PART II. COMPACTNESS REGAINED

5. *Admissibility*

In passing from $\mathscr{L}_{\omega\omega}$ to $\mathscr{L}_{\infty\omega}$ a very substantial gain in expressive power is achieved. As is to be expected, however, there is a considerable price to pay. Many of the very useful properties of $\mathscr{L}_{\omega\omega}$—most notably compactness—are no longer enjoyed by $\mathscr{L}_{\infty\omega}$. If we restrict our attention to $\mathscr{L}_{\omega_1\omega}$, then some of these properties are salvaged. For example, interpolation, and a reasonable form of completeness can be thus regained. Compactness, however, clearly still fails. To obtain an omitting types result, we considered countable fragments L_B of $\mathscr{L}_{\omega_1\omega}$. Though completeness looks even better in this framework, interpolation, for example, fails. Thus, while on the one hand we want to deal with parts of $\mathscr{L}_{\omega_1\omega}$ small enough to be manageable, on the other hand, we would nevertheless like them to be large enough to be closed, for example, under finding interpolants. For this latter consideration, it would be preferable if the pieces that we deal with were given in some absolute way, since then, using them to give bounds would be more meaningful from "the first-order" point of view. $L_{\omega_1\omega}$ itself, as a fragment of $\mathscr{L}_{\infty\omega}$, is given by cardinality conditions, and so is certainly not "first-order".

In order to introduce the notion that has proven fruitful in this respect, we will assume, first of all—without doing any of this explicitly—that the syntax and semantics of $\mathscr{L}_{\infty\omega}$ are given within set theory. That is, we assume that sentences are sets, structures are sets, satisfaction is a ternary relation between structures, formulas, and functions from variables, *etc.* For any transitive set B we will thus be able to define $L_B = L_{\infty\omega} \cap B$; that is, the formulas of L_B are those formulas of $L_{\infty\omega}$ in B. Mild assumptions on B will guarantee that L_B is a fragment in the sense we have been using. Somewhat stronger conditions will give us a great deal of closure, and, when combined with countability, will even give a form of compactness.

5.1. KP *and Admissible Sets*

An *admissible set* is a transitive set A, such that $\langle A, \in\rangle$ is a model of a certain theory KP, the initials standing for Kripke and Platek. Kripke [1964a, b] and Platek [1966] were engaged in trying to generalize recursion theory to the ordinals. They were following the earlier work of Takeuti [1960], [1965] and Tugué [1964] who were studying recursion on the set of all ordinals, and Kreisel–Sacks [1965] whose metarecursion theory, in turn, followed from earlier work of Kleene [1955b] on recursive ordinals and hyperarithmetic sets. For a more complete history, the reader should consult the introduction to Barwise [1975].

In order to present the theory KP, we must first recall the Lévy hierarchy of formulas of a language containing the binary relation symbol $\in$ and perhaps other symbols as defined in Lévy [1965]. The collection of Δ_0*-formulas* is the smallest collection of formulas containing the atomic formulas, closed under the boolean connectives of $\neg$, & and $\vee$, and under bounded quantification. (That is, if φ is a

Δ_0-formula and u and v are variables, $\exists u \in v\varphi$ and $\forall u \in v\varphi$ are Δ_0-formulas, where $\exists u \in v\varphi$ stands for $\exists u[u \in v \; \& \; \varphi]$, *etc.*). The Σ_1-*formulas* are formulas of the form $\exists v\varphi$, where φ is a Δ_0-formula. The collection of Σ-*formulas* is obtained from the Δ_0-formulas by closing under &, $\vee$, bounded quantifiers, and existential quantifiers. A relation on a structure is said to be Σ-definable, or simply Σ, if it can be defined by a Σ-formula. A relation is $\mathbf{\Sigma}$, if it can be defined by a Σ-formula using parameters. A relation is $\mathbf{\Pi}$ if its complement is $\mathbf{\Sigma}$, and is $\mathbf{\Delta}$ if it is both $\mathbf{\Sigma}$ and $\mathbf{\Pi}$. All other similar definitions should follow easily from this sample.

The reason that the above classes of formulas are important is related to the notion of an *end extension.* A structure $\langle B, F, \ldots\rangle$ is an end extension of a structure $\langle A, E, \ldots\rangle$, where E and F are binary, if $\langle A, E, \ldots\rangle$ is a submodel of $\langle B, F, \ldots\rangle$ and whenever $a \in A$ and $(c, a) \in F$, then $c \in A$. In words, elements of A do not get any new F-members in B. It is then quite easy to show inductively that Σ-formulas are preserved in going to end extensions. We call such formulas *persistent.* If we insist that all the structures involved be models of some theory T we arrive at the notion of *persistent relative to T.* A formula is *absolute relative to T* if it holds in a model of T iff it holds in any end extension which is a model of T. Clearly φ is absolute relative to T iff both φ and $\neg\varphi$ are persistent relative to T. There is a converse to the simple observation that Σ-formulas are persistent. Feferman and Kreisel [1966] (see Feferman [1968b]) have shown that if φ is persistent relative to T, then there is a Σ-formula ψ such that $T \vdash \varphi \leftrightarrow \psi$. Hence, if φ is absolute relative to T, then in T φ is provably equivalent to both a Σ- and a Π-formula.

We can now give a set of axioms for KP. First, there are the axioms of extensionality, pairing and union, and the foundation scheme for arbitrary formulas (since the set existence axioms are weak). In addition, we have the following two schemes:

$\mathbf{\Delta}_0$-Separation: $\exists v \forall x(x \in v \leftrightarrow x \in u \; \& \; \varphi(x))$, for each Δ_0-formula φ in which v does not occur free.

$\mathbf{\Delta}_0$-Collection: $\forall x \in u \, \exists y\varphi(x, y) \rightarrow \exists v \, \forall x \in u \, \exists y \in v\varphi(x, y)$, for each Δ_0-formula φ in which v does not occur free.

Now, a structure $\mathfrak{A} = \langle A, \in, \ldots\rangle$ is *admissible* if $\langle A, \in\rangle$ is transitive and $\langle A, \in, \ldots\rangle \models$ KP. A transitive set A is admissible if $\langle A, \in\rangle$ is an admissible structure. It is often of interest to consider structures $\langle A, \in, \mathscr{P}\rangle$, where $\mathscr{P}$ is the power set operation, and A is closed under power set. Even if $\langle A, \in\rangle$ is admissible and A is closed under power set, $\langle A, \in, \mathscr{P}\rangle$ need not be admissible. As a notational convention, we write L_A to denote such an admissible fragment even when considering an admissible structure $\mathfrak{A} = \langle A, \in, \ldots\rangle$.

For later use, we mention two classes of sets given by conditions weaker than admissibility. Transitive sets $\langle B, \in\rangle$ satisfying all axioms of KP–except perhaps that of $\mathbf{\Delta}_0$-collection—are called *rudimentary.* The *primitive recursive set functions* of Jensen–Karp [1971] contain certain innocuous functions, such as the zero function, the pairing function and the union function, and are closed under composition and recursion. Transitive sets closed under these functions are called *primitive recursively closed sets* and are easily seen to be rudimentary, though they are not necessarily admissible.

5.2. Some Admissible Sets

It will be useful to have some examples of admissible sets. The first example is from the set-theoretic point of view. For x a set, let TC(x) denote the transitive closure of x; that is, it is the smallest transitive set with x as a subset. For κ an infinite cardinal let $H(k) = \{x : |\mathrm{TC}(x)| < \kappa\}$, the set of all sets of hereditary cardinality less than κ. If κ is regular, then $H(\kappa)$ is easily seen to be admissible. If $\kappa = \aleph_0$ all axioms of ZF *except infinity* hold, while if $\kappa > \aleph_0$, all axioms *except perhaps power set* hold. (Note that $H(\beth_\omega)$ is closed under power set, but $(H(\beth_\omega)),$ $\in, \mathscr{P})$ is not admissible.) $H(\aleph_0)$ and $H(\aleph_1)$ are usually denoted by HF and HC, respectively. Assuming that the underlying language is coded appropriately, then $L_{H(\kappa)}$ is simply $L_{\kappa\omega}$.

The other example is of a more recursion-theoretic flavor. Let ω_1^{CK} denote the first non-recursive ordinal. That is, it denotes the first ordinal whose order type is not given by a recursive relation. Then, the set $L(\omega_1^{\mathrm{CK}})$ of all sets constructible before the ω_1^{CK}-th stage is an admissible set. In fact, it is the smallest admissible set containing ω. It is quite easy to see that no smaller set containing ω would be admissible. For a proof that it actually is admissible the reader should see Barwise [1975]. We note, for use later, that the subsets of ω in $L(\omega_1^{\mathrm{CK}})$ are exactly the hyperarithmetic sets.

An extremely important fact—and one about which we will have more to say in Section 5.4—is that for each set x there is a smallest admissible set containing x as an element. This set is denoted HYP(x). For B transitive, we let $o(B)$ denote the least ordinal not in B. Given an arbitrary set x—particularly if x happens to be some structure $\mathfrak{M}$—we can associate with x the ordinal $o(\mathrm{HYP}(x))$. As we shall see, this ordinal will have a strong model-theoretic relation to $\mathfrak{M}$.

5.3. Some Theorems of KP

KP is, of course, a weakened version of ZF, a version with separation and collection limited to Δ_0-formulas and power set totally eliminated. However, it turns out that collection actually follows for Σ-formulas, while separation holds for $\boldsymbol{\Delta}$ subsets. In addition, replacement holds for Σ-formulas, as does the reflection principle; that is, if φ is a Σ-formula, then $\varphi \leftrightarrow \exists u \varphi^{(u)}$ is a theorem of KP. As a consequence, every Σ-formula is equivalent to a Σ_1-formula.

In KP we can show that for any set x, its transitive closure TC(x) exists, and then prove the following scheme for *definition by* $\boldsymbol{\Sigma}$ *recursion*:

5.3.1 Lemma. *Suppose G is an* $(n + 2)$ *place* $\boldsymbol{\Sigma}$*-function. An* $(n + 1)$ *place* $\boldsymbol{\Sigma}$*-function may be defined by*:

$$F(x_1, \ldots, x_n, y) = G(x_1, \ldots, x_n, y, \{\langle z, F(x_1, \ldots, x_n, z)\rangle : z \in \mathrm{TC}(y)\}).$$

$\square$

There is an analogous scheme for relations, and it is given in

5.3.2 Lemma. *Suppose P, Q are Δ-relations of $(n+1)$ and $(n+2)$ places, respectively. An n place Δ-relation may be defined by*:

$$R(x_1, \ldots, x_n, 0) \leftrightarrow P(x_1, \ldots, x_n)$$

$$R(x_1, \ldots, x_n, y) \leftrightarrow Q(x_1, \ldots, x_n, y, \{z \in \mathrm{TC}(y)\colon \quad R(x_1, \ldots, x_n, z)\}). \quad \square$$

These schemes guarantee that certain important functions and relations are, respectively, Σ or Δ definable. For example, the usual operations of ordinal arithmetic or the rank of a set are Σ. In addition, by a straightforward argument it is possible to show that if $\langle X, \leq \rangle$ is a well-ordering of order type α and $\langle X, \leq \rangle$ is an element of the admissible set A, then $\alpha \in A$. Specifically, A can contain only well-orderings of order type $< o(A)$.

If $\langle B, E \rangle \models \mathrm{KP}$ and $b \in B$, then $\mathrm{TC}(b)$ will be well-founded in the sense of the real world V just in case the rank of b in the sense of $\langle B; E \rangle$ happens to be well-ordered in V. The set of all $a \in B$ which satisfy the condition (which is not expressible in $\langle B, E \rangle$, unless all elements of B satisfy the condition) is called the well-founded part of $\langle B, E \rangle$ and denoted $\mathrm{WF}(B, E)$. A result originating with Ville (see Barwise [1975]) states that if $\langle B, E \rangle \models \mathrm{KP}$, then $\mathrm{WF}(B, E)$ is isomorphic to an admissible set. This is often called the "truncation lemma".

