}$ is finite.

2.11. If $\mathbb{A}$ is a pure admissible set, $\mathbb{A} \neq \mathbb{HF}$, then $\omega \in \mathbb{A}$.

2.12. If $\mathbb{A}_{\mathfrak{M}}$ is admissible and $o(\mathbb{A}_{\mathfrak{M}}) = \omega$ then the pure part of $\mathbb{A}_{\mathfrak{M}}$ is $\mathbb{HF}$.

2.13. Prove that $\mathbb{HF}$ is a Δ_1 subset of any admissible set.

2.14. Let X be Σ_1 on $\mathbb{HF}$. Prove that X is Σ_1 on every admissible set.

2.15. Prove that $\mathrm{V}_M(\omega)$ is admissible iff M is finite.

2.16. Prove that $H(l) = \{n_1, \ldots, n_k\}$, where $l = 2^{n_1} + \cdots + 2^{n_k}$, $n_1 > \cdots > n_k$, is a Σ_1 operation of l.

2.17 Notes. Theorem 2.3 is a standard result of recursion theory, as is 2.5. Theorem 2.6 is due to Gordon [1970]. The class of extended first order formulas, introduced in 2.7, will be quite important in Chapters IV and VI when dealing with structures without much coding machinery built into them.

We conclude the notes to this section with a sketch of a proof of Theorem 2.6. The proof uses results from later chapters. We first show that every semi-search computable relation on $\mathfrak{M}$ is Σ_1 on $\mathbb{HF}_{\mathfrak{M}}$. The basic relation of the theory is

$$\{e\}\,(\vec{x}) \rightarrow y$$

and it is defined by means of a first order positive Σ inductive definition and so, by the main result of § VI.2, is Σ_1 on $\mathbb{HF}_{\mathfrak{M}}$.

To prove the other half, it suffices to show that some complete Σ_1 relation on $\mathbb{HF}_{\mathfrak{M}}$ is semi-search computable. Let T be the diagram of $\mathfrak{M}$ plus the axioms KPU coded up on M^* by means of the pairing function and let $S(x)$ iff "*x codes a sentence provable from T*".

It is implicit in Chapter V (and explicit in Chapter VIII) that S is a complete Σ_1 prediciate. But the relation "*p is a proof of x from axioms in T*" must be search computable (if the notion is to make any sense).

Hence the relation $\exists p$ ("*p is a proof of x from axioms in T*") is semi-search computable, since the semi-search computable relations are closed under $\exists$. Note that this gives another proof of 2.3 and 2.5.

3. Sets of Hereditary Cardinality Less Than a Cardinal κ

The next admissible set we come across is a simple generalization of $\mathbb{HF}_{\mathfrak{M}}$. Let κ be any infinite cardinal and define

$$H(\kappa)_M = \{a \in \mathbb{V}_M \mid \mathrm{TC}(a) \text{ has cardinality less than } \kappa\}.$$

In particular $H(\omega)_M = \mathbb{HF}_M$. If M is empty then we write $H(\kappa)$ for $H(\kappa)_M$. If κ is regular then we can also characterize $H(\kappa)_M$ as follows:

$$G(0) = 0;$$
$$G(\alpha+1) = \{a \subseteq M \cup G(\alpha) \mid \mathrm{card}(a) < \kappa\};$$
$$G(\lambda) = \bigcup_{\alpha<\lambda} G(\alpha), \text{ if } \lambda \text{ is a limit ordinal};$$
$$H(\kappa)_M = \bigcup_{\alpha} G(\alpha) = \bigcup_{\alpha<\kappa} G(\alpha).$$

For singular κ this characterization fails: a bad set sneaks into $G(\kappa+1)$, if not before (see Exercise 3.7). *We use the axiom of choice in this section.*

3.1 Theorem. *For all infinite cardinals κ, the set $H(\kappa)_{\mathfrak{M}} = (\mathfrak{M}; H(\kappa)_M, \in)$ is admissible. It is admissible above $\mathfrak{M}$ iff $\kappa > \mathrm{card}(M)$.*

The proof of this is not as simple as one might expect in the case when κ is a singular cardinal. For κ regular, though, it is a trivial result. We will return to the proof of 3.1 after Theorem 3.3.

3.2 Theorem. *Let κ be regular. If $(\mathfrak{M}; H(\kappa)_M, \in, \ldots)$ is a structure for $\mathsf{L}(\in, \ldots)$, then it is admissible.*

Proof. Just like for the case $\kappa = \omega$. In fact, we get full separation and full collection. $\square$

The next result, besides giving us a lot of new examples of admissible sets, also allows us to prove Theorem 3.1 for singular κ. By $\mathrm{card}(\mathsf{L}^*)$ we mean the cardinality of the set of symbols of L^*.

3.3 Theorem (A Löwenheim-Skolem Lemma). *Let $\mathsf{L}^* = \mathsf{L}(\in, \ldots)$ and let $\mathbb{A}_{\mathfrak{M}} = (\mathfrak{M}; A, \in, \ldots)$ be admissible. Let $A_0 \subseteq M \cup A$ be transitive and let κ be a cardinal with $\kappa \geqslant \mathrm{card}(\mathsf{L}^*) \cup \mathrm{card}(A_0)$. There is an admissible set $\mathbb{B}_{\mathfrak{N}} = (\mathfrak{N}; B, \in, \ldots)$ with the following properties:*

(i) $\mathfrak{N} \prec \mathfrak{M}$ *($\mathfrak{N}$ is an elementary submodel of $\mathfrak{M}$);*
(ii) $\mathrm{card}(N \cup B) \leqslant \kappa$*;*
(iii) $A_0 \subseteq N \cup B$*;*
(iv) *For any φ of L^* and any $x_1, \ldots, x_n \in A_0$, $\mathbb{B}_{\mathfrak{N}} \models \varphi[x_1, \ldots, x_n]$ iff $\mathbb{A}_{\mathfrak{M}} \models \varphi[x_1, \ldots, x_n]$; and*
(v) *In particular, $\mathbb{A}_{\mathfrak{M}} \equiv \mathbb{B}_{\mathfrak{N}}$ ($\equiv$ indicates elementary equivalence).*

Proof. (Note that it is not asserted that $\mathbb{B}_{\mathfrak{N}} \subseteq \mathbb{A}_{\mathfrak{M}}$!) Think of $\mathbb{A}_{\mathfrak{M}}$ as a single sorted structure

$$\mathfrak{A} = \langle M \cup A, M, A, \in, ---, \cdots \rangle,$$

where $\mathfrak{M} = \langle M, --- \rangle$. Find $\mathfrak{A}_1 \prec \mathfrak{A}$ with $A_0 \subseteq \mathfrak{A}_1$ and $\operatorname{card}(\mathfrak{A}_1) \leqslant \kappa$ by the usual Löwenheim-Skolem-Tarski Theorem. $\mathfrak{A}_1$ has the form:

$$\mathfrak{A}_1 = \langle N \cup A_1, N, A_1, \in \cap (N \cup A_1)^2, ---, \cdots \rangle.$$

Since there are no urelements in A_1, $\operatorname{clpse}(N \cup A_1) \subseteq V_N$. Let $B = \operatorname{clpse}(N \cup A_1) \cap V_N$ (i. e. B is the set of sets in $\operatorname{clpse}(N \cup A_1)$) and note that the set of urelements in $\operatorname{clpse}(N \cup A_1)$ is just N. Let $f = c_{N \cup A_1}$ in the notation of I.7. Since $N \cup A_1$ is extensional, f establishes an isomorphism between $\mathfrak{A}_1$ and a structure $\mathfrak{B} = \langle N \cup B; N, B, \in, ---, \cdots \rangle$, by the collapsing lemma. The isomorphism f is the identity for $x \in A_0$ by Lemma I.7.1. Let $\mathfrak{N} = \langle N, --- \rangle$ and $\mathbb{B}_{\mathfrak{N}} = (\mathfrak{N}; B, \in, \ldots)$, and all the properties of the theorem are clear. $\square$

Proof of 3.1. It remains to show that if κ is singular then $H(\kappa)_{\mathfrak{M}} = (\mathfrak{M}; H(\kappa)_M, \in)$ is admissible over $\mathfrak{M}$. Let κ^+ be the next cardinal $> \kappa$. The only axiom which is not immediate is Δ_0 Collection. ($H(\kappa)_M$ still satisfies full separation since $a \subseteq b \in H(\kappa)_M \Rightarrow a \in H(\kappa)_M$.) Suppose

(1) $\qquad \forall x \in a\, \exists y\, \varphi(x, y, z)$

is true in $H(\kappa)_{\mathfrak{M}}$, where $z \in H(\kappa)_{\mathfrak{M}}$. Now φ has only a finite number of symbols of L^* in it, so we may ignore the rest of L^* in what follows. Thus we assume $\operatorname{card}(\mathsf{L}^*) \leqslant \aleph_0 < \kappa$. Let $a, z \in X$, X transitive, $\operatorname{card}(X) < \kappa$; say $X = \mathrm{TC}(\{a, z\})$. Since (1) is true in $H(\kappa)_{\mathfrak{M}}$, it persists to $H(\kappa^+)_{\mathfrak{M}}$, which is admissible by 3.2. Using 3.3 we can get an admissible $\mathbb{A}_{\mathfrak{N}}$, with $\mathfrak{N} \subseteq \mathfrak{M}$, so that $X \subseteq \mathbb{A}_{\mathfrak{N}}$, $\operatorname{card}(\mathbb{A}_{\mathfrak{N}}) < \kappa$ and (1) holds in $\mathbb{A}_{\mathfrak{N}}$. By Δ_0 Collection in $\mathbb{A}_{\mathfrak{N}}$ there is a $b \in \mathbb{A}_{\mathfrak{N}}$ so that

(2) $\qquad \forall x \in a\, \exists y \in b\, \varphi(x, y)$

holds in $\mathbb{A}_{\mathfrak{N}}$. But $\mathbb{A}_{\mathfrak{N}} \subseteq_{\text{end}} H(\kappa)_{\mathfrak{M}}$, so (2) holds in $H(\kappa)_{\mathfrak{M}}$ by persistence. $\square$

3.4 Corollary. *Every infinite cardinal is an admissible ordinal. For every uncountable cardinal κ and $\beta < \kappa$, there is an admissible α where $\beta < \alpha < \kappa$.*

Proof. $\kappa = o(H(\kappa))$ proves the first assertion in view of 3.1. The second assertion follows from 3.3 by setting $\mathbb{A}_{\mathfrak{M}} = H(\kappa)$, $A_0 = \beta + 1$ and $\kappa = \operatorname{card}(\beta)$. $\square$

We could have also proved 3.1 by using the following result of Lévy [1965] (proved there for $M = 0$).

3.5 Theorem. *For all uncountable cardinals $\kappa < \lambda$ we have $H(\kappa)_{\mathfrak{M}} \prec_1 H(\lambda)_{\mathfrak{M}}$. That is, any Σ_1 sentence with constants from $H(\kappa)_{\mathfrak{M}}$ true in $H(\lambda)_{\mathfrak{M}}$ is already true in $H(\kappa)_{\mathfrak{M}}$.*

Proof. This is really just like the proof of 3.1. Suppose the formula $\exists y\, \varphi(x,y)$ holds in $H(\lambda)_{\mathfrak{M}}$, where $x \in H(\kappa)_{\mathfrak{M}}$. As in 3.1 we find an admissible set $\mathbb{A}_{\mathfrak{N}}$, with $\mathfrak{N} \subseteq \mathfrak{M}$, such that the formula holds in $\mathbb{A}_{\mathfrak{N}}$ and $\mathrm{card}(\mathbb{A}_{\mathfrak{N}}) < \kappa$. But then $\mathbb{A}_{\mathfrak{N}} \subseteq_{\mathrm{end}} H(\kappa)_{\mathfrak{M}}$; and so the formula holds in $H(\kappa)_{\mathfrak{M}}$ by persistence. □

One of the earliest generalizations of ordinary recursion theory on the integers goes back to papers of Takeuti where he defines recursive functions on ordinals less than some cardinal κ. When one looks for the analogue of $\mathbb{HF}$ for ordinal recursion theory on κ, the proper structure turns out to be $\mathrm{L}(\kappa)$, the set of sets constructible before κ, rather than $H(\kappa)$. The reason is that one needs to be able to code up the sets by ordinals in some way analogous to Lemma 2.4, if one is to prove a result like Theorem 2.3. We will study the constructible sets in § 5 and again in Chapter V.

3.6—3.7 Exercises

3.6. Let $\kappa < \lambda$ be infinite cardinals and let X be a transitive subset of $H(\lambda)$ with $\mathrm{card}(X) = \kappa$. Prove that there is an admissible set A of cardinality κ with $X \subseteq A$ such that $A \prec_1 H(\lambda)$, where $\prec_1$ is explained in I.8.10 and I.8.11. [Iterate 3.2.]

3.7. Let κ be a singular cardinal, let M be a set of urelements of cardinality κ and define G as above. Show that already in $G(2)$ there is a set not in $H(\kappa)_M$. [G is defined just before 3.1.]

