CHAPTER 2

GEOMETRIC PRELIMINARIES

Almost linear functions, approximate fundamental solutions,

and representation formulae. Harmonic coordinates.

2.1 OUTLINE OF THE CHAPTER

This chapter begins with a collection of basic estimates for Jacobi fields and some convexity results. We mostly follow the elegant presentation in [BK].

We then introduce the notion of almost linear functions on a manifold, the main technical innovation of [JK1]. Whereas standard coordinate functions, e.g. Riemannian normal coordinates, have only rather poor regularity properties (cf. the example in 2.8) due to the fact that they involve not only the distance function but also angular terms, almost linear functions will be constructed by only using the distance function, which admits a sufficient control through Jacobi field estimates. The basic idea is to use the Euclidean identity $2\langle x,p-q\rangle = |x-q|^2 - |x-p|^2$ $(p = -q)$ as a definition. These functions satisfy almost, i.e. up to a small error term, the usual characterizations of linear functions in Euclidean space, e.g. that the first derivatives are constant, the second ones vanish, or the Taylor expansion terminates after the second term. These error terms are inevitable due to the presence of curvature, conceptually considered as a measure of deviation from Euclidean space. Such error terms, however, generally are of lower order than the other terms which appear already in the Euclidean versions of the formulae and hence can be easily absorbed. In particular, we discuss approximate fundamental solutions of the Laplace and heat equation on manifolds and derive representation formulae. Almost linear functions permit to gain one order of differentiation in such formulae by enabling us to also approximate the derivatives of fundamental solutions.

Another application of almost linear functions is the construction of harmonic coordinates on manifolds with the help of a perturbation argument. They possess even better regularity properties, since, for instance, we can derive C^{α}-bounds for the corresponding Christoffel symbols in terms of curvature bounds only, not involving any curvature derivatives. They therefore seem to be optimally adapted to the concept of manifolds of bounded geometry. In the present notes, they will play an important role in the derivation of higher order a-priori estimates for harmonic maps.

Starting with section 2.6, all the results of this chapter are either taken from or inspired by [JK1].

2.2 JACOBI FIELD ESTIMATES

Let $c(s,t) = c_t(s)$ be a family of geodesics parametrized by t. s usually will be taken as the arc length parameter on each geodesic. $J_t(s) = \frac{\partial}{\partial t} c(s,t)$ is then a Jacobi field. It satisfies the equation

$$\frac{D}{\partial s}\frac{D}{\partial s} J_t(s) + R\left(\frac{\partial c}{\partial s}, J_t\right)\frac{\partial c}{\partial s} = 0 \tag{2.2.1}$$

which easily follows from $\frac{D}{\partial s}\frac{\partial}{\partial s} c = 0$ and the definition of the curvature tensor.

From (2.2.1) we see that the tangential component of a Jacobi field J, $J^{\tan} = \langle J, \frac{\partial c}{\partial s}\rangle J$ satisfies

$$\frac{D}{\partial s}\frac{D}{\partial s} J^{\tan} = 0$$

and is hence independent of the metric. In particular, $J^{\tan}$ is linear. In order to incorporate the tangential component in the estimates, we have to assume that we have curvature bounds

(2.2.2) $$\lambda \leq K \leq \mu\ , \quad \lambda \leq 0\ , \quad \mu \geq 0$$

i.e. a nonpositive lower and a nonnegative upper bound, or else to assume $J^{tan} = 0$.

We need some definitions:

$'$ always denotes a derivative with respect to s, while $^{\bullet}$ is the differentiation with respect to t.

We put

$$c_\rho(s) = \begin{cases} \cos(\sqrt{\rho}\ s) & \text{if } \rho > 0 \\ 1 & \text{if } \rho = 0 \\ \cosh(\sqrt{-\rho}\ s) & \text{if } \rho < 0 \end{cases}$$

and

$$s_\rho(s) = \begin{cases} \frac{1}{\sqrt{\rho}} \sin(\sqrt{\rho}\ s) & \text{if } \rho > 0 \\ s & \text{if } \rho = 0 \\ \frac{1}{\sqrt{-\rho}} \sinh(\sqrt{-\rho}\ s) & \text{if } \rho < 0 \end{cases}$$

Both functions solve the Jacobi equation for constant sectional curvature ρ, namely

(2.2.3) $$f'' + \rho f = 0$$

with initial values $f(0) = 1$, $f'(0) = 0$, or $f(0) = 0$, $f'(0) = 1$, resp.

c will always be a geodesic arc parametrized by s proportionally to arclength, and usually $|c'| = 1$ for simplicity.