Returning now to more model-theoretic concerns, suppose that A is admissible. Then, if the underlying vocabulary is Δ on A, so also will be the set of formulas of L_A and the set of sentences of L_A. The satisfaction relation will be Δ, while the quantifier rank of a formula will be given by a Σ-formula.

5.3.3 Application. Suppose $\mathfrak{M}$ is a structure, $m_1, \ldots, m_k \in M$, and α is an ordinal. It is quite easy to see that the function taking $(\mathfrak{M}, m_1, \ldots, m_k, \alpha)$ to $\sigma^{\alpha}_{\mathfrak{M}, m_1, \ldots, m_k}$ is defined by a Σ recursion; and so, in particular, the relation "$x = \sigma(\mathfrak{M})$" is Σ on any admissible set containing ω. Of course, this does not mean that an admissible set need be closed under σ.

Now, if $A = \mathrm{HC}$, and if φ is any sentence of $\mathscr{L}_{\omega_1\omega}$, then since, as was noted earlier, every countable structure $\mathfrak{M}$ (for a countable language) has its canonical Scott sentence $\sigma(\mathfrak{M})$ in $\mathscr{L}_{\omega_1\omega}$, the set $S = \{\sigma(\mathfrak{M})\colon \mathfrak{M} \models \varphi\}$ is Σ on HC and in one-to-one correspondence with the isomorphism types of countable models of φ. Now, by the general set-theoretic result of Mansfield [1975], S has cardinality $\leq \aleph_1$ or $= 2^{\aleph_0}$. This, of course, is simply the result of Morley [1970] on the weak form of Vaught's conjecture for $\mathscr{L}_{\omega_1\omega}$. The same argument also works for PC classes. On the other hand, it is known that the Vaught conjecture itself fails for PC classes. In fact, using the "truncation lemma", it is not difficult to see that the order types of the ordinals in countable models of KP must be of the form α or $\alpha + \eta \cdot \alpha$, where η is the order type of the rationals and α is a countable admissible ordinal. (To see that all the "non-standard" values are obtained one can appeal, for instance to Theorem 7.2.7. H. Friedman originally noted this for ZF in place of KP. However, by using KP, we get all the standard α immediately, which is what we need here). There are other proofs of this result and others relating to Vaught's conjecture. A good reference is Steel [1978]. More recently, Shelah (see Harrington–Makkai–Shelah [198?]) proved the Vaught conjecture for ω-stable theories in $\mathscr{L}_{\omega\omega}$.

5.3.4 Remark. A next step up from the theorems we have discussed would be Σ-separation. This principle is not provable in KP and is, in fact, quite strong. For example, if $\langle A, \in\rangle \models$ "Σ-separation", then it is easy to see that $\langle A, \in\rangle$ is a *β-model*, i.e., if $\langle A, \in\rangle \models$ "$\langle x, \leq\rangle$ is a well-ordering", then $\langle x, \leq\rangle$ really is a well-ordering. If $\langle A, \in\rangle \models$ "Σ-separation" and is *locally countable*. That is, if for each $a \in A$, there is some bijection from a into ω, then $\langle A, \in\rangle$ is *recursively inaccessible*, which means that if $a \in A$, then there is some admissible $\langle B, \in\rangle$ such that $a \in B \in A$. However, the smallest recursively inaccessible admissible set does not satisfy Σ-separation. For $\langle A, \in\rangle$ locally countable, it is shown in Nadel [1974b] that $\langle A, \in\rangle$ is a β-model iff $\langle A, \in\rangle$ is recursively inaccessible. Though the implication from right to left holds without local countability, there are β-models that are not recursively inaccessible; for example, consider HYP($L(\omega_1)$).

5.4. *Urelements*

When a model theorist studies a model $\mathfrak{M} = \langle M, R, \ldots\rangle$, the only structure he wants to consider is that imposed upon the elements of M by $R, \ldots$. The particular elements forming the universe M are irrelevant and regarded as atoms or urelements. Unfortunately, with ZF as metatheory, there are no urelements and M will consist of sets, each with its own internal structure. While this may not be aesthetically pleasing, in most instances the model theorist is able to simply ignore the fact. However, in the present rather sensitive context, this is not possible.

For the current purpose, there are two main considerations. First, the set HYP($\mathfrak{M}$) should depend only on the isomorphism type of $\mathfrak{M}$. In fact it would also be reasonable to expect that if $\mathfrak{M}$ and $\mathfrak{N}$ are potentially isomorphic, then so are $\langle \text{HYP}(\mathfrak{M}), \in\rangle$ and $\langle \text{HYP}(\mathfrak{N}), \in\rangle$. It should be apparent that even the first version would never be literally satisfied. One might then try to patch things up as follows: assign to each isomorphism type the intersection of all admissible sets containing models of that isomorphism type. This would work to some extent for countable structures (see Nadel–Stavi [1977]); but, as we shall mention later, even here there would be the difficulty that there need be no copy of $\mathfrak{M}$ in the intersection. However, suppose we consider even the simplest example of a structure, a set M with no relations or functions at all. Suppose M has cardinality $\beth_{\omega_1}$. Then any admissible set containing M must contain an uncountable ordinal. Clearly this would violate the stronger version.

The second consideration is that by allowing urelements, there are more admissible sets; and, consequently, a finer classification becomes possible. For example, so far HF is the only admissible set with ordinal ω. Allowing urelements will provide many others, and these will turn out to be a significant class which will be considered in more detail in Section 7.4.

Having presented some reasons why doing without urelements would cause problems, we go ahead and permit them from now on. This requires some changes in terminology and a slight modification of the axioms of KP to form the analogous theory KPU. We omit the precise details, all of which are carefully presented in Barwise [1975]. We also omit the precise construction of HYP($\mathfrak{M}$), which is via the Gödel operations beginning with $\mathfrak{M}$, a structure on urelements. Suffice it to say that HYP($\mathfrak{M}$) is the smallest admissible set containing $\mathfrak{M}$ as an element, and that the first consideration mentioned above holds in the strong version.

Now, having insisted on the need for urelements, we must confess that in terms of our presentation here—because we are considering admissibility more from the model-theoretic point of view than from the recursion theoretic, and we will be omitting most of the details—urelements will really not play a significant rôle, except in Section 5.5 and in our discussion of recursively saturated structures in Section 7.4. The results we will be considering usually carry over from admissible sets without urelements to the more general setting allowing urelements with little or no change. Thus, we will simply suppress mention of urelements except where they really do make a difference. However, there is one restriction that we should make clear at this point. In exchange for having additional admissible sets with ordinal ω, it is sometimes necessary to restrict the underlying vocabulary to be finite.

5.5. *The Pure Part of* HYP(M)

In this section we discuss some results concerning admissible sets with and without urelements. Assume that $\langle A, \in\rangle$ is an admissible set which may contain urelements. Those elements of A other than the urelements are called *sets*. Among the sets are distinguished the *pure sets* whose transitive closures do not contain urelements. We call admissible sets without urelements (that is, those containing only pure sets) *pure admissible sets*.

One urelement is like any other. And that is just the point. Consequently, distinct sets may only be distinguishable by reference to the specific urelements involved and might even be images of each other under some $\in$-automorphism. This cannot happen to pure sets. In some sense, then, pure sets have a real identity while arbitrary sets need not. This is especially evident in comparing elements from different admissible sets. For this reason, the set of pure sets in an admissible set A, denoted pp(A), the *pure part of* A, plays a special rôle. For example, the set of sentences of L_A would be taken to be a subset of pp(A), so that these sentences would form a subset of the sentences of $\mathscr{L}_{\infty\omega}$ as viewed from "the real world" where we need not have urelements.

The following easy result is from Barwise [1975].

5.5.1 Theorem. *If* $\langle A, \in\rangle$ *is admissible, then* $\langle \mathrm{pp}(A), \in\rangle$ *is a pure admissible set.* $\square$

The next result, which is due to Makkai (see Nadel–Stavi [1977]), gives some idea of the "internal" relation between pure sets and sets of urelements.

5.5.2 Theorem. *Let M be a countable structure on urelements. Then the following are equivalent:*

(i) *$\mathfrak{M}$ has only countably many automorphisms.*
(ii) HYP($\mathfrak{M}$) *contains a pure structure $\mathfrak{N}$ which is an isomorphic copy of $\mathfrak{M}$ and an isomorphism between $\mathfrak{M}$ and $\mathfrak{N}$.* $\square$

In Theorem 5.5.2, HYP($\mathfrak{M}$) might contain an isomorphic pure copy, but not an isomorphism. Moreover, HYP($\mathfrak{M}$) could be replaced by the class of sets constructible from $\mathfrak{M}$, or even hereditarily symmetric over $\mathfrak{M}$. The next result from Nadel–Stavi [1977] shows how pp(HYP($\mathfrak{M}$)) can be described without reference to urelements.

5.5.3 Theorem. pp(HYP($\mathfrak{M}$)) *is the smallest admissible set containing $\sigma^{\beta}_{\mathfrak{M}}$, for each $\beta \in$* HYP($\mathfrak{M}$).

A case can be made for using pp(HYP(x)) as a measure of the information contained in x. If we begin with a pure set x, rather than with a structure on urelements, then we denote by x^+ the smallest pure admissible set containing x as an element. The next result, which may be appreciated more after considering canonical Scott sentences again in Section 7.1, shows that $\mathfrak{M}$ and $\sigma(\mathfrak{M})$ contain about the same information.

5.5.4 Corollary. (i) *If $\sigma(\mathfrak{M}) \in$* HYP($\mathfrak{M}$), *then* pp(HYP($\mathfrak{M}$)) $= (\sigma(\mathfrak{M}))^+$;
(ii) *If $\sigma(\mathfrak{M}) \notin$* HYP($\mathfrak{M}$), *then* (pp(HYP($\mathfrak{M}$)) $\cup \{\sigma(\mathfrak{M})\})^+ = (\sigma(\mathfrak{M}))^+$. $\square$

Since admissible sets of the form pp(HYP($\mathfrak{M}$)) might have special properties, it is natural to ask which pure admissible sets can be represented as pp(HYP($\mathfrak{M}$)) for some $\mathfrak{M}$. First, some terminology is needed. An *ordinal* α is called *admissible* if $L(\alpha)$ is admissible. This is the same as saying that $\alpha = o(A)$ for some admissible set A. Sacks (see Friedman–Jensen [1968]) showed that a countable admissible ordinal is of the form ω_1^x for some $x \subseteq \omega$, where ω_1^x denotes Church–Kleene ω_1 relativized to x. In a similar spirit, Nadel–Stavi [1977] showed that every admissible $L(\alpha)$ is of the form pp(HYP($\mathfrak{M}$)) for some $\mathfrak{M}$, as well as some other representation theorems. Not all pure admissible sets could be represented as pp(HYP($\mathfrak{M}$)). An admissible set A is said to be *resolvable* iff there is a function $F: o(A) \to A$ such that $A = \bigcup_{\beta \in A} F(\beta)$, and $\langle A, \in, F \rangle$ is admissible. It is not difficult to see that if A is resolvable, we can always find F, such that, for each $\alpha < \beta \in A$, $F(\alpha) \in F(\beta)$, and $F(\alpha)$ is transitive. If F can be chosen $\mathbf{\Delta}$ on A, we call A, $\mathbf{\Delta}$-resolvable. Clearly pp(HYP($\mathfrak{M}$)) is resolvable, using $F(\beta) =$ pp($L(\beta, \mathfrak{M})$), but there are non-resolvable countable admissible sets. Nadel–Stavi [1977] asked if this is the only constraint. Using structures M motivated by Steel forcing in place of the simpler structures used in the partial result of Nadel–Stavi [1977], S. Friedman [1982a] has shown this to be the case.

5.6. Barwise Compactness

As we remarked earlier, compactness fails for $\mathscr{L}_{\omega_1\omega}$ even for the simplest infinitary fragments. However, the following variant of compactness does hold.