3.8 Notes. The technique of following an application of the Downward Lowenheim-Skolem Theorem with an application of the Collapsing Lemma (as in 3.3) is extremely important. In some sense, it goes back to Gödel's original proof that the GCH holds in L, the constructible universe. It was later used implicitly by Takeuti when proving, in our terminology, that uncountable cardinals are stable. Theorem 3.5 is due to Lévy [1965]. Theorem 3.1 is due to Kripke and Platek.

4. *Inner Models: The Method of Interpretations*

We assume that the reader understands the notion of an *interpretation*, say I, of one theory T_1 (formulated in a language L_1) in another theory T_2 (formulated in a possibly different language L_2). Readable accounts of this can be found in Enderton [1972] and Shoenfeld [1967]. We use φ^I for the interpretation of φ given by I. Thus φ is in L_1, φ^I is in L_2; and if φ is an axiom of T_1, then φ^I is a theorem of T_2. If $\mathfrak{M}$ is a model of T_2, then we use $\mathfrak{M}^{-I}$ for the L_1-structure given by $\mathfrak{M}$ and I; $\mathfrak{M}^{-I}$ is a model of T_1. Note that Enderton uses ${}^{\pi}\mathfrak{M}$ for our $\mathfrak{M}^{-I}$; while Shoenfeld doesn't make explicit the model theoretic counterpart of the syntactic transformation I.

We give a simple example. We can interpret Peano arithmetic in KPU by having I define

"natural number"	by	"finite ordinal",
"addition"	by	"ordinal addition",
"multiplication"	by	"ordinal multiplication",
"$x<y$"	by	"$x \in y$".

Then every axiom φ of Peano arithmetic (in $+, \cdot, <$) goes over to a theorem φ^I of KPU (formulated in $\mathsf{L}(\in, \ldots)$). If $\mathfrak{A}_{\mathfrak{M}} \models \mathrm{KPU}$ then $\mathfrak{A}_{\mathfrak{M}}^{-I} = \langle N', +, \cdot, < \rangle$ is the model of Peano arithmetic whose domain N' is the set of finite ordinals of $\mathfrak{A}_{\mathfrak{M}}$, and where $+, \cdot, <$ are the restrictions of the corresponding functions and relations of $\mathfrak{A}_{\mathfrak{M}}$ to N'. Rather than launch into a discussion of just how we use interpretations to construct admissible sets, we give a straight-forward illustration. The following result is a generalization of Theorem 1.5.

4.1 Theorem. *Let $\mathbb{A}_{\mathfrak{M}} = (\mathfrak{M}; A, \in)$ be admissible and let $\mathfrak{M}_0 \subseteq \mathfrak{M}$ be a substructure of $\mathfrak{M}$ whose universe M_0 is Σ_1 definable on $\mathbb{A}_{\mathfrak{M}}$. If $\mathbb{B}_{\mathfrak{M}_0} = (\mathfrak{M}_0; B, \in)$ is defined by $B = \{a \in A \mid \mathrm{sp}(a) \subseteq M_0\}$, then $\mathbb{B}_{\mathfrak{M}_0}$ is admissible over $\mathfrak{M}_0$.*

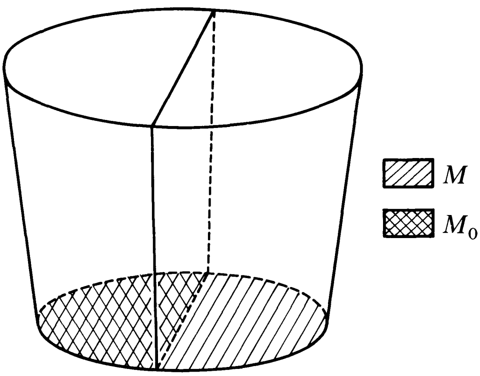

Fig. 4A. $\mathbb{B}_{\mathfrak{M}_0}$, the left half of $\mathbb{A}_{\mathfrak{M}}$

Proof. B is transitive so extensionality and foundation come for free. Pair, Union and Δ_0 Separation are routine. We prove Δ_0 Collection. Suppose $\mathbb{B}_{\mathfrak{M}_0}$ satisfies $\forall x \in a\, \exists y\, \varphi(x,y)$, where a and any other parameters in φ are in $\mathbb{B}_{\mathfrak{M}_0}$. For fixed $y \in \mathbb{B}_{\mathfrak{M}_0}$, we find in $\mathbb{A}_{\mathfrak{M}}$ that $\forall p \in \mathrm{sp}(y)\, \theta(p)$, where $\theta(p)$ is the Σ_1 formula defining $\mathfrak{M}_0$ in $\mathbb{A}_{\mathfrak{M}}$. Hence $\mathbb{A}_{\mathfrak{M}}$ satisfies the formula:

$$\forall x \in a\, \exists y\, [\varphi(x,y) \wedge \forall p \in \mathrm{sp}(y)\, \theta(p)].$$

By Σ Collection in $\mathbb{A}_{\mathfrak{M}}$, there is a b in $\mathbb{A}_{\mathfrak{M}}$ so that

(1) $$\forall x \in a\, \exists y \in b\, \varphi(x,y) \quad \text{and} \quad \forall y \in b\, \forall p \in \mathrm{sp}(y)\, \theta(p).$$

But then $\mathrm{sp}(b) \subseteq M_0$. So $b \in B$, and (1) holds in $\mathbb{B}_{\mathfrak{M}_0}$ by absoluteness. $\square$

Properly viewed, Theorem 4.1 is a trivial application of an interpretation I. If $\theta(p)$ defines $\mathfrak{M}_0$ then I, in effect, simply redefines:

"x is an urelement" by "x is an urelement $\wedge\, \theta(x)$",

"x is a set" by "x is a set $\wedge\, \forall p \in \mathrm{sp}(x)\, \theta(p)$",

and leaves $\in$ and the symbols of L unchanged. The proof that every axiom φ of KPU becomes a theorem φ^I of KPU$'$ is just like the proof of 4.1 (where KPU$'$ is KPU with axioms asserting θ is closed under any function symbols of L). Hence, for every model $\mathfrak{A}_{\mathfrak{M}}$ of KPU$'$, the structure $\mathfrak{A}_{\mathfrak{M}}^{-I}$ is also a model of KPU. In Theorem 4.1 we have $\mathbb{B}_{\mathfrak{M}_0} = \mathfrak{A}_{\mathfrak{M}}^{-I}$. In this example we don't gain much by looking at it from the point of view of interpretation, but we will in more complicated situations.

The interpretation we just used has some important features in common with most of the interpretations we use. They are what Shoenfield [1967, § 9.5] calls transitive $\in$-interpretations.

4.2 Definition. Let $\mathsf{L}^* = \mathsf{L}(\in)$ and let I be an interpretation of L^* into KPU (as formulated in L^*). I is a *transitive $\in$-interpretation* if I leaves the symbols of L and $\in$ unchanged and merely "cuts down on the urelements and sets" so that the following are provable in KPU:

(i) if (x is an urelement)^I then x is an urelement;

(ii) if (x is a set)^I, then x is a set and for all $y \in x$, (y is an urelement)^I or (y is a set)^I.

If I is a transitive $\in$-interpretation and $\mathfrak{A}_{\mathfrak{M}} \models \mathrm{KPU}$ then $\mathfrak{A}_{\mathfrak{M}}^{-I}$ is called the *inner submodel* of $\mathbb{A}_{\mathfrak{M}}$ given by I.

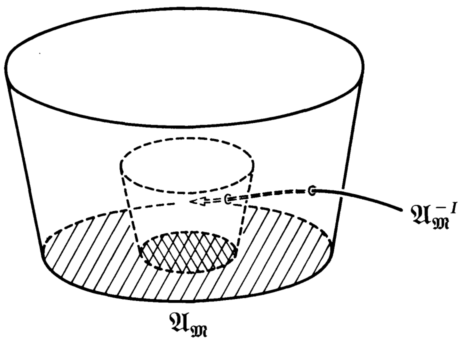

Fig. 4B. A model $\mathfrak{A}_{\mathfrak{M}}$ and an inner submodel

The conditions in 4.2 guarantee that $\mathfrak{A}_{\mathfrak{M}}^{-I} \subseteq_{\mathrm{end}} \mathfrak{A}_{\mathfrak{M}}$. Fig. 4B indicates the idea behind transitive $\in$-interpretations and inner submodels.

The following lemma is useful to keep in mind.

4.3 Lemma. *Let I be a transitive $\in$-interpretation.*

(i) $\mathrm{KPU} \vdash (\text{Extensionality})^I$;

(ii) *For each instance of foundation φ, we have* $\mathrm{KPU} \vdash \varphi^I$;

(iii) *For each Σ formula $\varphi(x)$:*

$$\mathrm{KPU} \vdash \mathrm{Urelement}(x)^I \vee \mathrm{Set}(x)^I \rightarrow [\varphi(x)^I \rightarrow \varphi(x)],$$

(iv) *For each Δ_0 formula φ:*

$$\mathrm{KPU} \vdash \mathrm{Urelement}(x)^I \vee \mathrm{Set}(x)^I \rightarrow [\varphi(x)^I \leftrightarrow \varphi(x)].$$

Proof. (i) $(\text{Extensionality})^I$ can be written as:

$$\mathrm{Set}^I(a) \wedge \mathrm{Set}^I(b) \wedge a \neq b \rightarrow \exists x\, [(\mathrm{Set}^I(x) \vee \mathrm{Urelement}^I(x)) \wedge \neg(x \in a \leftrightarrow x \in b)].$$

This follows immediately from property 4.2 (ii). To prove (ii) let φ be $\exists a\, \psi(a) \rightarrow \exists a\, [\psi(a) \wedge \neg \exists b \in a\, \psi(b)]$. Then φ^I states: If $\exists a\, [\mathrm{Set}^I(a) \wedge \psi^I(a)]$, then there is an a such that $\mathrm{Set}^I(a)$ and $\psi^I(a)$; but there is no b with $\mathrm{Set}^I(b)$ such that $b \in a$ and $\psi^I(b)$. This follows immediately by applying foundation to the formula: $\mathrm{Set}^I(a) \wedge \psi^I(a)$. Part (iii) follows model theoretically by the comment above about $\mathfrak{A}_{\mathfrak{M}}^{-I} \subseteq_{\text{end}} \mathfrak{A}_{\mathfrak{M}}$, for all $A_M \models \mathrm{KPU}$. It can also be proved directly by induction on φ. Part (iv) follows from (iii). $\square$

4.4 Exercise. Verify that the specific I defined on p. 56 is a transitive $\in$-interpretation.

5. *Constructible Sets with Urelements; $\mathbb{H}\mathrm{YP}_{\mathfrak{M}}$ Defined*

In this section we construct most of the more important admissible sets in one fell swoop by means of Gödel's hierarchy of constructible sets. For reasons which will become apparent, we restrict ourselves to the case where the language L has only a finite number of nonlogical symbols and where $\mathsf{L}^* = \mathsf{L}(\in)$. For simplicity we assume the symbols of L are relation symbols: a simple modification will extend the results to languages with function and constant symbols.

5.1 Apologia. There are two well known ways of defining the constructible sets in a theory without urelements, both developed by Gödel. The most intuitive is by iterating definability through the ordinals; the other uses some form of Gödel's $\mathscr{F}_1, \ldots, \mathscr{F}_8$. We have always preferred the former method but find ourselves forced to use the latter here. The reason is simple enough, but is one that doesn't arise in ZF. Many admissible sets $\mathbb{A}_{\mathfrak{M}}$ have ordinal $o(\mathbb{A}_{\mathfrak{M}}) = \omega$, i.e., are models of $\neg$Infinity, whereas natural ways of iterating first order definability need ω.

Even though we give up the iteration of full first order definability, we modify the usual approach (along lines used by Gandy [1975] and Jensen [1972]) via the $\mathscr{F}_i$'s to make it as similar to the definability approach as possible.

5.2 Assumption. *For the rest of* § 5 *we assume that* $\mathscr{F}_1,\ldots,\mathscr{F}_N$ *are* Σ_1 *operations (of two arguments each) introduced into* KPU *so that the following hold, where we define* $\mathscr{D}(b)=b\cup\{\mathscr{F}_i(x,y)\mid x,y\in b, 1\leqslant i\leqslant N\}$.

(i) $\mathscr{F}_1(x,y)=\{x,y\}$;

(ii) $\mathscr{F}_2(x,y)=\bigcup x$;

(iii) $\mathrm{KPU}\vdash \mathrm{sp}(\mathscr{F}_i(x,y))\subseteq \mathrm{sp}(x)\cup\mathrm{sp}(y)$, *for all* $i\leqslant N$;

(iv) $\mathrm{KPU}\vdash [\mathrm{Tran}(b)\to\mathrm{Tran}(\mathscr{D}(b))]$;

(v) *For each* Δ_0 *formula* $\varphi(x_1,\ldots,x_n)$ *with free variables among* $x_1,\ldots,x_n$ *and each variable* x_i, $i\leqslant n$, *there is a term* $\mathscr{F}$ *of* n *arguments built from* $\mathscr{F}_1,\ldots,\mathscr{F}_N$ *so that:*

$$\mathrm{KPU}\vdash \mathscr{F}(a,x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)=\{x_i\in a\mid \varphi(x_1,\ldots,x_n)\}.$$

There are many ways of fulfilling the assumptions. We will return to give a specific solution in § 6. Next, with 5.2 firmly in mind, we return to the development of set theory in KPU begun in Chapter I. First note that $\mathscr{D}$ is a Σ operation since $\mathscr{F}_1,\ldots,\mathscr{F}_N$ are. Define, in KPU, a Σ operation $\mathrm{L}(\cdot,\cdot)$ by recursion over the second argument:

$$\mathrm{L}(a,0)=\mathrm{TC}(a),$$

$$\mathrm{L}(a,\alpha+1)=\mathscr{D}(\mathscr{S}(\mathrm{L}(a,\alpha)))=\mathscr{D}(\mathrm{L}(a,\alpha)\cup\{\mathrm{L}(a,\alpha)\}),$$

$$\mathrm{L}(a,\lambda)=\bigcup\nolimits_{\alpha<\lambda}\mathrm{L}(a,\alpha)\quad\text{if}\quad \mathrm{Lim}(\lambda).$$

5.3 Definition. An object x is *constructible from* a, written $x\in\mathrm{L}(a)$, if $\exists\alpha\,[x\in\mathrm{L}(a,\alpha)]$. If x is constructible from 0, we say x is *constructible* and write $x\in\mathrm{L}$.

5.4 Lemma (of KPU). *For all sets* a *and ordinals* α*:*

(i) $a\in\mathrm{L}(a,1)$ *if* a *is transitive;*

(ii) $\mathrm{L}(a,\alpha)$ *is transitive;*

(iii) $\alpha<\beta$ *implies* $\mathrm{L}(a,\alpha)\subseteq\mathrm{L}(a,\beta)$*;*

(iv) $\mathrm{L}(a,\alpha)\in\mathrm{L}(a,\alpha+1)$*;*

(v) $x,y\in\mathrm{L}(a,\alpha)$ *implies* $\mathscr{F}_i(x,y)\in\mathrm{L}(a,\alpha+1)$, $1\leqslant i\leqslant N$*;*

(vi) $\alpha\in\mathrm{L}(a,\beta)$ *for some* β*;*

(vii) *An urelement* p *is in* $\mathrm{L}(a)$ *iff* $p\in\mathrm{sp}(a)$.

Proof. (i), (iii), (iv), (v) follow from the definition of $\mathrm{L}(a,\alpha)$ directly. Part (ii) is by induction on α using Assumption 5.2 (iv). Part (vii) is proved by showing that $p\in\mathrm{L}(a,\alpha)$ iff $p\in\mathrm{sp}(a)$ by induction on α (using 5.2 (iii)). This leaves (vi) which is also proved by induction on α. By the induction hypothesis we have $\forall\gamma<\alpha\,\exists\delta\,[\gamma\in\mathrm{L}(a,\delta)]$. So, by Σ Reflection, there is an ordinal λ such that

$\forall\gamma<\alpha\,\exists\delta<\lambda\,[\gamma\in L(a,\delta)]$. But then by (iii), every $\gamma<\alpha$ is in $L(a,\lambda)$; that is, $\alpha\subseteq L(a,\lambda)$. Now, applying Assumption 5.2(v) for the first time, we see that the set $b=\{x\in L(a,\lambda)\mid \mathrm{Ord}(x)\}$ is in $L(a,\lambda')$ for some $\lambda'\geqslant\lambda$. Since $L(a,\lambda)$ is transitive, b is an ordinal β and $\alpha\leqslant\beta$. Again, since $L(a,\lambda')$ is transitive, $\alpha\in L(a,\lambda')$, because either $\alpha=\beta$ or $\alpha\in\beta$. ▯

We now define a transitive $\in$-interpretation $\varphi^{L(a)}$ by the following:

$$(x \textit{ is a urelement})^{L(a)} \quad\text{is}\quad (x\in \mathrm{sp}(a)),$$

$$(x \textit{ is a set})^{L(a)} \quad\text{is}\quad (x \textit{ is a set } \wedge x\in L(a)),$$

leaving $\in$ and all symbols of the original language L unchanged. (We apologize for the two L's, but note that one is sanserif.) Note that this is indeed a transitive $\in$-interpretation in the sense of § 4.

5.5 Theorem. *For every axiom* φ *of* KPU$^+$, *we have* $\mathrm{KPU}\vdash\varphi^{L(a)}$.

Proof. We run through the axioms of KPU$^+$. Extensionality and Foundation follows from 4.3. Pair and Union follow from 5.2 (i), (ii), and 4.3 (iv). Δ_0 separation follows from 5.2 (v) and 4.3 (iv).

Δ_0 *Collection:* Suppose that $\varphi(x,y,z)$ is Δ_0. Working in KPU assume $a_0\in L(a)$, $z\in L(a)$ and $\forall x\in a_0\,\exists y\in L(a)\,[\varphi(x,y,z)^{L(a)}]$.

We suppress mention of z. Writing out $y\in L(a)$ and using 4.3 (iv) on $\varphi(x,y)$ we get $\forall x\in a_0\,\exists\alpha\,[\exists y\in L(a,\alpha)\,\varphi(x,y)]$. By Σ collection there is a β such that

$$\forall x\in a_0\,\exists\alpha<\beta\,[\exists y\in L(a,\alpha)\,\varphi(x,y)].$$

So, by 5.4 (iii), $\forall x\in a_0\,\exists y\in L(a,\beta)\,\varphi(x,y)$. Using 4.3 (iv) again, setting $b=L(a,\beta)$, we find:

$$[\forall x\in a_0\,\exists y\in b\,\varphi(x,y)]^{L(a)}.$$

Thus, the interpretation of Δ_0 Collection is provable.

Finally, we need to prove $[\exists b\,\forall x\,(x\in b\leftrightarrow\exists p\,(x=p))]^{L(a)}$. By Δ Separation it suffices to prove $[\exists b\,\forall p\,(p\in b)]^{L(a)}$. Let $b=\mathrm{TC}(a)=L(a,0)$. By definition, $b\in L(a,1)$ and $(x \textit{ is an urelement})^{L(a)}$ is just $x\in\mathrm{sp}(a)$; but $\mathrm{sp}(a)\subseteq b$. ▯

5.6 Definition. $L(\alpha)_{\mathfrak{M}}=(\mathfrak{M};\, L(M,\alpha)\cap\mathbb{V}_M,\in)$.

$L(\alpha)_{\mathfrak{M}}$ is a structure (for the language $\mathsf{L}^*=\mathsf{L}(\in)$) which may or may not be admissible. We use the intersection with $\mathbb{V}_M$ in 5.6 is just to take out the urelements in strict accord with our definition of structure for L^*.

5.7 Theorem. *If there is an admissible set* $\mathbb{A}=\mathbb{A}_{\mathfrak{M}}$ *above* $\mathfrak{M}$ *with* $o(\mathbb{A}_{\mathfrak{M}})=\alpha$, *then* $\mathrm{L}(\alpha)_{\mathfrak{M}}$ *is the smallest such. In other words* $\mathrm{L}(\alpha)_{\mathfrak{M}}$ *is admissible,* $\mathrm{L}(\alpha)_{\mathfrak{M}}\subseteq\mathbb{A}_{\mathfrak{M}}$, $M\in\mathrm{L}(\alpha)_{\mathfrak{M}}$ *and* $o(\mathrm{L}(\alpha)_{\mathfrak{M}})=\alpha$.

Proof. For $\beta<\alpha$, $\mathrm{L}(M,\beta)$ has the same meaning in $\mathbb{A}_{\mathfrak{M}}$ and $\mathbb{V}_{\mathfrak{M}}$ by absoluteness. Thus $\mathrm{L}(\alpha)_{\mathfrak{M}}$ is the inner model of $\mathbb{A}_{\mathfrak{M}}$ given by the interpretation defined above. Thus, in particular, $\mathrm{L}(\alpha)_{\mathfrak{M}}\subseteq\mathbb{A}_{\mathfrak{M}}$. By Theorem 5.5, $\mathrm{L}(\alpha)_{\mathfrak{M}}$ is admissible, and $M\in\mathrm{L}(\alpha)_{\mathfrak{M}}$. We see that $o(\mathrm{L}(\alpha)_{\mathfrak{M}})=\alpha$ from 5.4 (vi). □

If we had the option, the following definition would be printed in red. It introduces one of the principal objects of our study. Recall that $\mathbb{A}_{\mathfrak{M}}$ is admissible *above* $\mathfrak{M}$ if $\mathbb{A}_{\mathfrak{M}}\models\mathrm{KPU}^{+}$, the "+" being the part that gives "above".

5.8 Definition (The Next Admissible).
(i) $\mathbb{H}\mathrm{YP}_{\mathfrak{M}}=(\mathfrak{M};A,\in)$, where $A=\bigcap\{B\mid(\mathfrak{M};B,\in)$ is admissible above $\mathfrak{M}\}$.
(ii) $O(\mathfrak{M})=o(\mathbb{H}\mathrm{YP}_{\mathfrak{M}})$.

5.9 Theorem. (i) $\mathbb{H}\mathrm{YP}_{\mathfrak{M}}$ *is the smallest admissible set above* $\mathfrak{M}$.
(ii) $\mathbb{H}\mathrm{YP}_{\mathfrak{M}}=\mathrm{L}(\alpha)_{\mathfrak{M}}$ *for* $\alpha=O(\mathfrak{M})$.

Proof. We need only see that $\mathbb{H}\mathrm{YP}_{\mathfrak{M}}$ is *admissible* over $\mathfrak{M}$, since it is certainly contained in all other admissibles over $\mathfrak{M}$ with M an element. There is an admissible $\mathbb{A}_{\mathfrak{M}}$ with $M\in\mathbb{A}_{\mathfrak{M}}$ by 3.1. Let α be the least ordinal of the form $o(\mathbb{A}_{\mathfrak{M}})$, where $\mathbb{A}_{\mathfrak{M}}$ is admissible above $\mathfrak{M}$. Apply 5.7 to α and $\mathbb{A}_{\mathfrak{M}}$. □

We will study the structure of $\mathbb{H}\mathrm{YP}_{\mathfrak{M}}$ off and on in Chapters IV, VI, VII, VIII. For now we will simply state without proof, for the reader who understands the notions involved, that *if* $\mathcal{N}=\langle N,+,\cdot\rangle$ *is the usual structure of the natural numbers, then for any relation R on* $\mathcal{N}$, *R is hyperarithmetic iff* $R\in\mathbb{H}\mathrm{YP}_{\mathcal{N}}$, *and R is* Π_1^1 *on* $\mathcal{N}$ *iff R is* Σ_1 *on* $\mathbb{H}\mathrm{YP}_{\mathcal{N}}$. *Furthermore,* $O(\mathcal{N})=\omega_1^c=$ *the least nonrecursive ordinal.* Proofs will appear later.

For the next result recall that $\mathrm{L}(\alpha)=\mathrm{L}(0,\alpha)$; so $\mathrm{L}(\alpha)$ is a pure set. The proof is immediate from 1.8 and 5.7 with $M=0$.

5.10 Corollary. *An ordinal* α *is admissible iff* $\mathrm{L}(\alpha)$ *is a pure admissible set.*

An urelement free version of 5.9 is given below; the proof is similar.

5.11 Theorem. *Let a be a pure transitive set,* $A=\bigcap\{B\colon B$ *admissible,* $a\in B\}$. *Then A is admissible; it is of the form* $\mathrm{L}(a,\alpha)$ *for some admissible* α; *and it is the smallest admissible set with a as an element.*

5.12 Corollary. *If* α *is admissible and* $a\in\mathrm{L}(\alpha)$, *then* $\mathrm{L}(a,\alpha)=\mathrm{L}(\alpha)$.

Proof. Both $\mathrm{L}(a,\alpha)$ and $\mathrm{L}(\alpha)$ are the smallest admissible sets with a an element and ordinal α. □

The final results of this section will appear rather technical at present, but they are extremely important for much that is to follow.

5.13 Definition. Let $\mathbb{A}_{\mathfrak{M}}$ be admissible. Let $\varphi(v)$ be a Σ_1 formula with one free variable but with parameters from some set $X \subseteq \mathbb{A}_{\mathfrak{M}}$.

(i) If $\mathbb{A}_{\mathfrak{M}} \models \exists! v\, \varphi(v)$ and $\mathbb{A}_{\mathfrak{M}} \models \varphi[a]$, then $\varphi(v)$ is a Σ_1 *definition of a with parameters from X*.

(ii) If, in addition to (i), for every $\mathfrak{B}_{\mathfrak{M}} \supseteq_{\text{end}} \mathbb{A}_{\mathfrak{M}}$ which is a model of KPU we have $\mathfrak{B}_{\mathfrak{M}} \models \exists! v\, \varphi(v)$, then $\varphi(v)$ is a *good* Σ_1 definition of a with parameters from X.