5.6.1 Theorem (Barwise Compactness Theorem). *Let $\mathfrak{A}$ be a countable admissible structure and let T be a set of L_A sentences Σ definable on $\mathfrak{A}$. Suppose that each $T' \subseteq T$, $T' \in A$, has a model. Then T has a model.* $\square$

This result can be proved directly using the model existence theorem, or it can be obtained as a corollary to the extended Barwise completeness theorem which will be treated in Section 6.1. Barwise compactness resembles ordinary compactness, except that the theory T is restricted to be Σ on A, rather than arbitrary, while the hypothesis requires more than just finite sets being satisfiable. Nonetheless, Barwise compactness is a very powerful and important tool. It is safe to say that this result is what established admissible sets as an ongoing feature of model theory and started a second wave of interest in infinitary logic.

5.6.2 Remarks. It is easy to see that ordinary compactness for $\mathscr{L}_{\omega\omega}$ follows from Barwise compactness. The restriction to Σ-theories is really no restriction here since, for any set $X \subseteq \mathrm{HF}$, $\langle \mathrm{HF}, \in, X\rangle$ is admissible.

We will have more to say about Barwise compactness in Section 6.2 and will end this chapter with a brief application of it.

5.7. An Application of Barwise Compactness

In this section we will give a simple example of how Barwise compactness may be used. There are numerous applications to model theory. For a striking example of a more set-theoretic nature the reader should see Barwise [1971]. Barwise compactness is an especially potent tool used in conjunction with the omitting types theorem, as, for example, in Keisler [1971a, p. 58]. We will give a simple recursion-theoretic application which we will use later for model-theoretic purposes.

Kleene [1955b] gave an explicit definition of a recursive linear ordering that is well-ordered with respect to hyper-arithmetic subsets, but is not really well-ordered. Later, in Section 7.1 we will be interested in the canonical Scott sentence of such an ordering. We now will use Barwise compactness to show that such an ordering indeed exists. Once that is established, it is relatively simple to see what its order type could be. The object we construct is, by model-theoretic standards, quite refined, since we are insisting that it be recursive. Although Barwise compactness may seem at first glance to be much more restricted than ordinary compactness, the far greater expressive power of $\mathscr{L}_{\omega_1\omega}$ allows Barwise compactness to provide more subtle models than can be obtained from ordinary compactness.

Now, to begin the argument, let $A = L(\omega_1^{\text{CK}})$. We will use a language with a binary relation symbol $\in$, a constant symbol $\mathbf{a}$ for each $a \in A$, and an additional constant symbol $\mathfrak{M}$ (the symbols $\mathbf{a}$ are really expendable). Consider now a theory T in L_A that expresses the following:

(i) KP;
(ii) atomic diagram of $\langle A, \in\rangle$;
(iii) "every ordinal is recursive";
(iv) "$\mathfrak{M}$ is a recursive binary relation on ω which is a well-ordering";
(v) "$\mathfrak{M}$ has an initial segment of type $\boldsymbol{\alpha}$", $\alpha \in A$.

It is not difficult to see that T could be chosen to be $\boldsymbol{\Sigma}$ on A. It is also easy to see that every subset $T' \subseteq A$, $T' \in A$ has a model. Thus, T has a model $\mathscr{L} = \langle B, a, \mathfrak{M}\rangle_{a \in A}$. Finally, there is sufficient absoluteness to guarantee that M really is a recursive linear ordering with initial segment of type ω_1^{CK} and is also such that every hyper-arithmetic subset of ω has a least element. $\mathfrak{M}$ cannot really be well-ordered, since, if it were, it would be of order type some non-recursive ordinal.

6. *General Model-Theoretic Properties with Admissibility*

In this section we will deal with aspects of the model theory of L_A, for A admissible, where the syntax is somehow bound to the set A, but the models involved need not be.

6.1. *Barwise Completeness*

In Section 3.2 we introduced the notion of provability $\vdash_{L_B}$, and stated a completeness theorem for it in Theorem 3.2.1. Now, we would like to use a stronger notion of provability, a notion in which the proof itself—as well as the formulas in the proof—are elements of an admissible set A. In order for this stronger notion to be complete, however, we will need to modify the definition of proof slightly. Without going into all the details here (these can be found, for instance, in Barwise [1975]), we modify the clause for conjunctions by taking as a proof of $\psi \to \bigwedge \Phi$ a function f with domain Φ such that for each $\varphi \in \Phi$, $f(\varphi)$ is a set of proofs of $\psi \to \varphi$. Basically, this change is necessary because the axiom of choice need not hold within an admissible set. Let us denote this new notion of proof by $\vdash'_{L_A}$. It is then quite easy to see (using the axiom of choice in the universe) that for any sentence of L_A, $\vdash'_{L_A} \varphi$ iff $\vdash_{L_A} \varphi$. Finally, let $\vdash_A \varphi$ mean that there is some proof in A, in the sense of $\vdash'_{L_A}$, of φ. This is equivalent to saying "$\langle A, \in\rangle \models \vdash'_{L_A} \varphi$" since the notion that p is a proof of φ in the sense of $\vdash'_{L_A}$ is absolute for admissible sets. In particular, if T is a Σ_A-theory, that is, a theory in L_A Σ-definable on A, then $\{\varphi: T \vdash_A \varphi\}$ is Σ_A.

Barwise [1967] (also Barwise [1969b]) was able to prove

6.1.1 Theorem. *For any admissible A, and φ a sentence of L_A, $\vdash_{L_A} \varphi$ iff $\vdash_A \varphi$. Moreover, if T is a Σ_A-theory, then $T \vdash_{L_A} \varphi$ iff $T \vdash_A \varphi$.* □

Now, as an immediate consequence of Theorem 6.1.1 and the Karp completeness theorem (3.2.1) we have the following sharpening.

6.1.2 Theorem (Barwise Completeness Theorem). *Let A be a countable admissible set and φ a sentence of L_A, then $\models \varphi$ iff $\vdash_A \varphi$. Moreover, if T is a Σ_A-theory, then $T \models \varphi$ iff $T \vdash_A \varphi$.* □

We now obtain the following generalization of the fact that the set of valid sentences of $L_{\omega\omega}$ is r.e.

6.1.3 Corollary. *Let A be countable admissible and T a Σ_A-theory. Then $\{\varphi: \varphi$ is a sentence of L_A and $T \models \varphi\}$ is Σ_A.* □

6.1.4 Remarks. The Barwise completeness theorem must clearly fail in general for uncountable A, since the Karp completeness theorem already fails. In fact, the extended version is easily seen to fail, even for subsets of HC of power $\aleph_1$. The fact that Theorem 6.1.1 holds without cardinality restriction does show that provability is absolute for models of ZFC, and this allows us to finish the argument that was begun in the remarks of Section 3.2.2 that provability is equivalent to validity in boolean-valued extensions of the universe. If φ is boolean-valid, we simply pass to a universe in which φ is countable. In this universe φ is valid, and we now appeal to the Barwise completeness theorem.

Corollary 6.1.3 also fails for uncountable A. We will consider this subject further in Section 6.3.

There is a converse to the Barwise completeness theorem due to Stavi [1973] and extending partial results of Barwise [1967]. It is stated in reference to Theorem 6.1.1 instead, since, it then may hold for all cardinalities.

6.1.5. *Let B be a transitive primitively recursively closed set such that if $\vdash_{L_B} \varphi$, then $\vdash_B \varphi$. Then B is a union of admissible sets.* □

This result could be stated in a more general framework using certain classes of abstract provability predicates rather than the particular ones we have used. In contrast to Theorem 6.1.5, however, Stavi [1973] has shown that there is a countable transitive primitive recursively closed set A such that the set of valid sentences of L_A is Σ_1 on A, but A is not the union of admissible sets.

6.2. *Barwise Compactness* (Continued)

Recall that for A admissible, $S \subseteq A$, S is Σ on A iff S is Σ_1 on A. We say that a transitive set A is *Σ_1-compact* if L_A satisfies the Barwise compactness theorem for Σ_1 sets of sentences (rather than Σ). There are also relativized notions where additional predicates are mentioned. The next result is due to Barwise [1967] and shows that admissibility is the weakest assumption one can make to get Σ_1-compactness.

6.2.1 Theorem. *Suppose A is rudimentary. Then if A is Σ_1-compact, A is admissible.* $\square$

The subject of compactness for admissible fragments of $\mathscr{L}_{\infty\omega}$ will be considered in Section 6.3.

6.3. *Uncountable Admissible Sets*

In considering Barwise compactness on uncountable admissible sets, or in trying to determine the uncountable admissible fragments L_A for which the L_A validities are Σ_1 on A, there are basically two sorts of results. The first sort involves implications between these properties and other conditions that seek to strengthen the notion of admissibility. We will not pursue this line here. The interested reader should consult Barwise [1975] for an introduction to these matters. The second sort establishes the existence (in a "concrete" way) of uncountable admissible sets satisfying Barwise compactness, or on which the validities are Σ_1. Specialized results in this direction were obtained earlier by Barwise [1968], Chang–Moschovakis [1970], Green [1974], Karp [1972], Makkai [1974b], Nyberg [1974, 1976] and perhaps others. More recently, S. Friedman [1981] and Magidor–Shelah–Stavi [1984] have obtained more general treatments. Our presentation here is based upon the latter of these. The idea is simply to assume that in some reasonably nice way, the admissible set in question is the union of countably many "small" sets. For simplicity, we will assume our admissible sets are pure and give

6.3.1 Definition. Suppose $\mathfrak{A}$ is admissible. $S \subseteq A$, is said to be a *smallness predicate* for $\mathfrak{A}$ if

(i) S is Σ_1 on A;
(ii) if $x \in S$, then $\mathscr{P}(x) \in A$;
(iii) the relation $\{(x, \mathscr{P}(x)) \colon x \in S\}$ is Σ_1 on A.

A is said to have the *first decomposition property* (DP1) if for some smallness predicate S for $\mathfrak{A}$, every member of A is a countable union of members of S (in the real world).

6.3.2 Definition. A binary relation R on A is a *decomposition relation for A* if

(i) R is Σ_1 on A;
(ii) $\forall X\, \exists Y R(X, Y)$;
(iii) whenever $R(X, Y)$, then for some sequence $\langle X_n : n \in \omega \rangle$ such that $X_n \in Y$, and $\mathscr{P}(X_n) \subseteq Y$ for $n \in \omega$, $X = \bigcup \{X_n : n \in \omega\}$.

A is said to have the *second decomposition property* (DP2) if it has a decomposition relation. $\mathfrak{A}$ is said to have the decomposition property (DP) if $\mathfrak{A}$ has (DP1) and (DP2). A set $B \subseteq A$ is said to be *σ-small* if it is a countable union of elements of A. If $\mathfrak{A}$ is σ-small and has (DP), it is said to be *countably decomposable.*

Using the above notions, Magidor–Shelah–Stavi [1984] obtain their main result in the next theorem and its corollary.

6.3.3 Theorem. *Let A satisfy* (DP) *and assume $T \subseteq L_A$ is σ-small and Σ_1 on A_1. Then*

(i) *$\{\varphi \in L_A : T \models \varphi\}$ is Σ_1 on $\mathfrak{A}$;*
(ii) *If T has no model, then some $T_0 \subseteq T$, $T_0 \in A$ has no model.* □

6.3.4 Corollary. (i) *If A satisfies* (DP) *then $\{\varphi \in L_A : \models \varphi\}$ is Σ_1 on A.*
(ii) *If A is countably decomposable then A satisfies Barwise compactness and for a theory T, Σ_1 on A, $\{\varphi \in L_A : T \models \varphi\}$ is Σ_1 on A.* □

All of the specialized results on Barwise compactness and completeness alluded to above are consequences of Theorem 6.3.3 and Corollary 6.3.4, including the original results for A countable.

6.3.5 Examples. (i) If A is closed under (real) power set and the relation $\{\langle x, \mathscr{P}(x)\rangle : x \in A\}$ is Σ_1 on A, then, letting $S = A$ and $R = \{\langle x, \mathscr{P}(x)\rangle : x \in A\}$, we have the Barwise–Karp cofinality ω compactness theorem (see Barwise [1968] and Karp [1972]).