5.14 Theorem. *Let $M = \mathrm{sp}(a)$ where a is transitive and let α be the least ordinal such that $\mathbb{A} = (\mathfrak{M}; \mathrm{L}(a,\alpha) \cap \mathbb{V}_M, \in)$ is admissible. Every $x \in \mathbb{A}$ has a good Σ_1 definition on $\mathbb{A}$ with parameters from $a \cup \{a\}$.*

Proof. Let B be the set of $x \in \mathbb{A}$ which have good Σ_1 definitions on $\mathbb{A}$ with parameters from $\mathscr{S}(a)$. Note the following:

(1) $\mathscr{S}(a) \subseteq B$; and

(2) $x, y \in B$ implies $\mathscr{F}_i(x,y) \in B$ for $1 \leqslant i \leqslant N$.

For (2) we need the fact that $\mathscr{F}_i$ is Σ_1 definable in KPU without parameters, which was implicit in 5.2.

(3) *If $b \subseteq B$, then $\mathscr{D}(b) \subseteq B$.*

This follows from the fact that $\mathscr{D}(b) = b \cup \{\mathscr{F}_i(x,y) \mid x, y \in b, 1 \leqslant i \leqslant N\}$ and from (2). Next since $\mathrm{L}(\cdot,\cdot)$ is a Σ_1 operation of KPU we find:

(4) *If $\beta \in B$, then $\mathrm{L}(a,\beta) \in B$.*

We now prove, by induction on $\beta < \alpha$, that $\beta \in B$ and $\mathrm{L}(a,\beta) \subseteq B$.

Case 1. $\beta = 0$. 0 has a good Σ_1 definition and $\mathrm{L}(a,0) = a \subseteq B$ by (1).

Case 2. $\beta = \gamma + 1$. By induction hypothesis $\gamma \in B$ and $\mathrm{L}(a,\gamma) \subseteq B$. But if $\gamma \in B$ so is $\gamma + 1$. $\mathrm{L}(a,\gamma) \in B$ by (4). Thus

$$\mathscr{S}(\mathrm{L}(a,\gamma)) = \mathrm{L}(a,\gamma) \cup \{\mathrm{L}(a,\gamma)\} \subseteq B;$$

so $\mathrm{L}(a,\gamma+1) = \mathscr{D}(\mathscr{S}(\mathrm{L}(a,\gamma))) \subseteq B$, by (3).

Case 3. β is a limit ordinal. By the induction hypothesis we have $\beta \subseteq B$ and $\mathrm{L}(a,\beta) \subseteq B$, since $\beta = \{\gamma \mid \gamma < \beta\}$ and $\mathrm{L}(a,\beta) = \bigcup_{\gamma < \beta} \mathrm{L}(a,\gamma)$. Thus we need only prove $\beta \in B$. This, however, is the main point of the proof. By our choice of α and $\beta < \alpha$ we have $\mathrm{L}(a,\beta)_{\mathfrak{M}} = (\mathfrak{M}; \mathrm{L}(a,\beta) \cap \mathbb{V}_M, \in)$ is not admissible so there is a Δ_0 formula $\varphi(x,y,z)$ and there are objects $z, b \in \mathrm{L}(a,\beta)_{\mathfrak{M}}$ so that

(5) $\mathrm{L}(a,\beta)_{\mathfrak{M}} \models \forall x \in b\, \exists y\, \varphi(x,y,z)$, and

(6) $\mathrm{L}(a,\beta)_{\mathfrak{M}} \models \neg \exists c\, \forall x \in b\, \exists y \in c\, \varphi(x,y,z)$.

(Since $\mathrm{Lim}(\beta)$ holds, Δ_0 Collection is the only way for $L(a,\beta)_{\mathfrak{M}}$ to fail to be admissible by Exercise 5.16). Now $b,z\in B$, so they have good Σ_1 definitions $\sigma(u)$, $\psi(w)$ with parameters from $\mathscr{S}(a)$. Consider the following Σ formula $\theta(\beta)$:

$$\begin{aligned}&\mathrm{Ord}(\beta)\wedge\exists b\,\exists z\,[\sigma(b)\wedge\psi(z)\wedge\forall x\in b\,\exists y\in L(a,\beta)\,\varphi(x,y,z)\\&\qquad\wedge\forall\gamma<\beta\,\exists x\in b\,\forall y\in L(a,\gamma)\neg\varphi(x,y,z)]\,.\end{aligned}$$

Now clearly $\mathbb{A}\models\theta(\beta)$ so every end extension $\mathfrak{B}_{\mathfrak{M}}\models\theta(\beta)$. If $\mathfrak{B}_{\mathfrak{M}}\models\mathrm{KPU}$ then no "ordinal" of $\mathfrak{B}_{\mathfrak{M}}$ greater than β can satisfy θ by (5), (6). Similarly, no ordinal smaller can satisfy θ. Thus $\theta(\beta)$ defines β in every end extension of $\mathbb{A}_{\mathfrak{M}}$ satisfying KPU, so $\beta\in B$. $\square$

5.15 Corollary. *Every $a\in\mathbb{HYP}_{\mathfrak{M}}$ has a good Σ_1 definition on $\mathbb{HYP}_{\mathfrak{M}}$ with no parameters other than M and some $p_1,\ldots,p_k\in M$.*

5.16—5.20 Exercises

5.16. Let $M\subseteq\mathrm{sp}(a)$ and let λ be a limit ordinal. Show that $(\mathfrak{M};L(a,\lambda)\cap\mathbb{V}_M,\in)$ satisfies all the axioms of KPU except, possibly, Δ_0 Collection.

5.17. If κ is a cardinal, $\kappa>\mathrm{card}(M)$, then $L(\kappa)_{\mathfrak{M}}$ is admissible above $\mathfrak{M}$.

5.18. If κ,λ are uncountable cardinals, $\kappa<\lambda$ then $L(\kappa)_{\mathfrak{M}}\prec_1 L(\lambda)_{\mathfrak{M}}$.

5.19. $L(\kappa)$ is admissible for all cardinals $\kappa\geqslant\omega$.

5.20. Improve 5.4 (vi) by proving that $\alpha\in L(M,\alpha+\omega)$, assuming ω exists.

5.21 Notes. The constructible sets were first used by Gödel [1939] in his famous proof of the consistency of the generalized continuum hypothesis. In this paper, Gödel used iterated first-order definability. In the proof of Gödel [1940] the fundamental operations were introduced and used to generate the constructible sets. The approach to the constructible sets taken here borrows some ideas from Jensen [1972], but it is a little more complicated due to the presence of urelements and relations on them. We shall see that the complications only come up in fulfilling Assumption 5.2 in the next section.

6. Operations for Generating the Constructible Sets

We now turn to the task of finding $\mathscr{F}_1,\ldots,\mathscr{F}_N$ satisfying Assumption 5.2. We will see that we can get by with especially simple functions (substitutable functions). This will prove useful in understanding the sets constructible in ω steps.

The real strength of 5.2 resides in the requirement that for each Δ_0 formula $\varphi(x_1,\ldots,x_n)$, there is a term $\mathscr{F}$ built from the symbols $\mathscr{F}_1,\ldots,\mathscr{F}_N$ so that

$$\mathrm{KPU} \vdash \mathscr{F}(a,x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)=\{x_i \in a \mid \varphi(x_1,\ldots,x_n)\}.$$

We take care of this condition first.

We already have by 5.2:

$$(\mathscr{F}1) \qquad \mathscr{F}_1(x,y)=\{x,y\}, \quad \text{and}$$

$$(\mathscr{F}2) \qquad \mathscr{F}_2(x,y)=\bigcup x.$$

From these we obtain by various simple compositions the following:

$$\begin{aligned}\{x\} &= \mathscr{F}_1(x,x),\\ x\cup y &= \bigcup\{x,y\}=\mathscr{F}_2(\mathscr{F}_1(x,y),y),\\ \mathscr{S}(x) &= x\cup\{x\},\\ \langle x,y\rangle &= \{\{x\},\{x,y\}\},\\ \langle x_1,\ldots,x_n\rangle &= \langle x_1,\langle x_2,\ldots,x_n\rangle\rangle.\end{aligned}$$

The function $\mathscr{F}_2$ corresponds (in Lemma 6.1) to $\vee$ in Δ_0 formulas. To handle negations we need to define:

$$(\mathscr{F}3) \qquad \mathscr{F}_3(x,y)=x-y.$$

From this we get, by composition, $x\cap y=x-(x-y)=\mathscr{F}_3(x,\mathscr{F}_3(x,y))$.

The need to treat quantifiers leads us to the following more complicated functions:

$$(\mathscr{F}4) \qquad \mathscr{F}_4(x,y)=x\times y,$$

$$(\mathscr{F}5) \qquad \mathscr{F}_5(x,y)=\mathrm{dom}(x)=\{1^{\mathrm{st}}(z)\mid z\in x,\ z \text{ an ordered pair}\},$$

$$(\mathscr{F}6) \qquad \mathscr{F}_6(x,y)=\mathrm{rng}(x)=\{2^{\mathrm{nd}}(z)\mid z\in x,\ z \text{ an ordered pair}\},$$

$$(\mathscr{F}7) \qquad \mathscr{F}_7(x,y)=\{\langle u,v,w\rangle \mid \langle u,v\rangle\in x,\ w\in y\},$$

$$(\mathscr{F}8) \qquad \mathscr{F}_8(x,y)=\{\langle u,w,v\rangle \mid \langle u,v\rangle\in x,\ w\in y\}.$$

The functions $\mathscr{F}_7$, $\mathscr{F}_8$ are annoying. They arise from the peculiar nature of the ordered n-tuple. We tend to think of $\langle x_1,x_2,x_3,x_4\rangle$ as a rather symmetric object but it is, in fact, far from it. We can form it from x_1 and $\langle x_2,x_3,x_4\rangle$ (since it is just $\langle x_1,\langle x_2,x_3,x_4\rangle\rangle$) but we cannot form it from, say x_4 and $\langle x_1,x_2,x_3\rangle$ or from x_3 and $\langle x_1,x_2,x_4\rangle$ using $\mathscr{F}_1,\ldots,\mathscr{F}_6$. This accounts for the appearance of $\mathscr{F}_7$ and $\mathscr{F}_8$.

It now remains only to add the functions which correspond to atomic formulas:

$(\mathscr{F}9)$ $\qquad \mathscr{F}_{\mathsf{U}}(x,y) = \{z \in x \mid z \text{ is an urelement}\}$,

$(\mathscr{F}10)$ $\qquad \mathscr{F}_{=}(x,y) = \{\langle v,u\rangle \in y \times x \mid u = v\}$,

$(\mathscr{F}11)$ $\qquad \mathscr{F}_{\in}(x,y) = \{\langle v,u\rangle \in y \times x \mid u \in v\}$,

and for each relation symbol $\mathsf{R}(x_1,\ldots,x_n)$ of L an operation:

$(\mathscr{F}12)$—$(\mathscr{F}K)$ $\mathscr{F}_{\mathsf{R}}(x,y) = \{\langle p_n,\ldots,p_1,v\rangle \mid \langle p_n,\ldots,p_1\rangle \in x, \mathsf{R}(p_1,\ldots,p_n)$, and $v \in y\}$

In order to prove the desired result we prove something a little more general. It gives us a better inductive hypothesis in our proof which uses induction on Δ_0 formulas. For technical reasons, we have inverted the order of the variables in 6.1. For the same reason, there is an inversion taking place in lines $(\mathscr{F}10)$, $(\mathscr{F}11), \ldots, (\mathscr{F}K)$.

6.1 Lemma. *For every Δ_0 formula $\varphi(x_1,\ldots,x_n)$ with free variables among $x_1,\ldots,x_n$, there is a term $\mathscr{F}_\varphi$ built up from the symbols $\mathscr{F}_1,\ldots,\mathscr{F}_K$ so that*

$$\mathrm{KPU} \vdash \mathscr{F}_\varphi(a_1,\ldots,a_n) = \{\langle x_n,\ldots,x_1\rangle \in a_n \times \cdots \times a_1 \mid \varphi(x_1,\ldots,x_n)\}.$$

Proof. We treat $\mathsf{L}^* = \mathsf{L}(\in)$ as a single sorted language with symbols U (for urelement) and S (for set), $\in$, $=$, $\mathsf{R}_1,\ldots,\mathsf{R}_l$, and variables $x_1,x_2,x_3,\ldots$. Whenever we write a formula φ as $\varphi(x_1,\ldots,x_n)$ we mean that all the free variables of are among $x_1,\ldots,x_n$, but not all of these variables need actually appear as free variables in φ. For the purpose of this proof we need two special definitions. We call a formula of L^* an *orderly* formula if it satisfies the following condition: whenever a quantifier $\exists x_j$ or $\forall x_j$ occurs in φ, the index j is the largest index of all the free variables in the scope of the quantifier. By simply renaming bound variables systematically, we have:

(a) *Every Δ_0 formula of L^* is logically equivalent to an orderly Δ_0 formula with the same free variables.*

We call a formula $\varphi(x_1,\ldots,x_n)$ a termed-formula, or *t-formula*, if there is a term $\mathscr{F}_\varphi$ such that the conclusion of 6.1 holds. Note that there is a possible ambiguity here since a formula with free variables among x_1,x_2 is also a formula with free variables among x_1,x_2,x_3 and so could be written as $\varphi(x_1,x_2)$ or as $\varphi(x_1,x_2,x_3)$. To be completely precise, we should say that φ with free variables among $x_1,\ldots,x_n$ is a t-formula. Line (e) below will show us that we don't have to be this careful.

Our goal is to prove that every Δ_0 formula is a t-formula. We want to prove this by induction on Δ_0 formulas, but we must dispose of certain logical trivialities before we can treat even the atomic formulas. These trivialities are handled in $(b)-(j)$ below.

(b) *If* KPU $\vdash \varphi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n)$ *and* ψ *is a t-formula then so is* φ.

This last is clear. Combining (a) and (b) allows us to restrict attention to orderly Δ_0 formulas, so Lemma 6.1 follows finally from (z) below.

(c) *If* $\varphi(x_1, \ldots, x_n)$ *is* $\psi(x_1, \ldots, x_{n-1})$ *and* ψ *is a t-formula then so is* φ.

Define $\mathscr{F}_\varphi(a_1, \ldots, a_n) = a_n \times \mathscr{F}_\psi(a_1, \ldots, a_{n-1})$. This proves (c).

(d) *If* $\varphi(x_1, \ldots, x_n)$ *is* $\psi(x_1, \ldots, x_{n+1})$ *and* ψ *is a t-formula then so is* φ.

Note that $\{0\} = \{\mathscr{F}_3(a_1, a_1)\} = \mathscr{F}_1(\mathscr{F}_3(a_1, a_1), \mathscr{F}_3(a_1, a_1))$, so we may use $\{0\}$ inside terms. Define next:

$$\begin{aligned}\mathscr{F}_\varphi(a_1, \ldots, a_n) &= \operatorname{rng}(\mathscr{F}_\psi(a_1, \ldots, a_n, \{0\}))\\ &= \operatorname{rng}(\{\langle 0, x_n, \ldots, x_1\rangle \mid x_i \in a_i \text{ and } \psi(x_1, \ldots, x_n, 0)\})\\ &= \{\langle x_n, \ldots, x_1\rangle \in a_n \times \cdots \times a_1 \mid \varphi(x_1, \ldots, x_n)\}\,.\end{aligned}$$

This proves (d).

(e) *If* $\varphi(x_1, \ldots, x_n)$ *is* $\psi(x_1, \ldots, x_m)$ *and* ψ *is a t-formula, then so is* φ.

For $n > m$ this follows by induction on n using (c). For $m > n$ this follows by induction on $m - n$ using (d). For $m = n$ there is nothing to prove.

(f) *If* $\varphi(x_1, \ldots, x_n)$ *is a t-formula, so is* $\neg\varphi$.

Define $\mathscr{F}_{\neg\varphi}(a_1, \ldots, a_n) = a_n \times \cdots \times a_1 - F_\varphi(a_1, \ldots, a_n)$. This proves (f).

(g) *If* $\varphi(x_1, \ldots, x_n)$ *and* $\psi(x_1, \ldots, x_n)$ *are t-formulas so is* $\varphi \wedge \psi$.

Define $\mathscr{F}_{\varphi\wedge\psi}(a_1, \ldots, a_n) = \mathscr{F}_\varphi(a_1, \ldots, a_n) \cap \mathscr{F}_\psi(a_1, \ldots, a_n)$. This proves (g).

(h) *The t-formulas are closed under propositional connectives.*

This follows by (b), (e), (f) and (g). In the following we use $\varphi(x/y)$ to denote the result of replacing all free occurrences of y by x.

(i) *If* $\psi(x_1, \ldots, x_n)$ *is a t-formula and* $\varphi(x_1, \ldots, x_{n+1})$ *is* $\psi(x_1, \ldots, x_{n-1}, x_{n+1}/x_n)$, *then* φ *is a t-formula.*

If $n = 1$, define $\mathscr{F}_\varphi(a_1, a_2) = \mathscr{F}_\psi(a_2) \times a_1$. If $n > 1$, define:

$$\begin{aligned}\mathscr{F}_\varphi(a_1, \ldots, a_{n+1}) &= \mathscr{F}_8(\mathscr{F}_\psi(a_1, \ldots, a_{n-1}, a_{n+1}), a_n)\\ &= \{\langle x_{n+1}, \ldots, x_1\rangle \mid x_n \in a_n \text{ and}\\ &\qquad \langle x_{n+1}, x_{n-1}, \ldots, x_1\rangle \in \mathscr{F}_\psi(a_1, \ldots, a_{n-1}, a_{n+1})\}.\end{aligned}$$

(j) *If* $\psi(x_1,x_2)$ *is a t-formula and* $\varphi(x_1,\ldots,x_n)$ *is* $\psi(x_{n-1}/x_1,x_n/x_2)$, *then* φ *is a t-formula.*

This makes sense only if $n\geqslant 2$ and is non-trivial only if $n>2$. To prove (j) define:

$$\begin{aligned}\mathscr{F}_\varphi(a_1,\ldots,a_n)&=\mathscr{F}_7(\mathscr{F}_\psi(a_{n-1},a_n),a_{n-2}\times\ldots\times a_1)\\&=\{\langle x_n,\ldots,x_1\rangle\in a_n\times\ldots\times a_1\,|\,\langle x_n,x_{n-1}\rangle\in\mathscr{F}_\psi(a_{n-1},a_n)\}\,.\end{aligned}$$

In (k)—(v) we prove that atomic formulas are t-formulas.

(k) *For all n, if* $\varphi(x_1,\ldots,x_n)$ *is* $\mathsf{U}(x_n)$ *then* φ *is a t-formula.*

For (k) define $\mathscr{F}_\varphi(a_1,\ldots,a_n)=\mathscr{F}_{\mathsf{U}}(a_n,a_n)\times a_{n-1}\times\cdots\times a_1$.

(l) $(x_1=x_2)$ *is a t-formula by* ($\mathscr{F}$ 10).

(m) $(x_n=x_{n+1})$ *is a t-formula by* (l) *and* (j).

(n) $(x_n=x_m)$ *is a t-formula for all* $m>n$.

This follows by induction on m using (m) for the base and (i) for the induction step.

(p) $(x_n=x_m)$ *is a t-formula for all* n,m.

For $n<m$, this is (n). For $n=m$, take $\mathscr{F}_\varphi(a_1,\ldots,a_n)=a_n\times\cdots\times a_1$. For $n>m$, note that $(x_n=x_m)$ iff $(x_m=x_n)$, so the result follows from (b) and (n).

(q) $(x_1\in x_2)$ *is a t-formula by* ($\mathscr{F}$ 11).

(r) $(x_{n+1}\in x_{n+2})$ *is a t-formula by* (q) *and* (j).

(s) *If* $\varphi(x_1,\ldots,x_n)$ *is* $(x_i\in x_j)$, *then* φ *is a t-formula.*

Let $\psi(x_1,\ldots,x_{n+2})$ be $(x_i=x_{n+1})\wedge(x_j=x_{n+2})\wedge(x_{n+1}\in x_{n+2})$, so that ψ is a t-formula by (p), (r), (e), (q). Hence we define:

$$\begin{aligned}&\mathscr{F}_\psi(a_1,\ldots,a_n,a_i,a_j)\\&=\{\langle x_{n+2},\ldots,x_1\rangle\in a_j\times a_i\times a_n\times\cdots\times a_1\,|\,x_i=x_{n+1},x_j=x_{n+2},x_i\in x_j\}\end{aligned}$$

We now use $\mathscr{F}_6$ to obtain the proof of (s):

$$\mathscr{F}_\varphi(a_1,\ldots,a_n)=\mathrm{rng}\,\mathrm{rng}(F_\psi(a_1,\ldots,a_n,a_i,a_j)))\,.$$

(t) *If* $\varphi(x_1,\ldots,x_{k+m})$ *is* $\mathsf{R}(x_{k+1},\ldots,x_{k+m})$, *where* R *is an m-ary relation symbol of* L *and* $k>1$, *then* φ *is a t-formula.*

Define $\mathscr{F}_\varphi(a_1, \ldots, a_{k+m}) = \mathscr{F}_{\mathsf{R}}(a_{k+m} \times \cdots \times a_{k+1}, a_k \times \cdots \times a_1)$. This proves (t).

(u) *If* R *is an m-ary relation symbol of* L *and* $\varphi(x_1, \ldots, x_n)$ *is* $\mathsf{R}(x_{i_1}, \ldots, x_{i_m})$, *then* φ *is a t-formula.*

Let $\psi(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ be $\mathsf{R}(x_{n+1}, \ldots, x_{n+m}) \wedge (x_{i_1} = x_{n+1}) \wedge \cdots \wedge (x_{i_m} = x_{n+m})$. Thus ψ is a t-formula by (t), (p), (e) and (g). Define

$$\mathscr{F}_\varphi(a_1, \ldots, a_n) = \mathrm{rng}^m(F_\psi(a_1, \ldots, a_m, a_{i_1}, \ldots, a_{i_m})),$$

where we apply rng m-times. This proves (u).

(v) *All atomic formulas are t-formulas.*

The only ones not covered by earlier cases are those of the form $\mathsf{S}(x_i)$, but $\mathsf{S}(x_i) \leftrightarrow \neg \mathsf{U}(x_i)$ so this follows from (b), (f) and (k). We have not only shown that every atomic formula is a t-formula, but also that the t-formulas are closed under propositional connectives. We now turn to bounded quantifiers.

(w) *If* $\psi(x_1, \ldots, x_{n+1})$ *is a t-formula and* $\varphi(x_1, \ldots, x_n)$ *is* $\exists x_{n+1} \in x_j \psi(x_1, \ldots, x_{n+1})$, *then* φ *is a t-formula.*

Let $\theta(x_1, \ldots, x_{n+1})$ be $(x_{n+1} \in x_j)$ so $\psi \wedge \theta$ is a t-formula $\sigma(x_1, \ldots, x_{n+1})$. Note that

$$\mathscr{F}_\sigma(a_1, \ldots, a_n, \bigcup a_j) = \{\langle x_{n+1}, \ldots, x_1\rangle \mid x_{n+1} \in x_j, x_i \in a_i \text{ for } 1 \leqslant i \leqslant n, \text{ and } \psi(x_1, \ldots, x_{n+1})\}.$$

So we may define $\mathscr{F}_\varphi$ by $\mathscr{F}_\varphi(a_1, \ldots, a_n) = \mathrm{rng}(\mathscr{F}_\psi(a_1, \ldots, a_n, \bigcup a_j))$. This proves (w).

(x) *If* $\psi(x_1, \ldots, x_k)$ *is a t-formula and* $\varphi(x_1, \ldots, x_n)$ *is* $\exists x_k \in x_j \psi(x_1, \ldots, x_k)$, *where* $k > n$, *then* φ *is a t-formula.*

The proof of (x) is just like that for (w) except we must apply rng $k-n$ times.

(y) *If* $\psi(x_1, \ldots, x_n)$ *is a t-formula and* $\varphi(x_1, \ldots, x_n)$ *is* $\forall x_k \in x_j \psi$, *where* $k > n$, *then* $\varphi(x_1, \ldots, x_n)$ *is a t-formula.*

This follows from (b), (f) and (x) since

$$\forall x_k \in x_j \psi \leftrightarrow \neg \exists x_k \in x_j \neg \psi .$$

(z) *All orderly* Δ_0 *formulas are t-formulas by* (v), (h), (x) and (y). □

6.2 Corollary. $\mathscr{F}_1, \ldots, \mathscr{F}_K$ *satisfy Assumption* 5.2(v).

Proof. Let $\varphi(x_1, \ldots, x_n)$ be a Δ_0 formula. We need a term $\mathscr{F}$ so that

$$\mathscr{F}(a, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = \{x_i \in a \mid \varphi(x_1, \ldots, x_n)\}.$$

But we can form this set from $\mathscr{F}_\varphi(\{x_1\},\dots,\{x_{i-1}\},a,\{x_{i+1}\},\dots,\{x_n\})$ by using $\mathscr{F}_6$ (rng) $n-i$ times and then $\mathscr{F}_5$ (dom). ☐

It may seem discouraging, but we are not through yet because $\mathscr{F}_1,\dots,\mathscr{F}_K$ do not give us the transitivity condition demanded by 5.2(iv). Recall that we want to show that Tran(b) implies Tran($\mathscr{D}(b)$), where:

$$\mathscr{D}(b)=b\cup\{\mathscr{F}_i(x,y)\mid x,y\in b,\ 1\leqslant i\leqslant N\}\,.$$

This reduces to showing that for $1\leqslant i\leqslant N$ we have:

(*) $\qquad b$ transitive and $x,y\in b$ implies $\mathrm{TC}(\mathscr{F}_i(x,y))\subseteq\mathscr{D}(b)$.