(ii) Let A be an admissible set containing some element b, such that in the sense of A, every element of A has cardinality at most the cardinality of b, and that for some sequence $\langle b_n : n \in \omega \rangle \in A$, where $\bigcup \{\mathscr{P}(b_n) : n \in \omega\} \in A$, $b = \bigcup \{b_n : n \in \omega\}$. Then, if we take $S = \{x \in A : x$ has cardinality at most b_n in the sense of A, for some $n \in \omega\}$ and $R = \{(X, Y) : \exists f \in A\, [f$ is a function from a subset of b onto X and $Y = \{f''Z : Z \in \bigcup \{\mathscr{P}(b_n) : n \in \omega\}\}$ we obtain Makkai's compactness theorem, Makkai [1974b], which generalizes Green [1974].

To what extent is the above decomposition property necessary? Magidor–Shelah–Stavi [1984] gives the following partial converse to Corollary 6.3.4(i).

6.3.6 Theorem. *Assume that $V = L$, then for $\alpha > \omega$ admissible, $\{\varphi \in L_\alpha : \models \varphi\}$ is Σ_1 on $\langle L_\alpha, \in\rangle$ iff $\langle L_\alpha, \in\rangle$ satisfies* (DP).

The situation for Barwise compactness is more complicated. Results of Barwise [1975] and Stavi [1978] show, for example, that for κ regular, there is a closed unbounded subset of $\alpha < \kappa$ such that $\langle L_\alpha, \in\rangle$ satisfies Barwise compactness. The idea here is that, for "soft" reasons, there are many $\langle L_\alpha, \in\rangle$ satisfying Barwise compactness, and some of these will not be countably decomposable. Magidor–Shelah–Stavi [1984] realized that by strengthening the notion of Barwise compactness to *stable* Σ_1*-compactness*, where we call $\mathfrak{A}$ stably Σ_1-compact if all admissible expansions of $\mathfrak{A}$ satisfy Barwise compactness, a result would be forthcoming. And this result we give in

6.3.7 Theorem. *Assume that $V = L$. Let A be an admissible structure of the form $\langle L_\alpha, \in, R_1, \ldots, R_n\rangle$. Then $\mathfrak{A}$ is stably Σ_1-compact iff either A is countably decomposable or α is a weakly compact cardinal.* $\square$

There is an analogous result for the second part of Corollary 6.3.4(ii) and other interesting results which the reader can find in Magidor–Shelah–Stavi [1984].

6.4. Interpolation

In Section 3.2 we mentioned that $\mathscr{L}_{\omega_1\omega}$ satisfies interpolation. However, countable fragments L_B of $L_{\omega_1\omega}$ do not, in general, satisfy interpolation. Barwise [1967] has nevertheless shown that for A countable admissible, L_A does satisfy interpolation; and, hence, its consequences such as Beth definability. His proof in Barwise [1975] is similar to the consistency property proof of the Lopez-Escobar interpolation theorem for $L_{\omega_1\omega}$, except that in order to show the set under consideration is a consistency property, it is necessary to appeal to the Barwise completeness theorem. We point out here that no analogous appeal is needed in the earlier result.

There is a converse result due to H. Friedman (see Makowsky–Shelah–Stavi [1976]).

6.4.1. Theorem. *Let A be a transitive primitive recursively closed set. If L_A is Δ-closed, then A is the union of admissible sets.* $\square$

6.5. Hanf Numbers

Barwise [1967] was able to obtain a finer Hanf number result for countable admissible fragments L_A. The results we state here are for Σ_1-theories rather than single sentences and originate in Barwise–Kunen [1971].

6.5.1 Theorem. *Let A be countable admissible and T a Σ_A-theory. If, for each $\beta < o(A)$, T has a model of cardinality at least $\beth_\beta$, then T has a model of each infinite cardinality.* $\square$

There is a generalization of Theorem 6.4.1 to arbitrary admissible sets, but for these some preliminary discussion is required.

6.5.2 Definition. Let T be a Σ_A-theory for some admissible fragment A. Assume the vocabulary has among its relation symbols a binary relation symbol $<$. T is said to *pin down the ordinal* α if

(i) For any model M of T, $<^{\mathfrak{M}}$ is a well-ordering of its field and
(ii) T has a model with $<^{\mathfrak{M}}$ of order type α.

The least ordinal not pinned down by some Σ_A-theory T is denoted $h_{\Sigma}(A)$.

6.5.3 Theorem. *Let A be admissible and $\kappa = |A|$. The Hanf number (for Σ_A-theories) of L_A is* $\sup\{\beth_\beta(\kappa): \beta < h_{\Sigma}(A)\}$.

In Section 7.2 we will give a very short proof of the following important fact.

6.5.4 Theorem. *Let A be countable admissible, then $h_{\Sigma}(A) = o(A)$.* □

See Chapter IX for information about the size of $h_{\Sigma}(A)$ for A uncountable.

6.6. Global Definability

In Theorem 4.2.2 we mentioned an interesting local definability result. Here, we give an important global definability result of Makkai [1977b]. The version we will give first appeared in Barwise [1975], and it involves Σ_1^1-sentences of $\mathscr{L}_{\omega_1\omega}$ which are simply sentences of the form $\exists\bar{Q}\varphi$ where $\bar{Q}$ is a set of symbols and φ is a sentence of $\mathscr{L}_{\omega_1\omega}$. The semantics is the obvious one.

6.6.1 Theorem. *Let $\exists\bar{Q}\varphi(P, \bar{Q})$ be a Σ_1^1-sentence of the countable admissible fragment $L_A(\tau)$. For a countable structure $\mathfrak{M}$ define $S(\mathfrak{M}) = \{P: \mathfrak{M} \models \exists\bar{Q}\varphi(P, \bar{Q})\}$. The following are equivalent*:

(i) *For each countable $\mathfrak{M}$, $|S(M)| = \aleph_0$.*
(ii) *For each countable $\mathfrak{M}$, $|S(M)| < 2^{\aleph_0}$.*
(iii) *There is a sentence ψ of $L_A(\tau)$ of the form*

$$\bigvee \exists y_1 \ldots y_{j_i} \forall x_1, \ldots, x_k[P(x_1, \ldots, x_k) \leftrightarrow \psi_i(x_1, \ldots, x_k, y_1, \ldots, y_{j_i})],$$

which is a logical consequence of $\varphi(P, \bar{Q})$, where each ψ_i contains only symbols of τ not in $\bar{Q} \cup \{P\}$. □

The proof of this result is somewhat involved and uses the interpolation theorem. It has a number of important corollaries, all of which can be found in Barwise [1975].

6.7. Omitting Types Revisited

We now continue the thread that we began spinning in Section 1.5. Barwise [1981] has shown that the facts that $\mathscr{L}_{\omega_1\omega}$ and $\mathscr{L}_{\omega_1\omega}(Q_1)$ each satisfies an omitting types theorem as well as Barwise completeness and compactness results are not isolated events, but rather are part of a general result of the type described at the end of Section 1.5. Recall that we assume that the logic $\mathscr{L}^*$ satisfies the substitution axiom and admits only first-order variables. In the subsequent discussion we will assume that $\mathfrak{R}, \mathfrak{S}_1, \ldots, \mathfrak{S}_m$ are relation symbols, and $c_1, \ldots, c_n$ are constant symbols not contained in the vocabulary under consideration, while "true" is some valid sentence in that vocabulary.

Barwise [1981] generalizes the notion of an omitting types theorem by the following string of definitions.

6.7.1 Definition. A sentence $\varphi(R)$ of L^* is said to be a *test sentence* if for all structures $\mathfrak{M}$, $(\mathfrak{M}, \bigcup_{n<\omega} R_n) \models^* \varphi(R)$ implies there is some $n < \omega$ such that $(\mathfrak{M}, \mathfrak{R}_n) \models^* \varphi(R)$. A *test set* is a set of test sentences.

For $\mathscr{L}_{\omega\omega}$ the relevant test set is just the set of all sentences of the form $\exists\vec{x}[\varphi(\vec{x})\ \&\ R(\vec{x})]$, for $\varphi \in \mathscr{L}_{\omega\omega}$, while for $\mathscr{L}(Q_1)$ it is the set of sentences of the form $S\ \exists\vec{x}(\varphi(\vec{y}, \vec{x})\ \&\ R(\vec{x}))$ where S is a string of $\exists y_i$'s and Qy_j's and φ is a sentence of $\mathscr{L}(Q_1)$.

6.7.2 Definition. (i) For any theory T of $\mathscr{L}^*$ and set $\Sigma(c_1, \ldots, c_n)$ of $\mathscr{L}^*$-sentences, we say that T *accepts* $\Sigma(c_1, \ldots, c_n)$ if there is a model of $T \cup \{\forall x_1 \ldots x_n \bigvee \Sigma\}$.

(ii) T *locally accepts* $\Sigma(c_1, \ldots, c_n)$ with respect to a test set $\mathscr{T}$ if for all $\varphi(R) \in \mathscr{T}$, if $T \cup \{\varphi(\text{true}/R)\}$ has a model, so does $T \cup \{\varphi(\sigma/R)\}$ for some $\sigma \in \Sigma$.

(iii) $\mathscr{L}^*$ has the *Omitting Types Property* (OTP) with respect to a test set $\mathscr{T}$ if for all theories T of $\mathscr{L}^*$ and all countable sets $\{\Sigma_i(c_i, \ldots, n_i): i < \omega\}$, if T locally accepts each Σ_i, then T accepts all the Σ_i simultaneously; that is to say, there is some

$$M \models T \cup \{\forall x_1, \ldots, x_{n_i} \bigvee \Sigma_i(x_1, \ldots, x_{n_i}): i < \omega\}.$$

(iv) $\mathscr{L}^*$ has the OTP if $\mathscr{L}^*$ has the OTP for some test set $\mathscr{T}$.

We need one final definition before the results can be stated.

6.7.3 Definition. Let $\mathscr{T}$ be a test set. By a *$\mathscr{T}$-closed fragment* of $\mathscr{L}^*_{\omega_1\omega}$ we mean a sublogic L^*_B which contains $\mathscr{L}^*$, is closed under subformulas, satisfies the substitution axiom, if $\bigwedge \Phi \in L^*_B$ so is $\bigvee \{\neg\varphi: \varphi \in \Phi\}$, and such that if $\varphi(R) \in \mathscr{T}$ and $\varphi(\bigvee \{\psi_i : i < \omega\}/R) \in L^*_B$ then $\bigvee \{\varphi(\psi_i/R): i < \omega\} \in L^*_B$. A $\mathscr{T}$-closed fragment L^*_B is said to be *countable* if for each countable vocabulary τ, $\mathscr{L}^*_B(\tau)$ is countable.

6.7.4 Theorem. *Let $\mathscr{L}^*$ be $\aleph_0$-compact and have the* OTP *with respect to $\mathscr{T}$. Let L_B^* be a countable $\mathscr{T}$-closed fragment of $\mathscr{L}^*_{\omega_1\omega}$. Then $\mathscr{L}_B^*$ has the* OTP *with respect to the set $\mathscr{T}_B$ of $\mathscr{L}_B^*$ sentences of the form $\varphi(R, \psi_1/S_1, \ldots, \psi_n/S_n)$, where $\varphi(R, S_1, \ldots, S_n) \in \mathscr{T}$, and $\psi_1, \ldots, \psi_n$ are sentences of L_B^*.* $\square$

This result follows easily from the proof of the next completeness result, a result which gives an alternate axiomatization for $\mathscr{L}_{\omega_1\omega}$.

6.7.5 Theorem. *Let $\mathscr{L}^*$ be an $\aleph_0$-compact logic and $\mathscr{L}^*$ have the* OTP *with respect to the test set $\mathscr{T}$. Then the following proof system is complete for $\mathscr{L}^*_{\omega_1\omega}$:*

Axioms:

(A1) *For each $\varphi(R) \in \mathscr{T}$, all sentences of the form*

$$\varphi(\bigvee \{\psi_i : i < \omega\}/R) \rightarrow \bigvee \{\varphi(\psi_i/R) : i < \omega\}.$$

(A2) *All valid sentences of $\mathscr{L}^*$.*

(A3) *All sentences of $\mathscr{L}^*_{\omega_1\omega}$ of the form*

$$\bigwedge \{\psi_i : i < \omega\} \rightarrow \psi_j, \qquad j < \omega.$$

(A4) *All sentences of $\mathscr{L}^*_{\omega_1\omega}$ of the form*

$$\bigvee \{\psi_i : i < \omega\} \rightarrow \neg \bigwedge \{\neg \psi_i : i < \omega\}.$$

Rules:

(R1) *Modus ponens.*

(R2) *Generalization.*

(R3) *From $\varphi \rightarrow \psi_i$ for all $i < \omega$ infer $\varphi \rightarrow \bigwedge \{\psi_i : i < \omega\}$.*

(R4) *From $\varphi(\mathfrak{R}_1, \ldots, \mathfrak{R}_k)$ infer $\varphi(\sigma_1/\mathfrak{R}_1, \ldots, \sigma_k/\mathfrak{R}_k)$, for all formulas $\varphi(\mathfrak{R}_1, \ldots, \mathfrak{R}_k) \in \mathscr{L}^*$, $\sigma_1, \ldots, \sigma_k \in \mathscr{L}_{\omega_1\omega}$.* $\square$

We will not give a complete proof of Theorem 6.6.5 (the reader should consult Barwise [1981] for this), but will merely sketch the main lines of argument. The proof is based on an idea from Keisler [1970]. Beginning with L_B^*, we first form $L_B^*(\tau')$ in which we allow finitely many occurrences of some countable set of new constants. Then, for each infinite disjunction $\bigvee \Phi_i(c_1, \ldots, c_{n_i})$ of $L_B^*(\tau')$, we add a new unary relation symbol R_i which will interpret $\bigvee \Phi_i(c_1, \ldots, c_{n_i})$. The vocabulary obtained by adding on these R_i's will be called τ''. The idea is that for each ψ of $L_B^*(\tau')$ we will define some $\psi^{\#}$ of $L^*(\tau'')$ which will play the part of ψ and will be "finitary" also. Specifically, $\psi^{\#}$ is defined inductively by the following four clauses:

(i) if θ is in $\mathscr{L}^*(\tau')$, $\theta^{\#} = \theta$.

(ii) $(\bigvee \Phi_i(c_1, \ldots, c_n))^{\#} = R_i(c_1, \ldots, c_n)$.

(iii) $(\bigwedge \Phi)^{\#} = \neg(\bigvee \{\neg \varphi : \varphi \in \Phi\})^{\#}$.

(iv) $\varphi(\sigma_1/R_1, \ldots, \sigma_k/\mathfrak{R}_k)^{\#} = \varphi(\sigma_1^{\#}/\mathfrak{R}_1, \ldots, \sigma_k^{\#}/\mathfrak{R}_k)$.

An appeal to the $\aleph_0$-compactness of $\mathscr{L}_B^*$ is then made in order to prove the key lemma to follow, where $T \cup \{\varphi\}$ is a set of sentences of $\mathscr{L}_B(\tau')$, $T^\# = \{\theta^\#: \theta \in T\}$, and $\vdash$ denotes provability in the above system:

$$(\#) \qquad T^\# \models \varphi^\# \quad \text{iff} \quad T \vdash \varphi.$$

Next, by making use of ($\#$) and the fact that $\mathscr{L}^*$ satisfies the OTP with respect to $\mathscr{T}$ (recall that $\vdash$ depends on $\mathscr{T}$), we can then prove Theorem 6.7.5. Finally, by adding admissibility, Barwise [1981] obtains the result given in.

6.7.6 Theorem. *Let $\mathscr{L}^*$ be $\aleph_0$-compact and have the* OTP *with respect to the test set $\mathscr{T}$. Let $L_{\mathbb{A}}^*$ be a countable admissible fragment where the admissible structure $\mathbb{A} = (A, \in, \ldots)$ has the property that the set of valid sentences of L^* and the set $\mathscr{T}$ are each Σ_1 definable on $\mathbb{A}$. Then*:

(i) *The set of valid sentences of $L_{\mathbb{A}}^*$ is Σ_1 on $\mathbb{A}$;*
(ii) *$L_{\mathbb{A}}^*$ is Σ_1-compact; that is to say, if $T \subseteq L_{\mathbb{A}}^*$ is Σ on $\mathbb{A}$, and if every $T_0 \subseteq T$ with $T_0 \in \mathbb{A}$ has a model, then T has a model;*
(iii) *If $\varphi \in L_{\mathbb{A}}^*$, then the least ordinal not pinned down by φ is in $\mathbb{A}$.*

The proofs of all parts of the above follow from Theorem 6.7.5 in the same way that the analogous results for $\mathscr{L}_A$ follow from the Karp completeness theorem for $\mathscr{L}_{\omega_1\omega}$, the hypothesis on the validities of $\mathscr{L}^*$ being required to manage the axioms of the form (A2).

7. *"Harder" Model Theory with Admissibility*

In this section we will consider aspects of the model theory of countable admissible fragments L_A in which the structures themselves are restricted to the set A or its environs.

7.1. Scott Sentences and Admissible Sets

Suppose A is admissible and $\mathfrak{M}$ is a structure with $\mathfrak{M} \in A$. How much can we say about $\mathfrak{M}$ or its complete $\mathscr{L}_{\infty\omega}$ theory $\mathrm{th}_{\infty\omega}(\mathfrak{M})$ by just knowing its complete L_A-theory, $\mathrm{th}_A(\mathfrak{M})$? The first result asserts that $\mathrm{th}_A(\mathfrak{M})$ tells you all you need to know to distinguish $\mathfrak{M}$ from other structures $\mathfrak{N} \in A$.

7.1.1 Theorem. *Suppose A is admissible and $\mathfrak{M}$, $\mathfrak{N} \in A$ with $\mathfrak{M} \equiv_{L_A} \mathfrak{N}$. Then $\mathfrak{M} \equiv_{\infty\omega} \mathfrak{N}$.* $\square$

7.1.2 Corollary. *Suppose A is countable admissible and $\mathfrak{M}$, $\mathfrak{N} \in A$ with $M \equiv_{L_A} \mathfrak{N}$. Then $\mathfrak{M} \cong \mathfrak{N}$.* $\square$

For the easy proof of Theorem 7.1.1 see Nadel [1974b] where a slightly weaker hypothesis is used. Scott's theorem then easily gives Corollary 7.1.2. From Theorem 7.1.1 we can obtain the better bounds promised in Section 4.2. Specifically, the formula φ in Theorems 4.2.1 and 4.2.2 can be taken to be in $\mathrm{HYP}(\mathfrak{M})$. For the remainder of this section let us ignore all structures $\mathfrak{M}$ such that $o(\mathrm{HYP}(\mathfrak{M})) = \omega$ since the questions we consider are of no interest for them.

Can Theorem 7.1.1 be improved by dropping the restriction that $\mathfrak{N} \in A$, or—even better—by showing that $\mathfrak{M}$ has a Scott sentence in A; or—still better—that the canonical Scott sentence $\sigma(A)$ is in A? Any of the possibilities would actually imply Vaught's conjecture for $\mathscr{L}_{\omega_1\omega}$. The results are due to Sacks, Harnik–Makkai [1976], Makkai [1977b], and Steel [1978] who showed that Vaught's conjecture holds for sentences whose models have these properties.

However, all of these possible strengthenings fail to hold, as the following example will show. It has long been known (see Nadel [1974b]) that there is a recursive ordering $\mathfrak{M}$ of order type $\omega_1^{\mathrm{CK}} + \omega_1^{\mathrm{CK}} \cdot \eta$. (In fact, the example in Section 5.7 can be strengthened to provide this.) $\mathfrak{M}$ is obviously an element of $L(\omega_1^{\mathrm{CK}})$; and, moreover, it can be shown that $\mathfrak{M} \equiv_{\omega_1^{\mathrm{CK}}} (\omega_1^{\mathrm{CK}}, <)$. This latter fact follows from general results found in Karp [1965] or from a more specialized argument given in Nadel [1974b], an argument which is based on the fact that $L(\omega_1^{\mathrm{CK}})$ "thinks" that $\mathfrak{M}$ is well-ordered.

On the positive side, by applying Theorem 7.1.1 to expansions of $\mathfrak{M}$ by finitely many constants, we easily obtain the following result of Nadel [1974b] on Scott heights.

7.1.3 Theorem. *Let A be admissible and suppose $\mathfrak{M} \in A$. Then $\mathrm{SH}(\mathfrak{M}) \leq o(A)$, whence $\sigma(\mathfrak{M})$ has quantifier rank at most $o(A) + \omega$, and is in $\mathrm{HYP}(A)$.* ☐

Let us call a structure $\mathfrak{M}$ such that $\mathrm{SH}(\mathfrak{M}) < o(\mathrm{HYP}(\mathfrak{M}))$ *tame*. Otherwise, they will be termed, *wild*. It is easy to see that $\mathfrak{M}$ is tame iff $\sigma(\mathfrak{M}) \in \mathrm{HYP}(\mathfrak{M})$. Practically speaking, one has to go out of one's way to find a wild structure. On the other hand, there are not many positive results saying that various types of structures are tame. We mention three. Nadel [1974b] shows that every scattered linear ordering is tame. Nadel [1974a] shows that if $\varphi \in L_A$, A countable, and φ has only finitely many non-isomorphic countable models, then, for every $\mathfrak{M} \models \varphi$, $\mathrm{SH}(\mathfrak{M}) < o(A)$; and, if φ is countable in the sense of A, then $\sigma(\mathfrak{M}) \in A$. Thus, in the above situation, if $o(A) = \omega_1^{\mathrm{CK}}$, then every model of φ is tame. Finally, if $\mathfrak{M}$ is countable and has $< 2^{\aleph_0}$ automorphisms, then $\mathfrak{M}$ is tame (see Nadel [1974b]).

Now we state the result of Nadel [1974b], a result which was alluded to earlier in Section 4.1 and which, in some sense, helps justify the choice of $\sigma(\mathfrak{M})$ as the "canonical" Scott sentence of $\mathfrak{M}$.

7.1.4 Theorem. *Let A be countable admissible with ω, $\mathscr{L} \in A$. Suppose $\varphi \in A$ is a Scott sentence of some model $\mathfrak{M}$ (not necessarily in A). Then $\sigma(\mathfrak{M}) \in A$. In fact, $\mathrm{SH}(\mathfrak{M})$ is at most the quantifier rank of $\varphi + \omega$.*

We will return for additional comments on wild structures after discussing Gregory's result on uncountable models in Section 7.3.

7.2. Löwenheim–Skolem Results and Σ_A-saturated Models

In this section we briefly treat some downward Löwenheim–Skolem or, alternatively, "basis"—results, that are more subtle than the standard results dealing only with cardinality. We assume all theories T mentioned are consistent.

7.2.1 Theorem. *Let A be admissible and L_B a countable fragment of $L_{\infty\omega}$ in the sense of A. Let $T \in A$ be a complete L_B theory. Then T has a model $\mathfrak{M} \in A$.* ∎

The proof of the above result is straightforward and can be found in Nadel [1974b]. If L_B is not required to be countable in the sense of A the result does not hold, nor does it hold if T is not required to be complete. In the latter case, a model can always be found in A^+, even if T is a theory in L_A which is Σ on A, so long as A is countable in A^+. If the theory T in Theorem 7.2.1 happens to have a prime model, then a prime model can be found in A. Instead of looking for a model in a set A, we can also try to find one in a "fattening" of A, that is to say, in a set $B \supseteq A$ such that $o(A) = o(B)$. The next result, which is in this direction, is due to Barwise–Schlipf [1976], Nadel [1974a], and Ressayre [1977] and is only one aspect of an equivalence we shall discuss later.

7.2.2 Theorem. *Suppose A is countable admissible and T is a Σ_A-theory. Then there is a countable admissible set $B \supseteq A$, with $o(A) = o(B)$ and a model $\mathfrak{M} \models T$, with $\mathfrak{M} \in B$.* ∎

In the special case that $o(A) = \omega$, $\mathfrak{M}$ will be a model on urelements. Otherwise, $\mathfrak{M}$ could be composed of urelements or sets. (The results in Nadel [1974a] and Ressayre [1977] were formulated before the re-introduction of urelements.) We can give now a very short proof of Theorem 6.4.3 as we promised earlier.