The only functions among $\mathscr{F}_1,\dots,\mathscr{F}_K$ for which condition (*) could fail are those involving n-tuples. To satisfy (*) for these functions define, for each $n\geqslant 2$, functions $\mathscr{G}_n^1,\mathscr{G}_n^2$, and $\mathscr{H}_1,\mathscr{H}_2,\mathscr{H}_3$ by:

$$\begin{aligned}
\mathscr{G}_n^1(x,y)&=\langle x_n,\dots,x_1,y\rangle &&,\ \text{if}\ \ x=\langle x_n,\dots,x_1\rangle\\
&=0 &&,\ \text{otherwise};\\[1ex]
\mathscr{G}_n^2(x,y)&=\{x_n,\langle x_{n-1}\dots x_1,y\rangle\} &&,\ \text{if}\ \ x=\langle x_n\dots x_1\rangle\\
&=0 &&,\ \text{otherwise};\\[1ex]
\mathscr{H}_1(x,y)&=\langle x,y\rangle;\\[1ex]
\mathscr{H}_2(x,y)&=\langle u,y,v\rangle &&,\ \text{if}\ \ x=\langle u,v\rangle\\
&=0 &&,\ \text{otherwise};\\[1ex]
\mathscr{H}_3(x,y)&=\{u,\langle y,v\rangle\} &&,\ \text{if}\ \ x=\langle u,v\rangle\\
&=0 &&,\ \text{otherwise.}
\end{aligned}$$

6.3 Definition. Let J be the largest number of places of a symbol of L. The functions $\mathscr{F}_1,\dots,\mathscr{F}_N$ use to generate L consist of $\mathscr{F}_1,\dots,\mathscr{F}_K$ together with $\mathscr{G}_n^1,\mathscr{G}_n^2$, for all $n\leqslant J$, plus $\mathscr{H}_1,\mathscr{H}_2,\mathscr{H}_3$.

6.4 Theorem. *The functions* $\mathscr{F}_1,\dots,\mathscr{F}_N$ *satisfy Assumption* 5.2.

Proof. We need to see that condition (*) holds for those functions involving n-tuples. Let us check $\mathscr{F}_7$ in some detail.

Suppose x,y are in the transitive set b. Let us list the members of $\mathrm{TC}(\mathscr{F}_7(x,y))$ which are not in b, together with the reason they are in $\mathscr{D}(b)$. Recall that $\mathscr{F}_7(x,y)=\{\langle u,v,w\rangle\mid\langle u,v\rangle\in x,\ w\in y\}$

Members of $\mathrm{TC}(\mathscr{F}_7(x,y))$	*Excuse for appearing in* $\mathscr{D}(b)$
$\langle u,v,w\rangle$ with $\langle u,v\rangle\in x, w\in y$	$\mathscr{G}_2^1(\langle u,v\rangle,w)$
$\{u\}$	$\mathscr{F}_1(u,u)$
$\{u,\langle v,w\rangle\}$	$\mathscr{G}_2^2(\langle u,v\rangle,w)$
$\langle v,w\rangle$	$\mathscr{H}_1(v,w)$
$\{v\}$	$\mathscr{F}_1(v,v)$
$\{v,w\}$	$\mathscr{F}_1(v,w)$.

Anything else in $\mathrm{TC}(\mathscr{F}_7(x,y))$ is in b, since b is transitive. $\mathscr{F}_8$ and the $\mathscr{F}_R$ are similar. The others are simpler. ▯

6.5—6.7 Exercises

6.5. Show that each of $\mathscr{F}_{K+1},\ldots,\mathscr{F}_N$ can be written as a term in $\mathscr{F}_1,\ldots,\mathscr{F}_K$. [Hint: This is fairly easy using 6.1.]

6.6. Define $\mathrm{L}'(a,\lambda)$ using only $\mathscr{F}_1\ldots\mathscr{F}_K$. Show that for limit ordinals λ, $\mathrm{L}'(a,\lambda)=\mathrm{L}(a,\lambda)$. The only point of using $\mathscr{F}_{K+1},\ldots,\mathscr{F}_N$ was to make each $\mathrm{L}(a,\alpha)$ transitive.

6.7. Verify condition (*) in the proof of 6.2 for $\mathscr{F}_8$.

6.8 Notes. The proof of 6.1 is one of the few places where the addition of urelements and relations on them causes extra work. Neither space nor memory permit us to list all the people who have found gaps in earlier proofs of this lemma.

When used in a class or seminar, section 6 should be supplemented with coffee (*not* decaffeinated) and a light refreshment. We suggest *Heatherton Rock Cakes.* (*Recipe:* Combine 2 cups of self-rising flour with 1 *t.* allspice and a pinch of salt. Use a pastry blender or two cold knives to cut in 6 *T* butter. Add $\frac{1}{3}$ cup each of sugar and raisins (or other urelements). Combine this with 1 egg and enough milk to make a stiff batter (3 or 4 *T* milk). Divide this into 12 heaps, sprinkle with sugar, and bake at 400 °F. for 10—15 minutes. They taste better than they sound.)

7. First Order Definability and Substitutable Functions

The functions $\mathscr{F}_1,\ldots,\mathscr{F}_N$ defined in 6.3 are actually quite simple compared with some Σ operations we might have used to satisfy Assumption 5.2. We will exploit this to prove the following theorem; the first corollary is of special importance.

7.1 Theorem. *Let $\mathfrak{M}=\langle M, R_1, \ldots, R_l\rangle$, let a be transitive in V_M with $M\subseteq a$. Let $A=a\cap V_M$ and let $\mathbb{A}_{\mathfrak{M}}=(\mathfrak{M}; A, \in)$. Then a relation S on $\mathbb{A}_{\mathfrak{M}}$ is first-order definable using parameters from a iff $S\in \mathrm{L}(a,\omega)$.*

7.2 Corollary. *If $O(\mathfrak{M})=\omega$, then the relations on $\mathfrak{M}$ in $\mathbb{H}\mathrm{YP}_{\mathfrak{M}}$ are just the first-order relations.*

Proof. If $o(\mathbb{H}\mathrm{YP}_{\mathfrak{M}})=\alpha$ then $\mathbb{H}\mathrm{YP}_{\mathfrak{M}}=\mathrm{L}(\alpha)_{\mathfrak{M}}$. □

7.3 Corollary. *The relations on $\mathrm{L}(a,\alpha)$ in $\mathrm{L}(a,\alpha+\omega)$ are the relations first order definable over $(\mathfrak{M}_0; \mathrm{L}(a,\alpha)\cap V_{M_0}, \in)$, where $\mathfrak{M}_0$ is the substructure of $\mathfrak{M}$ with domain $\mathrm{Sp}(a)$.*

Proof. Apply 7.1, reading $\mathrm{L}(a,\alpha)$ for a and $\mathfrak{M}_0$ for $\mathfrak{M}$. □

We begin the proof of 7.1 by studying substitutable functions.

7.4 Definition. A Σ operation symbol F of n-arguments is *substitutable* if the Δ_0 formulas are closed under substitution by F; that is, if for each Δ_0 formula $\varphi(w, v_1, \ldots, v_k)$, there is a Δ_0 formula $\psi(u_1, \ldots, u_n, v_1, \ldots, v_k)$ not involving F so that $\mathrm{KPU}\vdash \varphi(\mathsf{F}(\vec{u}), \vec{v})\leftrightarrow\psi(\vec{u}, \vec{v})$.

7.5 Lemma. (i) *The substitutable operations are closed under composition.*

(ii) *If $\mathrm{KPU}\vdash\forall\vec{u}(\mathsf{F}(\vec{u})$ is a set), then F is substitutable iff for each Δ_0 formula φ, the formula $\exists x\in \mathsf{F}(\vec{u})\varphi(x,\vec{v})$ is equivalent (in KPU) to a Δ_0 formula $\psi(\vec{u},\vec{v})$.*

(iii) *If F is substitutable, so is G defined by $\mathsf{G}(x,\vec{y})=\{\mathsf{F}(z,\vec{y})\mid z\in x\}$.*

Proof: (i) is more or less obvious. For example, if $\varphi(\mathsf{F}(u))\leftrightarrow\psi(u)$ and $\psi(\mathsf{G}(x))\leftrightarrow\theta(x)$, then $\varphi(\mathsf{F}(\mathsf{G}(x)))\leftrightarrow\psi(\mathsf{G}(x))\leftrightarrow\theta(x)$.

The necessity in (ii) is a special case of Definition 7.4. To prove the other half note that

$$\begin{aligned}
y\in \mathsf{F}(\vec{x}) &\quad\text{iff}\quad \exists z\in \mathsf{F}(\vec{x})(y=z),\\
a=\mathsf{F}(\vec{x}) &\quad\text{iff}\quad \forall z\in a(z\in \mathsf{F}(\vec{x}))\wedge\forall z\in \mathsf{F}(\vec{x})(z\in a),\\
\mathsf{F}(\vec{x})\in a &\quad\text{iff}\quad \exists b\in a[\mathsf{F}(\vec{x})=b],\\
p=\mathsf{F}(\vec{x}) &\quad\text{iff}\quad p\neq p,\\
\mathsf{R}(\ldots\mathsf{F}(\vec{x})\ldots) &\quad\text{iff}\quad x_1\neq x_1.
\end{aligned}$$

So all atomic formulas involving F are Δ_0. A simple induction on Δ_0 formulas, using the hypothesis of (ii), shows that F is substitutable in each of them.

To prove (iii) note that $\exists u\in \mathsf{G}(x,\vec{y})\varphi(u)\leftrightarrow\exists z\in x\,\varphi(\mathsf{F}(z,\vec{y}))$; so G is substitutable by (ii). □

7.6 Lemma. *Each of the operations $\mathscr{F}_1, \ldots, \mathscr{F}_N, \mathscr{D}$ is substitutable.*

Proof. We run through a few cases, using 7.5 (ii) quite heavily.

$\mathscr{F}_1$: $\exists u \in \{x,y\}\, \varphi(u) \leftrightarrow \varphi(x) \vee \varphi(y)$;

$\mathscr{F}_2$: $\exists u \in \bigcup x\, \varphi(u) \leftrightarrow \exists z \in x\, \exists u \in z\, \varphi(u)$;

$\mathscr{H}_1$: $\exists z \in \langle x,y \rangle\, \varphi(z) \leftrightarrow \varphi(\{x\}) \vee \varphi(\{x,y\})$, which is Δ_0 since $\mathscr{F}_1$ is substitutable;

$\mathscr{F}_4$: $\mathscr{F}_4(x,y) = x \times y = \{\langle u,v \rangle \mid u \in x, v \in y\} = \bigcup \{\{\langle u,v \rangle \mid u \in x\} \mid v \in y\}$.

Thus we see that $\mathscr{F}_4$ is substitutable, since $\mathscr{F}_2$ and $\mathscr{H}_1$ are, by composition and 7.5 (iii).

$\mathscr{F}_5$: $\exists u \in \mathrm{dom}(x)\, \varphi(u) \leftrightarrow \exists u, v \in \bigcup\bigcup x\, [\langle u,v \rangle \in x \wedge \varphi(u)]$, so $\mathscr{F}_5$ follows from $\mathscr{H}_1$.

$\mathscr{D}$: $\exists x \in \mathscr{D}(b)\, \varphi(x) \leftrightarrow \exists x \in b\, \varphi(x) \vee \exists y, z \in b\, [\bigvee_{i \leqslant n} \varphi(\mathscr{F}_i(x,y))]$.

The other $\mathscr{F}_i$ are just as routine. □

For the remainder of the section fix $\mathfrak{M}, a$ and $\mathbb{A}_{\mathfrak{M}}$ as in the statement of the theorem to be proved, Theorem 7.1.

7.7 Lemma. *For every element $x \in \mathrm{L}(a,\omega)$ there is a term $\mathscr{F}$ in the symbols $\mathscr{F}_1, \ldots, \mathscr{F}_N, \mathscr{D}$ and $y_1, \ldots, y_m \in a \cup \{a\}$ such that $x = \mathscr{F}(y_1, \ldots, y_m)$.*

Proof. Note that $\mathrm{L}(\alpha, n) = \mathscr{D}\mathscr{S}(\mathscr{D}\mathscr{S}(\ldots(a)\ldots))$ for n repetitions of $\mathscr{D} \circ \mathscr{S}$ ($\mathscr{S}$ is a term in $\mathscr{F}_1, \mathscr{F}_2$ as we saw in § 6) so each $\mathrm{L}(a,n)$ is of the appropriate form. We now show that each $x \in \mathrm{L}(a,n)$ is of the appropriate form by induction on n. Since $\mathrm{L}(a,\omega) = \bigcup_{n<\omega} \mathrm{L}(a,n)$ the result follows.