Proof of Theorem 6.5.4. First, modifying the example we discussed in Section 1.3, we define by induction formulas $\psi_\alpha(x)$ in the vocabulary of linear orderings that express that the predecessors of x have order type α. Note that the formulas ψ_α can be found in L_A whenever $\alpha \in A$. This already shows that $h_\Sigma(A) \geq o(A)$.

Now, suppose $h_\Sigma(A) > o(A)$. In particular, suppose the Σ_A-theory T pins down some $\alpha > o(A)$. Consider the Σ_A-theory $T' = T \cup \{\exists x \psi_\beta(x) \colon \beta < o(A)\}$. T' is clearly consistent by Barwise compactness and by Theorem 7.2.2. T' has a model $\mathfrak{M}$ in some admissible set B with $o(B) = o(A)$. Now, if $<^{\mathfrak{M}}$ is a well-ordering, then T' insists it have type at least $o(B)$. However, we observed in Section 5.3 that an admissible set C cannot contain a well-ordering of order type $\geq o(C)$, and so $<^{\mathfrak{M}}$ is not well-ordered. ∎

The three papers Barwise–Schlipf [1976], Nadel [1974a] and Ressayre [1977] were written with different purposes in mind and employed different terminology. We will try to employ the terminology that seems to be in current use. The next definition, which is due to Ressayre [1977], with modifications by Harnik and Makkai, appears, at first glance to be more complicated than one might expect. We will point out the reason for this presently.

7.2.3 Definition. Let A be admissible. A structure $\mathfrak{M}$ is said to be *Σ_A-saturated* if, for each $m_1, \ldots, m_k \in M$ it satisfies

(i) if $\Gamma(x_1, \ldots, x_k, v)$ is a Σ_A type in L_A, then

$$\mathfrak{M} \models \left(\bigwedge_{\Gamma' \subseteq \Gamma, \Gamma' \in A} \exists v \Gamma'(m_1, \ldots, m_k, v) \right)$$

$$\rightarrow \exists v \bigwedge \Gamma(m_1, \ldots, m_k, v);$$

(ii) if $I \in A$, q is Σ_A, and for each $i \in I$, q_i denotes $\{\varphi\colon \varphi$ is a formula of L_A in the free variables $x_1, \ldots, x_k$, such that $(i, \varphi) \in q\}$, then

$$\mathfrak{M} \models \left(\bigwedge_{(q' \subseteq q, q' \in A)} \bigvee_{i \in I} \bigwedge q'_i \right) \rightarrow \bigvee_{i \in I} \bigwedge q_i.$$

Condition (i) alone is what one might expect as the definition. Models satisfying (i) alone are sometimes called Σ_A-compact. Both conditions are needed, however, to prove Theorem 7.2.6, which explains much of the importance of Σ_A-saturated models.

7.2.4 Theorem. *Let A be countable admissible and let T be a consistent Σ_A-theory. Then T has a Σ_A-saturated model.*

The proof of this result can be obtained from the proof of Lemma 8.2.2. ☐

7.2.5 Definition. Let A be admissible and suppose $(\mathfrak{M}, m_1, \ldots, m_k)$ is a structure for a vocabulary $\tau \in A$. $\mathfrak{M}$ is said to be *Σ_A-resplendent* if, whenever $\tau' \supseteq \tau$, $\tau' \in A$ and T is a Σ_A-theory in $L_A(\tau')$ consistent with the $L_A(\tau)$ theory of $(\mathfrak{M}, m_1, \ldots, m_k)$, then $(\mathfrak{M}, m_1, \ldots, m_k)$ can be expanded to a model of T. If the expansion can always be taken to be itself Σ_A-saturated we say $\mathfrak{M}$ is *strongly Σ_A-resplendent.*

7.2.6 Theorem. *Let A be countable admissible. If $\mathfrak{M}$ is a countable Σ_A-saturated structure, then $\mathfrak{M}$ is Σ_A-resplendent. In fact, $\mathfrak{M}$ is strongly Σ_A-resplendent.*

In building the expansion, condition (i) in the definition of Σ_A-saturation is used to realize types and witness existential formulas, while condition (ii) is needed to handle disjunctions. We can obtain, with little difficulty, the converse of Theorem 7.2.6, which holds without any cardinality restrictions. Now, we can relate Σ_A-saturated structures to the earlier Löwenheim–Skolem results. This result was first obtained by Ressayre [1977], with Schlipf [1977] examining the case in which A = HF.

7.2.7 Theorem. *Let A be admissible and $\mathfrak{M}$ a Σ_A-saturated model. Then there is some admissible $B \supset A$ with $o(B) = o(A)$ such that $\mathfrak{M} \in B$.* ☐

To prove Theorem 7.2.7 we use, for the countable case, strong Σ_A-resplendency to build a model of KP around $\mathfrak{M}$, with standard ordinals the same as A, and then use Ville's result to take its well-founded part. Lévy's absoluteness gives the general result.

Does Theorem 7.2.7 have a converse? The answer is "almost". Nadel [1974a] and Ressayre [1977] were able to show that countable $\mathfrak{M}$ satisfying the conclusion of Theorem 7.2.7 were almost Σ_A-resplendent. The problem occurs because Σ_A sets need not be Σ_B sets. This does not occur if A is Σ_B; for instance, if $A = L(\alpha)$, for some α. More recently, Adamson [1978] has been able to find a complete converse, by slightly strengthening the notion of "fattening" used.

Most often in practice, rather than use the property of Σ_A-saturation directly, we use instead the properties given in Theorems 7.2.6 and 7.2.7. However, Σ_A-saturation has a distinct advantage over the other two notions: It is easy to see that Σ_A-saturation is preserved under the union of an L_A-elementary chain. This point is quite important for the proof of the main result of the next section.

7.3. Uncountable Models

As we noted earlier, a consistent sentence of $L_{\omega_1\omega}$ with an infinite model need not have an uncountable model. The following important result is from Gregory [1973] and it tells us when certain countable theories have uncountable models.

7.3.1 Theorem. *Let A be countable admissible and suppose T is a Σ_A-theory of L_A. Then the following are equivalent*:

(i) *T has an uncountable model*
(ii) *There are models of T $\mathfrak{M}$, $\mathfrak{N}$ such that $\mathfrak{M} \prec_{\neq L_A} \mathfrak{N}$.*

Using the results of the previous section Ressayre was able to give a proof of Theorem 7.3.1, a proof which was much simpler that Gregory's original argument and which we can present quite briefly. The difficult direction is in showing that (ii) implies (i). The idea here is to build an L_A-elementary chain of countable models whose union will be the desired uncountable model. Using (ii), the fact that T is Σ_A and the appropriate expansion theory, it is possible to find Σ_A-saturated models $\mathfrak{M}_0 \models T$ and $\mathfrak{M}_1$ such that $\mathfrak{M}_0 \prec_{\neq L_A} \mathfrak{M}_1$. Now, using strong resplendency, we can find a Σ_A-saturated $\mathfrak{M}_2$ such that $\mathfrak{M}_1 \prec_{\neq L_A} \mathfrak{M}_2$. This shows how to take care of any successor stage in the chain. To manage limit stages, we need only use the fact that the union of an L_A-elementary chain of Σ_A-saturated models is Σ_A-saturated.

The requirement that T is Σ_A in Theorem 7.3.1 is necessary, as was shown by an example of Gregory mentioned in Gregory [1970].

We will now return to the subject of Scott sentences for a few additional remarks. Since most familiar structures were tame, and wild structures were only found with difficulty, various conjectures concerning wild structures naturally arose from this limited experience.

Of the original wild structures $\mathfrak{M}$, each had a proper $L_{\mathrm{HYP}(\mathfrak{M})}$ elementary submodel, and so $\mathrm{Th}_{L_{\mathrm{HYP}(\mathfrak{M})}}(\mathfrak{M})$ had an uncountable model. It was thought that this might always be the case. However, Makkai [1981] has given a counterexample. He also gives an example of a sentence φ of $\mathscr{L}_{\omega_1\omega}$ with models of Scott height cofinal in ω_1, but no uncountable model.

Again, for all the original wild structures it was the case that $\bigwedge \mathrm{Th}_{L_{\mathrm{HYP}(\mathfrak{M})}}(\mathfrak{M})$ was not a Scott sentence. Makkai [1981] also gives a counterexample to the obvious conjecture here as well. It should be pointed out that the examples mentioned above, even with the alternate proofs by Shelah, are quite complicated.

7.4. Recursively Saturated Models

We now specialize our consideration of Σ_A-saturated models to the case in which $A = \mathrm{HF}$, the case originally considered by Barwise–Schlipf [1976]. In particular, τ will now be finite.

Here, condition (ii) in the definition holds automatically, since I must be finite, and so the definition looks more like what we might have first guessed. Furthermore, Σ_{HF} is essentially the same as r.e. in the sense of ordinary recursion theory. Thus, a structure is Σ_{HF}-saturated iff every r.e. 1-type over the model is realized. By Craig's theorem, this becomes no weaker if we restrict to recursive types. In fact, such models are called *recursively saturated.* On the other hand, a recursively saturated model will realize every type over the model r.e. in the complete theory of any simple expansion of the model by finitely many constants.

The notion of Σ_{HF}-resplendent is actually equivalent to the weaker looking condition on $\mathfrak{M}$, that if $\mathfrak{M}$ is a τ-structure and R is relation symbol not in τ such that for some $\mathfrak{N} \succ \mathfrak{M}$, $\mathfrak{N} \models \exists R\varphi(R)$, then $\mathfrak{M} \models \exists R\varphi(R)$, where φ is any sentence, possibly with parameters from M. Without admitting parameters, the notion becomes strictly weaker for $\mathfrak{M}$ uncountable. For $\mathfrak{M}$ countable, the parameters are not necessary.

The corresponding condition on fattenings is that $o(\mathrm{HYP}(\mathfrak{M})) = \omega$, and so, of course, we must have $\mathfrak{M}$ a model on urelements.

Finally, it follows from our earlier discussion, that these three conditions are equivalent. We should also point out that, from Theorem 7.1.1, it follows that recursively saturated models are ω-homogeneous.

It has been noticed that the class of recursively saturated models appears in certain natural applied situations. For example, Barwise–Schlipf [1975] showed that the recursively saturated models of Peano arithmetic are exactly those models that can be expanded to models of Δ_1^1-PA, a certain natural fragment of analysis. Lipshitz–Nadel [1978] show that if $\langle A, +, \cdot\rangle$ is a model of Peano arithmetic, then both $\langle A, +\rangle$ and $\langle A, \cdot\rangle$ must be recursively saturated. If $\langle A, +\rangle$ is a countable recursively saturated model of Presburger arithmetic, then resplendency allows us to expand it to a model of Peano. This is not true in the uncountable case; but, as shown in Nadel [1980b] for groups of cardinality $\aleph_1$, recursive saturation together with a simple group theoretic condition is enough, at least for the "integer" version of Presburger arithmetic, and is also necessary.

The notion of recursive saturation has already become an object of great interest and many results have been forthcoming concerning it. While space does not permit its further consideration here, it is safe to say that recursive saturation seems likely to enter the permanent repertory of the model theorist.

It seems especially fitting to end our study of $\mathscr{L}_{\omega_1\omega}$ with the topic of recursive saturation, which, after all, can be expressed quite simply in $\mathscr{L}_{\omega\omega}$. The investigation of finitary logic led to the investigation of infinitary logic, which in turn engendered the study of admissible sets, a study which has since come back to enrich the study of $\mathscr{L}_{\omega\omega}$.

8. *Extensions of $\mathscr{L}_{\omega_1\omega}$ by Propositional Connectives*

The objective of this concluding section is threefold. First, there is the matter of considering propositional connectives other than simple conjunction and disjunction. The second objective will be achieved as a by-product. In the course of obtaining the results we will have occasion to employ techniques which help to illustrate some of the ideas of the earlier sections. The third objective, which we will consider first, involves more abstract considerations, namely the problem of characterizing $\mathscr{L}_{\omega_1\omega}$.