For $n=0$ we have $\mathrm{L}(a,0) = a$, since a is transitive, so the result is trivial. If $x \in \mathrm{L}(a,n+1) - \mathrm{L}(a,n)$, then $x = \mathrm{L}(a,n)$ or $x = \mathscr{F}_i(z,y)$ for some $y, z \in \mathrm{L}(a,n) \cup \{\mathrm{L}(a,n)\}$. The first case is taken care of by the first part of the proof. If $x = \mathscr{F}_i(y,z)$ with $y, z \in \mathrm{L}(a,n) \cup \{\mathrm{L}(a,n)\}$, then y, z are of the appropriate form. Hence, x is also of the correct form. □

7.8 Lemma. *If $\varphi(x_1, \ldots, x_n, y)$ is Δ_0 without parameters, then the relation*

$$\{(x_1, \ldots, x_n) \in \mathbb{A}_{\mathfrak{M}} \mid \mathrm{L}(a,\omega) \models \varphi(x_1, \ldots, x_n, a)\}$$

is first-order definable over $\mathbb{A}_{\mathfrak{M}}$.

Proof. A trivial induction on Δ_0 formulas; just replace $\forall x \in a$ by $\forall x$, etc. □

Proof of Theorem 7.1. Suppose $S \subseteq a^n$, $S \in \mathrm{L}(a,\omega)$. Then, by 7.7, there is a term $\mathscr{F}$ in $\mathscr{F}_1, \ldots, \mathscr{F}_N, \mathscr{D}$ such that $S = \mathscr{F}(x_1, \ldots, x_k, a)$ for some $x_1, \ldots, x_k \in a$. But then $S(y_1, \ldots, y_n)$ iff $\langle y_1, \ldots, y_n \rangle \in \mathscr{F}(x_1, \ldots, x_k, a)$.

The right hand side is equivalent to a Δ_0 formula $\varphi(y_1, \ldots, y_n, x_1, \ldots, x_k, a)$ by the substitutability of $\mathscr{F}$ (using 7.6 and 7.5 (i)) and $\langle\ \rangle$. The relation $\varphi(y_1, \ldots, y_n, x_1, \ldots, x_k, a)$ is definable on $\mathbb{A}_{\mathfrak{M}}$, and hence S is definable using the parameters $x_1, \ldots, x_k$. The converse is trivial since every definable relation S on $\mathbb{A}_{\mathfrak{M}}$ is Δ_0 on $\mathrm{L}(a,1)$ and so is in $\mathrm{L}(a,\omega)$ by, say, Exercise 5.16. □

7.9—7.10 Exercises

7.9. F is effectively substitutable if the ψ of 7.4 can be found effectively from φ. Show that each $\mathscr{F}_1, \ldots, \mathscr{F}_N, \mathscr{D}$ is effectively substitutable. [Use Church's Thesis.]

7.10. Verify that the effective version of 7.8 holds.

7.11 Notes. It seems to be an open problem whether the converse of 7.2 is true in general. The study of substitutable functions goes back to Levy [1965]. He called them "admissible terms", terminology clearly inadmissible in our context. They were used by Gandy [1975] and Jensen [1972] (written later than Gandy [1975]) to prove the urelementless version of Corollary 7.3. Gandy called them "substitutable", Jensen called them "simple".

8. The Truncation Lemma

Recall (from I.9.5) that a binary relation E on a set X is well founded iff for all nonempty $Y \subseteq X$ there is an $x \in Y$ such that for all $y \in Y$ we have $\neg(yEx)$. The notion is what we have tried to capture in the axiom of foundation, but of course we fail since it is just not expressible in the first-order language of set theory. A *nonstandard* model of KPU is one of the form $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, E, \ldots)$, where E is not well founded; the other models are the standard, or intended models since, by the next result, they are isomorphic to admissible sets. The proof is essentially the same as that of I.9.6.

8.1 Proposition. *If* $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, E, \ldots)$ *is a well-founded model of extensionality then, it is isomorphic to a structure of the form* $\mathbb{B}_M = (\mathfrak{M}; B, \in, \ldots)$ *with* $M \cup B$ *transtitive. Both* $\mathbb{B}_{\mathfrak{M}}$ *and the isomorphism* f *are unique, and* f *satisfies*

$$f(p) = p, \qquad \text{for} \quad p \in M;$$

$$f(a) = \{f(b) \mid bEa\}, \quad \text{for} \quad a \in A.$$

Now let $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, E)$ be any structure and let $\mathscr{W} = \{\mathfrak{B}_{\mathfrak{M}} \subseteq_{\text{end}} \mathfrak{A}_{\mathfrak{M}} \mid \mathfrak{B}_{\mathfrak{M}}$ is well founded$\}$. Assume $\mathscr{W} \neq 0$, which is the case iff $\mathfrak{A}_{\mathfrak{M}} \models \exists x \forall x\, (y \notin x)$.

8.2 Lemma. *There is a largest* $\mathfrak{B}_{\mathfrak{M}} \in \mathscr{W}$ *(one which is an end extension of all other members of* $\mathscr{W}$*).*

Proof. Let $\mathfrak{B}_{\mathfrak{M}}$ be the union of all structures in $\mathscr{W}$. It is easy to check that $\mathfrak{B}_{\mathfrak{M}} \subseteq_{\text{end}} \mathfrak{A}_{\mathfrak{M}}$ and $\mathfrak{C}_{\mathfrak{M}} \subseteq_{\text{end}} \mathfrak{B}_{\mathfrak{M}}$ for all $\mathfrak{C}_{\mathfrak{M}} \in \mathscr{W}$. To see that $\mathfrak{B}_{\mathfrak{M}}$ is well founded, let X be a non empty subset of $M \cup B$. We must find an $x \in X$ such that $y \in X$ implies $\neg yEx$. Since $\mathfrak{B}_{\mathfrak{M}}$ is the union of $\mathscr{W}$, there is a $\mathfrak{C}_{\mathfrak{M}} \in \mathscr{W}$ such that $X' = X \cap (M \cup C)$ is nonempty. Since $\mathfrak{C}_{\mathfrak{M}}$ is well founded there is an $x \in X'$ such that $y \in X'$ implies $\neg yEx$. But yEx implies $y \in M \cup C$ for all $y \in M \cup A$ (by $\mathfrak{C}_{\mathfrak{M}} \subseteq_{\text{end}} \mathfrak{A}_{\mathfrak{M}}$), so we have $\neg yEx$ for all $y \in X$. $\square$

8.3 Definition. The largest well-founded $\mathfrak{B}_{\mathfrak{M}}$ such that $\mathfrak{B}_{\mathfrak{M}} \subseteq_{\text{end}} \mathfrak{A}_{\mathfrak{M}}$ is called the *well-founded part* of $\mathfrak{A}_{\mathfrak{M}}$ and is denoted by $\mathscr{W}\!f(\mathfrak{A}_{\mathfrak{M}})$.

Note that this makes sense whether or not $\mathfrak{A}_{\mathfrak{M}}$ is not well founded. If $\mathfrak{A}_{\mathfrak{M}}$ is well founded, then $\mathscr{W}\!f(\mathfrak{A}_{\mathfrak{M}}) = \mathfrak{A}_{\mathfrak{M}}$. If $\mathfrak{A}_{\mathfrak{M}}$ is a model of extensionality, so is $\mathscr{W}\!f(\mathfrak{A}_{\mathfrak{M}})$, since $\mathscr{W}\!f(\mathfrak{A}_{\mathfrak{M}}) \subseteq_{\text{end}} \mathfrak{A}_{\mathfrak{M}}$. In this case *we often identity $\mathscr{W}\!f(\mathfrak{A}_{\mathfrak{M}})$ with the unique transitive structure isomorphic to it*, as given by 8.1. We make this identification in the next result, for example, which is an example of one way in which KPU is better behaved than stronger theories like ZF. It gives us a new method of constructing admissible sets, which accounts for its occurrence in this chapter.

8.4 Truncation Lemma. *Let* $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, E, \ldots)$ *and* $\mathbb{B}_{\mathfrak{M}} = (\mathfrak{M}; B, \in, \ldots)$ *be* L^*-*structures with* $\mathfrak{A}_{\mathfrak{M}} \models \text{KPU}$ and $\mathbb{B}_{\mathfrak{M}} \subseteq_{\text{end}} \mathfrak{A}_{\mathfrak{M}}$, *where* $(\mathfrak{M}; B, \in) = \mathscr{W}\!f(\mathfrak{M}; A, E)$. *Then* $\mathbb{B}_{\mathfrak{M}}$ *is admissible over* $\mathfrak{M}$.

Proof. We need to show that the hypotheses of Lemma I.8.9 are satisfied, for then we get all the axioms of KPU except Foundation true in $\mathbb{B}_{\mathfrak{M}}$. But $\mathbb{B}_{\mathfrak{M}}$ is well founded, so it certainly satisfies Foundation. First note:

(1) If $a \in A$ and $a_E \subseteq B$, then $a \in B$.

This follows from the maximality of $\mathbb{B}_{\mathfrak{M}} \in \mathscr{W}$.

(2) If $a \in B$ and $\mathfrak{A}_{\mathfrak{M}} \models \text{rk}(a) = \alpha$, then $\alpha \in B$.

This follows by $\in$ induction on a, using (1), since $\mathfrak{A}_{\mathfrak{M}} \models \alpha = \sup\{\text{rk}(x) + 1 \mid x \in a\}$.

(3) If $\alpha \in B$ and $\mathfrak{A}_{\mathfrak{M}} \models \text{rk}(a) = \alpha$, then $a \in B$.

This follows by induction on α using (1). Thus we see that if $\mathfrak{A}_{\mathfrak{M}} \models \text{rk}(a) = \alpha$, then $a \in B$ iff $\alpha \in B$.

(4) There is no sup in $\mathfrak{A}_{\mathfrak{M}}$ for the ordinals of $\mathbb{B}_{\mathfrak{M}}$.

This follows from (1). Thus, we have what we need to apply I.8.9. □

We have worded 8.4 in a roundabout way because of the functions which might appear in the list.... The universe of $\mathscr{W}\!f(\mathfrak{M}; A, E)$ might not be closed under them. Perhaps it is worth stating a special case of 8.4, the one we usually apply. It follows at once from 8.4.

8.5 Corollary. *If* $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, E)$ *is a model of* KPU *then its wellfounded part is an admissible set over* $\mathfrak{M}$.

8.6 Theorem. *Let* $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$. *The admissible set* $\mathbb{H}\text{YP}_{\mathfrak{M}}$ *is the intersection of all models* $\mathfrak{A}_{\mathfrak{M}}$, *well-founded or not, of* KPU^+. *More accurately, given any model* $\mathfrak{A}_{\mathfrak{M}}$ *of* KPU^+, *there is a unique embedding of* $\mathbb{H}\text{YP}_{\mathfrak{M}}$ *onto an initial substructure of* $\mathfrak{A}_{\mathfrak{M}}$.

Proof. By 8.5, $\mathscr{Wf}(\mathfrak{A}_{\mathfrak{M}})$ is admissible above $\mathfrak{M}$ and hence $\mathbb{HYP}_{\mathfrak{M}} \subseteq \mathscr{Wf}(\mathfrak{A}_{\mathfrak{M}}) \subseteq_{\text{end}} \mathfrak{A}_{\mathfrak{M}}$, the first inclusion being correct up to the unique embedding discussed above. □

Recall that $O(\mathfrak{M})$ is, by definition, $o(\mathbb{HYP}_{\mathfrak{M}})$. Structures $\mathfrak{M}$ such that $O(\mathfrak{M}) = \omega$ are going to play an interesting role in our study of admissible sets and structures. We call such structures *recursively saturated.* This terminology will be justified in Chapter IV (cf. Definition IV.5.1 and Theorem IV.5.3). In the next theorem we use the truncation lemma to prove that there are lots of recursively saturated structures; that is, structures $\mathfrak{M}$ with $O(\mathfrak{M}) = \omega$.

8.7 Theorem. *For every structure $\mathfrak{M} = \langle M, R_1, \ldots, R_k \rangle$ there is a recursively saturated elementary extension $\mathfrak{N}$ of $\mathfrak{M}$ of the same cardinality.*

Proof. Consider $\mathbb{HYP}_{\mathfrak{M}}$ as a single-sorted structure of the form:

$$\mathfrak{A} = \langle M \cup A, M, A, R_1, \ldots, R_l, \in \rangle,$$

and let $\mathfrak{B} = \langle N \cup B, N, B, R'_1, \ldots, R'_l, E \rangle$ be an elementary extension with non-standard natural numbers. This exists by the ordinary Compactness Theorem. Let $\mathfrak{N} = \langle N, R'_1, \ldots, R'_l \rangle$, and let $\mathfrak{B}_{\mathfrak{N}} = (\mathfrak{N}; B, E)$, which is a model of KPU$^+$. The well-founded part of $\mathfrak{B}_{\mathfrak{N}}$ is an admissible set $\mathbb{B}'_{\mathfrak{N}}$ with $N \in \mathfrak{B}'$, since $\mathrm{rk}(N) = 1$. Also $o(\mathbb{B}'_{\mathfrak{N}}) = \omega$, since $\mathfrak{B}_{\mathfrak{N}}$ has non-standard integers. Thus $o(\mathbb{HYP}_{\mathfrak{N}}) = \omega$ by Theorem 5.9. The cardinality considerations are routine. □

This shows that we cannot expect $\mathfrak{M} \prec \mathfrak{N}$ and $O(\mathfrak{N}) = \omega$ together to imply $O(\mathfrak{M}) = \omega$.