The reader has no doubt been already struck by Lindström's characterizations of $\mathscr{L}_{\omega\omega}$ as a maximal logic satisfying various sets of conditions in Chapter II. $\mathscr{L}_{\infty\omega}$ can also be characterized as a maximal logic in several different ways, ways that are described in Chapters III and XVII. Can $\mathscr{L}_{\omega_1\omega}$ be characterized in this way? It is obvious how to characterize $\mathscr{L}_{\omega_1\omega}$ as a minimal logic, but not as a maximal logic. A natural question to ask would be whether $\mathscr{L}_{\omega_1\omega}$ is the maximal logic whose syntax lives on HC and which satisfies certain basic model theoretic properties, such as interpolation, some natural completeness result, and perhaps some others. The results of Section 8.3 will show that this would not seem to be the case.

8.1. Propositional Connectives

Our presentation in the remainder of this section is based on Harrington [1980] which continues earlier work of H. Friedman [1977] and unpublished work of Kunen. We will be concerned with the logic obtained by adding to $\mathscr{L}_{\omega_1\omega}$ a new countable propositional connective.

First, we add to the definition of the formulas of $\mathscr{L}_{\omega_1\omega}$ the clause

(*) if φ_i is a formula for $i < \omega$, in some fixed finite set of free variables, then so is $C(\langle\varphi_i : i \in \omega\rangle)$.

The semantics corresponding to this clause depends on the choice of a fixed function $P\colon \mathscr{P}(\omega) \to \{0, 1\}$. We denote the resulting logic by $\mathscr{L}(P)$. Specifically, we have the clause

$$(*) \qquad \mathfrak{M} \models_P C(\langle \varphi_i \colon i \in \omega \rangle) \quad \text{iff} \quad P(\{i\colon \mathfrak{M} \models_P \varphi_i\}) = 1.$$

Though the syntax of $\mathscr{L}(P)$ looks rather different, it is easy to see that $\mathscr{L}(P)$ is a sublogic of $\mathscr{L}_{\infty\omega}$. In fact, it is a sublogic of $\mathscr{L}_{(2^{\omega})^+\omega}$.

There is a natural proof system for $\mathscr{L}(P)$ which is obtained from the usual Hilbert-style proof system for $\mathscr{L}_{\omega_1\omega}$ by adding the following axioms:

1a. $\bigwedge(\{\varphi_i\colon i \in X\} \cup \{\neg\varphi_i\colon i \in \omega \backslash X\}) \to C(\langle \varphi_i\colon i \in \omega\rangle)$, for each $X \subseteq \omega$ such that $P(X) = 1$;
1b. $\bigwedge(\{\varphi_i\colon x \in X)\} \cup \{\neg\varphi_i\colon i \in \omega \backslash X\}) \to \neg C(\langle \varphi_i\colon i \in \omega\rangle)$, for each $X \subseteq \omega$ such that $P(X) = 0$;
2. $\bigwedge\{\varphi_i \leftrightarrow \varphi_i'\colon i \in \omega\} \to (C(\langle \varphi_i\colon i \in \omega\rangle) \leftrightarrow C(\langle \varphi_i'\colon i \in \omega\rangle))$, for each pair of sequences $\langle \varphi_i\colon i \in \omega\rangle$, $\langle \varphi_i'\colon i \in \omega\rangle$ of formulas.

We write $\vdash_P$ for provability in this system and reserve $\vdash$ for provability in our standard system for $\mathscr{L}_{\omega_1\omega}$. We use $\models_P$ and $\models$ in a similar way for validity in the two logics as well as for satisfaction. We say that P—or, more properly $\mathscr{L}(P)$—*is complete* if for every sentence φ of $\mathscr{L}(P)$, $\vdash_P \varphi$ iff $\models_P \varphi$.

Just as for $\mathscr{L}_{\omega_1\omega}$, since each rule of proof has only countably many hypotheses, if $\vdash_P \varphi$, then φ has a countable proof. This point will be essential for what comes later and so we simply require that proofs be countable. As usual, one direction in completeness is easy to verify, that is, that, $\vdash \varphi$ implies $\models \varphi$.

It will be necessary to consider *partial proposition connectives*, which are simply (partial) functions from a subset of $\mathscr{P}(\omega)$ to $\{0, 1\}$. If D is a derivation in the above system, then there is a natural associated partial propositional connective P_D defined so that $P_D(X) = 1$ if some axiom of type 1a for X is used in D, and $P_D(X) = 0$ if some axiom of type 1b for X is used. Otherwise, $P_D(X)$ is undefined.

Very much as in Section 6.6, the general technique employed here will be to treat the extra connective as a new atomic formula. Specifically, writing $\langle \varphi_i\rangle$ in place of the longer $C(\langle \varphi_I\colon i \in \omega\rangle)$, for each $\langle \varphi_i\rangle$ we introduce a new relation symbol $\mathfrak{R}_{\langle \varphi_i\rangle}$ of the appropriate number of places. Given a formula ψ of $\mathscr{L}(P)$, we define ψ^* in such a way that $\psi^* = \psi$ for ψ atomic, $(C(\langle \varphi_i\colon i \in \omega\rangle))^* = \mathfrak{R}_{\langle \varphi_i\rangle}$, and so that $*$ commutes with the other connectives and quantifiers. Notice also that ψ^* is always a formula in $\mathscr{L}_{\omega_1\omega}$. A structure for the new relation symbols will be called an *expanded structure*.

There is a small technical problem which must be overcome before we proceed: Not every $\mathscr{L}_{\omega_1\omega}$ formula in the new symbols is of the form ψ^*, for some formula of $\mathscr{L}(P)$ (in the original symbols). This arises because if, for example, $\mathfrak{R}_{\langle \varphi_i\rangle}(x)$ is 1-place and τ is a term we may form $\mathfrak{R}_{\langle \varphi_i\rangle}(\tau)$ and this will not be of the form ψ^*. However, $\mathfrak{R}_{\langle \varphi_i\rangle}(\tau)$ "ought to be equivalent" to $R_{\langle \varphi_i(\tau)\rangle}(x)$. We make this official by adding a set of axioms Γ to this effect, for each $\mathfrak{R}_{\langle \varphi_i\rangle}$ and appropriate sequence of terms. Then, relative to Γ, each φ in $\mathscr{L}_{\omega_1\omega}$ is equivalent to some ψ^*, where the $\mathscr{L}(P)$-formula ψ is found by tracing back through the recursive definition of $*$.

We denote this ψ by $\varphi^{\#}$; and, similarly, we let $T^{\#} = \{\theta^{\#} : \theta \in T\}$, for a set T of $\mathscr{L}_{\omega_1\omega}$ formulas. We use a similar convention for derivations.

By a *fragment* we will mean a subclass $\mathscr{F}$ of the formulas of $\mathscr{L}(P)$ that is closed under subformulas such that if $C(\langle\varphi_i : i \in \omega\rangle)$ and $C(\langle\varphi_i' : i \in \omega\rangle)$ are in $\mathscr{F}$, so is the corresponding axiom of type 2. Now, given a fragment $\mathscr{F}$, we let $S(\mathscr{F})$ be the collection of all ψ^* such that ψ in $\mathscr{F}$ is an instance of the axiom scheme 2. An expanded structure is called an *$\mathscr{F}$-structure* if it is a model of $S(\mathscr{F})$. If P is a partial propositional connective, we let $S(P, \mathscr{F})$ be the collection of all ψ^* such that ψ in $\mathscr{F}$ is an instance of the axiom scheme 1. An $\mathscr{F}$-structure $\mathfrak{M}$ gives rise to a partial propositional connective $P_{\mathfrak{M}}$ as follows: Suppose $\mathfrak{M} \models \varphi_i[\bar{a}]$ iff $i \in X$. Then let $P_{\mathfrak{M}}(X) = 1$ if $\mathfrak{M} \models \mathfrak{R}_{\langle\varphi_i\rangle}(\bar{a})$, and let $P_{\mathfrak{M}}(X) = 0$ if $\mathfrak{M} \models \neg\mathfrak{R}_{\langle\varphi_i\rangle}(\bar{a})$. Since $\mathfrak{M}$ is an $\mathscr{F}$-structure, $P_{\mathfrak{M}}$ is well-defined. The next result mentions some basic facts about the notions we have just introduced. These facts are easy to check.

8.1.1 Lemma. *Let $\mathscr{F}$ be a fragment and $\mathfrak{M}$ an $\mathscr{F}$-structure, then*

(i) *suppose $T \subseteq \mathscr{F}$ is a set of sentences and $\mathfrak{M} \models T^*$. Then, for any propositional connective $P \supseteq P_{\mathfrak{M}}$, $\mathfrak{M} \models_P T$;*
(ii) *if P is a partial propositional connective, then P and $P_{\mathfrak{M}}$ are compatible; that is to say, $P \cup P_{\mathfrak{M}}$ is a partial propositional connective, iff $\mathfrak{M} \models S(P, \mathscr{F})$;*
(iii) *for $T \subseteq \mathscr{F}$, and P a partial propositional connective, if D is a derivation in $\mathscr{L}_{\omega_1\omega}$ from $T^* \cup S(\mathscr{F}) \cup S(P, \mathscr{F})$, then $D^{\#}$ is a derivation from T in $\mathscr{L}(P)$, with $P_D^{\#} \subseteq P$;*
(iv) *if D is a derivation in $\mathscr{L}(P)$ using axioms $\alpha_0, \alpha_1, \ldots$, then D^* is a derivation in $L_{\omega_1\omega}$ from $\alpha_0^*, \alpha_1^*, \ldots$.* $\square$

8.2. The Main Lemma

The next result deals with $\mathscr{L}_{\omega_1\omega}$ and is the main lemma we will need to derive the desired results about $\mathscr{L}(P)$. It mixes omitting types with Σ_A-saturated models and its proof—which we will only sketch here—will nevertheless fill in some earlier omissions.

8.2.1 Definition. Let A be an admissible structure and let Φ be a type over L_A. We say that Φ is *semi-complete over A* iff $\Phi \cup \{\neg\varphi : \varphi \in \Phi\}$ is Δ on A.

It is obvious that complete types are semi-complete. If a semi-complete type Φ is principal over a Σ_A-theory, then Φ is Δ on A.

8.2.2 Lemma. *Let $\mathfrak{A}$ be a countable admissible structure, T a consistent Σ_A-theory, and Γ a collection of L_A types, each semi-complete over A, such that no member of Γ is Δ on $\mathfrak{A}$ and $|\Gamma| < 2^{\aleph_0}$. Then there is a Σ_A-saturated countable model of T which omits all the types in Γ.*

Proof. For each $f \in 2^\omega$, we build a countable Σ_A-saturated model $\mathfrak{M}_f$ of T such that for $f \neq g$, the only semi-complete types realized in both $\mathfrak{M}_f$ and $\mathfrak{M}_g$ are Δ

over A (and hence not in Γ). Thus, since $|\Gamma| < 2^{\aleph_0}$ and any $\Phi \in \Gamma$ can be realized in at most one $\mathfrak{M}_f$, some $\mathfrak{M}_f$ must omit all types in Γ.

Let D be a countable set of new constant symbols to use in the ensuing Henkin construction. For each $\alpha \in 2^{<\omega}$, we construct by induction a theory T_α satisfying the following conditions:

(i) T_α is a consistent Σ_A-theory, involving only finitely many constants from D;
(ii) $T_\phi = T$ and for $\alpha \subseteq \beta$, $T_\alpha \subseteq T_\beta$;
(iii) For each step of a complete Henkin construction, there is some $n \in \omega$ such that for all $\alpha \in 2^n$, T_α has carried out this step.
(iv) For each Σ_A-type $\Phi(\vec{x})$ that mentions only finitely many constants from D, there is an $n \in \omega$ such that for all $\alpha \in 2^n$, if $T_\alpha \cup \Phi(\vec{x})$ is consistent, then there are constants $\mathbf{d}_1, \ldots, \mathbf{d}_k \in D$ such that $\Phi(\mathbf{d}_1, \ldots, \mathbf{d}_k) \in T_\alpha$.
(v) For each $I \in A$, q, and i as in Definition 7.2.3(ii), there is some $n \in \omega$ such that for all $\alpha \in 2^n$, if for some $i \in I$, $T_\alpha \cup \{\bigwedge q_i\}$ is consistent, then there are constants $\mathbf{d}_1, \ldots, \mathbf{d}_k \in D$ such that $q_i(\mathbf{d}_1, \ldots, \mathbf{d}_k) \subseteq T_\alpha$.
(vi) For each sequence of variables $\vec{x} = x_1, \ldots, x_k$ and collection F of formulas in the free variables $\vec{x}$ closed under negation and Δ on $\mathfrak{A}$, and each $\mathbf{c}_1, \ldots, \mathbf{c}_k, \mathbf{d}_1, \ldots, \mathbf{d}_k$ from D, there are infinitely many $n \in \omega$ such that for all $\alpha, \beta \in 2^n$, if $\alpha \neq \beta$, then either (1) for all $\varphi \in F$, $T_\alpha \models \varphi(\mathbf{c}_1, \ldots, \mathbf{c}_k)$ or $T_\alpha \models \neg\varphi(\mathbf{c}_1, \ldots, \mathbf{c}_k)$ or (2) for some $\varphi \in F$, $T_\alpha \models \varphi(\mathbf{c}_1, \ldots, \mathbf{c}_k)$ but $T_\beta \models \neg\varphi(\mathbf{d}_1, \ldots, \mathbf{d}_k)$.