Finally, we use 8.5 to get a rather technical looking results. The real content of 8.8 will emerge gradually throughout the book.

8.8. Proposition. *Let S be an n-ary relation on a structure $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$. If S is Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$ then there is a Σ_1 formula $\varphi(x_1, \ldots, x_n, p_1, \ldots, p_k, \mathfrak{M})$, with only constants $p_1, \ldots, p_k \in M$ such that for all $q_1, \ldots, q_n \in M$ the following are equivalent:*

(i) $S(q_1, \ldots, q_n)$;

(ii) $\mathbb{HYP}_{\mathfrak{M}} \models \varphi(\vec{q}, \vec{p}, M)$;

(iii) *For all models of* KPU$^+$ *of the form* $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, E)$ *we have* $\mathfrak{A}_{\mathfrak{M}} \models \varphi(\vec{q}, \vec{p}, M)$.

Proof. By 6.4 every $a \in \mathbb{HYP}_{\mathfrak{M}}$ can be defined by a Σ_1 formula with constants from $M \cup \{M\}$. Thus we may replace any of these a's by its definition to get a φ of the appropriate kind such that (i)$\Longleftrightarrow$(ii). Since $\mathbb{HYP}_{\mathfrak{M}} \models \mathrm{KPU}^+$, we see that (iii)$\Rightarrow$(ii). To see that (ii)$\Rightarrow$(iii) note that any such $\mathfrak{A}_{\mathfrak{M}}$ is (isomorphic to) an end extension of $\mathbb{HYP}_{\mathfrak{M}}$, by 8.6. Hence if $\varphi(q, p, M)$ holds in $\mathbb{HYP}_{\mathfrak{M}}$, it holds in $\mathfrak{A}_{\mathfrak{M}}$, since it is Σ_1. Of course, we need to know that the isomorphism is the identity on $M \cup \{M\}$, but this follows from 8.1. □

8.9—8.15 Exercises

8.9. Let $\mathfrak{M}=\langle M,R_1,\ldots,R_l\rangle$ be such that $\mathfrak{M}\prec\mathfrak{N}$, and $\mathrm{card}(\mathfrak{M})=\mathrm{card}(\mathfrak{N})$ implies $\mathfrak{M}\cong\mathfrak{N}$ (equivalently, $\mathrm{Th}(\mathfrak{M})$ is $\mathrm{card}(\mathfrak{M})$-categorical). Show that $o(\mathbb{HYP}_{\mathfrak{M}})=\omega$, and hence the relations S on $\mathfrak{M}$ in $\mathbb{HYP}_{\mathfrak{M}}$ are just the ones first-order definable over $\mathfrak{M}$.

8.10. Let $\mathfrak{M}=\langle M,=\rangle$ be infinite. Show that a subset $X\subseteq M$ is in $\mathbb{HYP}_{\mathfrak{M}}$ iff X or $M-X$ is finite.

8.11. (F. Ville) Suppose α is not admissible and $\langle \mathrm{L}(\alpha),\in\rangle\subseteq_{\mathrm{end}}\langle A,E\rangle$, where $\langle A,E\rangle\models\mathrm{KP}$. Show that, up to a unique isomorphism, $\langle \mathrm{L}(\beta),\in\rangle\subseteq_{\mathrm{end}}\langle A,E\rangle$, where β is the least admissible ordinal greater than α.

8.12. Use the notation of 8.11. Let S be a relation on $\mathrm{L}(\alpha)$, S Σ_1 on $\mathrm{L}(\beta)$. Find a Σ_1 formula $\varphi(x_1,\ldots,x_n,a)$ with $\vec{a}\in\mathrm{L}(\alpha)$ and no other constants such that the following are equivalent:

(i) $S(\vec{x})$;

(ii) $\mathrm{L}(\beta)\models\varphi(\vec{x},\vec{a})$;

(iii) For all models $\mathfrak{A}=\langle A,E\rangle$ of KP if $\langle\mathrm{L}(\alpha),\in\rangle\subseteq_{\mathrm{end}}\mathfrak{A}$, then $\mathfrak{A}\models\varphi(\vec{x},\vec{a})$.

[Hint: Find a good Σ_1 definition of α to get rid of $\mathrm{L}(\alpha)$ in φ.]

8.13. If $\mathfrak{A}_{\mathfrak{M}}=(M;A,E,P)$ is a model for KPU + Power and $(\mathfrak{M};B,\in)=\mathcal{W}\!f(\mathfrak{M};A,E)$, then $\mathbb{B}_{\mathfrak{M}}=(\mathfrak{M},B,\in,P\restriction B)$ is admissible and a model of Power.

8.14. Show that the well-founded part of a model $\mathfrak{A}_{\mathfrak{M}}$ of KPU + Beta need not satisfy Beta. (Not for the beginner.) The well-founded part of a model $\langle A,E\rangle$ of all of ZF need not satisfy Beta.

8.15. (For those familiar with Π_1^1.) Let $\mathfrak{N}=\langle N,+,\cdot\rangle$ and let S be a relation on $\mathfrak{N}$. Show that if S is Σ_1 on $\mathbb{HYP}_{\mathfrak{N}}$, then S is Π_1^1.

8.16 Notes. The history of the Truncation Lemma is more complicated than the lemma itself. Starting from the fact that every ω-model of second-order arithmetic contains all hyperarithmetic sets of natural numbers, Mlle. F. Ville generalized this by proving Exercise 8.11. This was in 1966 and her proof remains unpublished. Barwise [1969] generalized this to obtain a $\mathrm{V}=\mathrm{L}$ or $\mathrm{V}=\mathrm{L}(x)$ version of the Truncation Lemma. It is not clear to the present author who first thought of the trick (used back in Lemma I.8.9) that allows the full result to go through.

9. *The Lévy Absoluteness Principle*

We have been rather free wheeling with our metatheory, for example in § 1 and § 3 of this chapter. We used the power set axiom, results on cardinal numbers and even the axiom of choice (in the guise of the Downward Löwenheim-Skolem theorem) in § 3. It should be clear, though, that everything we have done could be formalized within ZFC, Zermelo-Fraenkel set theory with choice. (Given a structure $\mathfrak{M}=\langle M,\ldots\rangle$ for example, with $M\in\mathbb{V}$, we can define $\mathbb{V}_{\mathfrak{M}}$ as a class in $\mathbb{V}$ without difficulty as long as we remember that $\in_M$ is distinct from $\in$.) Weaker theories would suffice; but, because it is familiar to almost everyone, *we fix* ZFC *as our metatheory for this book*, unless some other theory like KPU is specified the way it was in Chapter I.

The following version of the Löwenheim-Skolem Theorem, implicit in 3.4, will be of considerable use to us in what follows, though we usually use the simple parameter-free version given in 9.2.

9.1 Theorem. *Let $\varphi(x_1,\ldots,x_n,y_1,\ldots,y_m)$ be a Π formula in the language of* ZFC *(with only $\in$ and $=$) with the free variables only as shown. The following sentence is a theorem of* ZFC:

$$\forall y_1,\ldots,y_m\in H(\aleph_1)\,[\forall x_1,\ldots,x_n\in H(\aleph_1)\;\varphi(\vec{x},\vec{y})\rightarrow\forall x_1,\ldots,x_n\;\varphi(\vec{x},\vec{y})]\,.$$

9.2 Corollary. *Let $\varphi(x)$ be a Π formula in the language of* ZFC *with only the one free variable x. Then* $\mathrm{ZFC}\vdash\forall x\in H(\aleph_1)\;\varphi(x)\rightarrow\forall x\;\varphi(x)$.

Proof of 9.1. *Since* $\mathrm{KP}\subseteq\mathrm{ZFC}$ *we may assume $\varphi(x,y)$ is* Π_1, that is, of the form $\forall z\,\psi(\vec{x},\vec{y},z)$, where ψ is Δ_0 by I.4.3. We work within ZFC and prove the sentence in question by contraposition. Let $y_1,\ldots,y_m\in H(\aleph_1)$, and suppose there are $x_1,\ldots,x_n$, such that $\neg\varphi(\vec{x},\vec{y})$, i.e. there is a z such that $\neg\psi(\vec{x},\vec{y},z)$. Pick $\kappa\geqslant\aleph_1$ so large that $x_1,\ldots,x_n,z\in H(\kappa)$. Then, since $\neg\psi$ is Δ_0 we have $\langle H(\kappa),\in\rangle\models\neg\psi(\vec{x},\vec{y},z)$ by absoluteness. By 3.4, $\langle H(\aleph_1),\in\rangle\prec_1\langle H(\kappa),\in\rangle$, so we find $\langle H(\aleph_1),\in\rangle\models\exists x_1,\ldots,x_n z\,\neg\psi(\vec{x},\vec{y},z)$. Pick $x_1,\ldots,x_n\in H(\aleph_1)$ so that $\langle H(\aleph_1),\in\rangle\models\exists z\neg\psi(\vec{x},\vec{y},z)$. Then by Lemma I.4.2, $\exists z\neg\psi(\vec{x},\vec{y},z)$ is true, which means that $\neg\varphi(\vec{x},\vec{y})$. Since $x_1,\ldots,x_n\in\mathrm{H}(\aleph_1)$, this proves our result. □

We conclude this section with a simple example of the use of the Absoluteness Principle.

9.3 Proposition. *Let $\mathfrak{M}=\langle M\rangle$ be a structure with no relations. If $X\subseteq M$ is constructible from $\mathfrak{M}$, $X\in\mathrm{L}(\mathfrak{M})$, then X or $M-X$ is finite.*

Proof. The statement to be proved has the form:

$$\forall M\forall X\forall\alpha\,[X\subseteq M\wedge X\in\mathrm{L}(M,\alpha)\rightarrow X\textit{ is finite}\vee M-X\textit{ is finite}].$$

In ZF, or even in KPU + Infinity, this is a Π statement (by use of P_ω from I.9) so it suffices to prove it for countable M and α. We may assume M is infinite since otherwise the result is trivial.

Let σ be any one-one map of M onto M. We can extend σ to an automorphism $\bar{\sigma}$ of $\mathbb{V}_M$ onto $\mathbb{V}_M$ by recursion on $\in$:

$$\bar{\sigma}(p) = \sigma(p),$$
$$\bar{\sigma}(a) = \{\bar{\sigma}(x) \mid x \in a\}.$$

Note that $\bar{\sigma}(\mathscr{F}_i(x,y)) = \mathscr{F}_i(\bar{\sigma}(x), \bar{\sigma}(y))$, whenever $1 \leqslant i \leqslant \mathscr{N}$, by inspection. A simple proof by induction shows that $\bar{\sigma}(\mathrm{L}(M,\alpha)) = \mathrm{L}(M,\alpha)$ for all α.

Now suppose that M and α are countable but that there is an $X \in \mathrm{L}(M,\alpha)$ with $X \subseteq M$ such that X and $M - X$ are both infinite. Then, for any $Y \subseteq M$ with Y and $M - Y$ infinite, there is a one-one map σ mapping X onto Y so that $\bar{\sigma}(X) = Y$. But then $X \in \mathrm{L}(M,\alpha)$ implies $\bar{\sigma}(\mathrm{L}(M,\alpha))$; so $Y \in \mathrm{L}(M,\alpha)$. But there are $2^{\aleph_0}$ such X, whereas $\mathrm{L}(M,\alpha)$ is countable. $\square$

9.4—9.7 Exercises

9.4. Let ZFU$^+$ be KPU$^+$ plus full separation, full collection, Power and Infinity. Prove that for each $\varphi \in \mathrm{ZFU}^+$, we have $\mathrm{ZFU} \vdash \varphi^{\mathrm{L}(M)}$.

9.5. Show that if M is as in 9.3 then $\mathrm{L}(M)$ is a model of ZFU$^+$ plus "all subsets of M are finite or cofinite". This shows that choice fails very badly in this particular $\mathrm{L}(M)$.

9.6. Let $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$ and let σ be an automorphism of $\mathfrak{M}$. Extend σ to a $\bar{\sigma}: \mathbb{V}_{\mathfrak{M}} \to \mathbb{V}_{\mathfrak{M}}$ as in 9.3. Show that $\bar{\sigma}: \mathrm{L}(\alpha)_{\mathfrak{M}} \to \mathrm{L}(\alpha)_{\mathfrak{M}}$, one-one and onto, for all α.

9.7 Notes. The Lévy Absoluteness Principle was first proved by Lévy [1965]. See the notes from § 3 for more details on the general argument. One of the main features of this book (at least from our point of view) is the systematic use of the Lévy Absoluteness Principle to simplify results by reducing them to the countable case. This is particularly true of Part *B* of the book.

We will see, as a by product of § V.8, that the axiom of choice is not needed in the proof of 9.1. See the proof of V.8.10, in particular.