Using the fact that T and the types in Γ are Σ, Barwise completeness allows us to carry through a construction with the above properties. Now, for each $f \in 2^\omega$, $\bigcup \{T_{f \upharpoonright n}: n \in \omega\}$ is a complete Henkin theory by (iii) and so gives rise to a countable model $\mathfrak{M}_F$ of T. Conditions (iv) and (v) guarantee that $\mathfrak{M}_F$ is Σ_A-saturated. (Observe that to verify part (ii) in the definition of Σ_A-saturation, we must appeal to some property of admissibility such as Σ-refection). Finally, condition (vi) guarantees that if $f \neq g$, and $\mathfrak{M}_f$ and $\mathfrak{M}_g$ realize some type Φ semi-complete over $\mathfrak{A}$, then Φ is Δ on $\mathfrak{A}$. ☐

8.3. $\mathscr{L}(P)$'s with Nice Properties

Armed with Lemma 8.2.2, we are now able to begin our construction of logics $\mathscr{L}(P)$ which are complete and enjoy other desirable properties. For P a partial propositional connective and $\mathfrak{A} = \langle A, \in, P \upharpoonright A\rangle$ a countable admissible structure, $\mathscr{L}(P) \cap A$ is a fragment and $S(\mathscr{L}(P) \cap A)$ and $S(P \upharpoonright A, \mathscr{L}(P) \cap A)$ are each $\Sigma_{\mathfrak{A}}$.

8.3.1 Lemma. *Let P be a partial propositional connective with $|P| < 2^{\aleph_0}$ and let $\mathfrak{A} = \langle A, \in, P \upharpoonright A\rangle$ be a countable admissible structure. Suppose T is a set of $\mathscr{L}(P)$ sentences of A, Σ on $\mathfrak{A}$. Then either*

(i) *there is an $\mathscr{L}(P)$ derivation $D \in A$ of a contradiction from T with $P_D \subseteq P$ or*
(ii) *there is a countable $\Sigma_{\mathfrak{A}}$-saturated $\mathscr{L}(P) \cap A$-structure $\mathfrak{M}$ such that $\mathfrak{M} \models T^*$ and $P_{\mathfrak{M}}$ and P are compatible.*

Proof. Suppose that the $\Sigma_{\mathfrak{A}}$-theory $T' = T^* \cup S(\mathscr{L}(P) \cap A) \cup S(P \upharpoonright A, \mathscr{L}(P) \cap A)$ is consistent. Let Γ be the set of all types Φ of the form $\Phi = \{\varphi_i^*: i \in X\} \cup \{\neg\varphi_i^*: i \in \omega \backslash X\}$ where $X \in (\mathrm{dom}\, P)\backslash A$ and $\langle\varphi_i\rangle \in A$. Then each Φ is semi-complete but not Δ on A, since $X \notin A$. $|\Gamma| < 2^{\aleph_0}$ since $|P| < 2^{\aleph_0}$. Now, by Lemma 8.2.2, we obtain $\mathfrak{M}$ as in option (ii) since our choice of Γ prevents $P_{\mathfrak{M}}$ from clashing with P.

If, on the other hand, T' is inconsistent, then since T' is just a Σ_A-theory of $\mathscr{L}_{\omega_1\omega}$, we may apply Barwise compactness to obtain an $\mathscr{L}_{\omega_1\omega}$-derivation D in A of a contradiction from T'. Now, Lemma 8.1.1(iii) gives us option (i). ☐

8.3.2 Theorem. *There is a complete $\mathscr{L}(P)$.*

Proof. We will build an increasing chain of partial propositional connectives P_ζ, $\zeta < 2^{\aleph_0}$ such that $P_0 = \phi$, $P_\lambda = \bigcup \{P_\zeta: \zeta < \lambda\}$ for λ a limit, and such that $|P_\zeta| \le |\zeta \cdot \omega|$ for all $\zeta < 2^{\aleph_0}$. P will then be $\bigcup \{P_\zeta: \zeta < 2^{\aleph_0}\}$.

First, we enumerate all sentences of $\mathscr{L}(P)$ as $\langle\varphi_{\zeta+1}: \zeta < 2^{\aleph_0}\rangle$. Suppose we have already constructed P_ζ. Choose a countable A such that $\varphi_{\zeta+1} \in A$ and $(A, \in, P_\zeta \upharpoonright A)$ is admissible (this is no problem using, for example, the downward Löwenheim–Skolem theorem). Now, applying Lemma 8.3.1 there is a partial propositional connective P' compatible with P_ζ such that either $P' \supseteq P_D$ for some $\mathscr{L}(P)$-derivation D of $\neg\varphi_{\zeta+1}$, or $P' \supseteq P_{\mathfrak{M}}$ for some $\mathscr{L}(P) \cap A$-structure $\mathfrak{M} \models \varphi_{\zeta+1}$. We then take $P_{\zeta+1} = P_\zeta \cup P'$. It is easy to see that $P = \{P_\zeta: \zeta < 2^{\aleph_0}\}$ will be complete. ☐

8.3.3 Remarks. At each successor step $\zeta + 1$ of the construction there would be no problem in fixing P' arbitrarily on some X not in the domain of P_ζ. This would allow us to construct $2^{(2^{\aleph_0})}$ different complete P's.

Now that we know complete P's exist, the next result sheds a great deal of light on the problem of characterizing $\mathscr{L}_{\omega_1\omega}$ as a maximal "nice" logic whose syntax "lives" on HC, a goal that we mentioned at the outset of this section.

8.3.4 Theorem. *Let P be a complete propositional connective and let $\mathfrak{A} = \langle A, \in, P \upharpoonright A\rangle$ be a countable admissible structure. Then $\mathscr{L}(P) \cap A$ satisfies each of the following:*

(i) *Extended Barwise completeness.*
(ii) *Barwise compactness.*
(iii) *Interpolation.*

Proof. Suppose $T \subseteq \mathscr{L}(P) \cap A$ is Σ_1 on A and inconsistent. Then, since P is complete, there is some derivation D in $\mathscr{L}(P)$ of a contradiction from T. Now, if we apply Lemma 8.3.1 to $P \upharpoonright A \cup P_D$ then option (i) must hold, since otherwise we have the contradictory situation that $\mathfrak{M} \models T^*$ and P_D and $P_{\mathfrak{M}}$ are compatible, whence $\mathfrak{M} \models S(P_D, \mathscr{L}(P) \cap A)$, and $\mathfrak{M} \models \neg \bigwedge T^*$ since D was a derivation of a contradiction from T.

(ii) Barwise compactness for $\mathscr{L}(P) \cap A$ now follows immediately as usual from extended Barwise completeness.

(iii) Harrington [1980] describes two different proofs of interpolation for $\mathscr{L}(P) \cap A$.

The first is via a cut-free proof system for $\mathscr{L}(P)$. The second also gives a new proof of interpolation for $\mathscr{L}_A$ as well. It makes heavy use of the details of the construction of HYP($\mathfrak{M}$) and so is beyond the scope of our presentation here. In particular, it uses the fact that every element of HYP($\mathfrak{M}$) is denoted by some "term" with parameters from M and, furthermore, that the behavior of Δ_0-formulas over HYP($\mathfrak{M}$) is already "mirrored" back in $\mathfrak{M}$. A rather detailed treatment of these matters can be found in the final section of Nadel–Stavi [1977]. $\square$

8.3.5 Remark. An alternate approach to $\mathscr{L}(P)$ was given earlier (although not published) by Kunen. It involves the notion of a *selective ultrafilter* on ω which is an ultrafilter $\mathscr{U}$ on ω having the property that if $f: \omega \to \omega$ then either $f^{-1}(n) \in \mathscr{U}$ for some $n \in \omega$ or $f \upharpoonright X$ is 1–1 on some $X \in \mathscr{U}$. Though the existence of selective ultrafilters on ω follows from the continuum hypothesis or Martin's axiom, Kunen has shown that it is independent of ZFC.

Given an ultrafilter $\mathscr{U}$ on ω, we define a propositional connective $P_{\mathscr{U}}$ by $P_{\mathscr{U}}(X) = 1$ iff $X \in \mathscr{U}$. Kunen strengthens the usual proof system for $L_{\omega_1\omega}$ by adding the following axioms where $\langle \varphi_i : i \in \omega \rangle$ and $\langle \varphi_i^j : i \in \omega \rangle$ are any sequences of $\mathscr{L}(P)$ formulas:

1′. $(\bigwedge \{\bigvee \{\varphi_i^j : j \le n\} : i \in X\}) \to \bigvee \{C(\langle \varphi_i^j : i \in \omega \rangle) : j \le n\}$, for each $X \in \mathscr{U}$ and $n \in \omega$; and
2′. $\neg C(\langle \varphi_i : i \in \omega \rangle) \leftrightarrow C(\langle \neg \varphi_i : i \in \omega \rangle)$.

Now, with respect to this proof system, Kunen shows that $P_{\mathscr{U}}$ is complete iff $\mathscr{U}$ is selective. Kunen is also able to prove a Barwise completeness and compactness theorem, as well as interpolation for admissible fragments $\mathscr{L}(P) \cap A$, but only under the added hypothesis that every member of $\mathscr{U}$ that is Σ on $\mathfrak{A}$ has a subset in $\mathscr{U}$ that is Δ on $\mathfrak{A}$. This extra hypothesis is actually necessary. Kunen's proof is naturally more set-theoretic, and we will not go into it here. It can, however, be found in Harrington [1980].

8.3.6 Exercise. A good review of the material in this section, as well as of much that is in the entire chapter, can be had by working out the following problems. It is assumed that A is as in Theorem 8.3.4.

(i) Prove that the Hanf number for $\mathscr{L}(P) \cap A$ is $\beth_{o(A)}$.
(ii) State and prove an omitting types theorem for $\mathscr{L}(P) \cap A$.

Appendix

In this short section we will briefly note some of the major omissions of our article and give some references for each.

We have said nothing at all about the work done on categoricity theory for $\mathscr{L}_{\omega_1\omega}$. The interested reader should consult Keisler [1971a], Kierstead [1980], and Shelah [1975c].

Some work has been done on model-theoretic forcing in $\mathscr{L}_{\omega_1\omega}$. The reader who is interested in this aspect of the subject might want to consult Keisler [1973] and Lee–Nadel [1977].

Game sentences are closely connected to the subject of this article. Relevant information is available in Vaught [1973b], Harnik–Makkai [1976] and, to some extent, in Chapter X of the present volume. The reader should also be aware of Makkai [1977a] in which game sentences play a very basic rôle in the presentation of the general theory of admissible fragments. Another important connection involved here is that between $\mathscr{L}_{\omega_1\omega}$ and descriptive set theory.

Venturing off more in the direction of recursion theory proper, we come to the subject to inductive definability, the study of which could naturally be begun with Chapter X of the present volume. More "classical" recursion theory on admissible sets has become an object of much interest, and a study of this area might well begin by consulting Barwise [1975] and Shore [1977].

Finally, information about the "soft model-theoretic" aspects of the logics we have considered, including the relevant Lindström type results, can be found in Chapters III and XVII of the present work.