LEMMA 2.2.1 *Assume* $K \leq \mu$ *and* $|c'| = 1$, *and either* $\mu \geq 0$ *or* $J^{tan} \equiv 0$.

Let $f_\mu := |J(0)|\ c_\mu + |J|'(0)\ s_\mu$ *be the solution of* $f'' + \mu f = 0$ *with the same initial conditions as* $|J|$.

If $f_\mu(s) > 0$ *for* $s \in (0,\sigma)$, *then*

(2.2.4) $$\langle J,J'\rangle f_\mu \geq \langle J,J\rangle f'_\mu \quad \textit{on} \quad (0,\sigma)$$

(2.2.5) $$1 \leq \frac{|J(s_1)|}{f(s_1)} \leq \frac{|J(s_2)|}{f(s_2)} \quad \textit{if} \quad 0 < s_1 \leq s_2 < \sigma$$

(2.2.6) $$|J(0)|\, c_\mu(s) + |J|'(0)\, s_\mu(s) \leq |J(s)| \quad \textit{for} \quad s \in (0,\sigma) .$$

Proof $$|J|'' + \mu|J| = |J|^{-1}(-\langle R(c',J)\; c',J\rangle + \mu\langle J,J\rangle)$$

$$+ |J|^{-3}(|J'|^2|J|^2 - \langle J,J'\rangle^2) \geq 0 .$$

Hence

$$(|J|'f_\mu - |J|f'_\mu)' = |J|''f_\mu - |J|f''_\mu \geq 0 .$$

Since $|J|(0) = f_\mu(0)$, $|J|'(0) = f'_\mu(0)$, (2.2.4) follows. Then

$$\left(\frac{|J|}{f_\mu}\right)' = \frac{1}{f_\mu^2}\,(|J|'f_\mu - |J|\cdot f'_\mu) \geq 0 ,$$

since it vanishes at 0 and has nonnegative derivative.

(2.2.5) again follows from the initial conditions, and (2.2.5) implies (2.2.6).

LEMMA 2.2.2 *Assume* $K \leq \mu$, *and either* $\mu \geq 0$ *or* $J^{\tan} = 0$, *and* $|K| \leq \Lambda^2$, $J(0) = 0$, $|c'| = 1$, $c_\mu \geq 0$ *on* $(0,\sigma)$.

Then

(2.2.7) $$|J(s) - sJ'(s)| \leq |J(t)| \cdot \frac{1}{2}\Lambda^2 s^2 .$$

Proof Let P be a parallel vector field along c , and $s \in (0,\sigma)$.

$$|\langle J(s) - sJ'(s),\, P(s)\rangle'| = |s\langle R(c',J)c',P\rangle(s)|$$

$$\leq \Lambda^2\, s|J(s)|$$

$$\leq \Lambda^2\, s|J(\sigma)|\,\frac{s_\mu(s)}{s_\mu(\sigma)} \qquad \text{by (2.2.5)}$$

$$\leq \Lambda^2 \, s|J(\sigma)| \ , \quad \text{since} \quad c_\mu \geq 0 \quad \text{on } [0,\sigma]$$

and (2.2.7) follows by integration of this inequality.

q.e.d.

Instead of prescribing $J(0)$ and $J'(0)$, one can also prescribe $J(0)$ and $J(\rho)$ for $\rho < \pi/\sqrt{\mu}$. For example, since we showed in the proof of Lemma 2.2.1 that $|J|'' + \mu|J| \geq 0$, we conclude, assuming $|c'| = 1$ again,

$$(2.2.8) \quad \sin(\sqrt{\mu}\rho)\ |J(s)| \leq \sin(\sqrt{\mu}s)\ |J(\rho)| + \sin(\sqrt{\mu}(\rho-s))\ |J(0)| \ .$$

We shall also need the following estimate of Jäger-Kaul [JäK2].

LEMMA 2.2.3 *Suppose* $K \leq \mu$, $|c'| = 1$, *and* $0 < \rho < \pi/\sqrt{\mu}$ *in case* $\mu > 0$. *If* X *is a Jacobi field along* c *with*

$$\langle X,c'\rangle = 0 \ ,$$

then

$$(2.2.9) \quad \langle X,X'\rangle \Big|_0^\rho \geq \frac{s_\mu'(\rho)}{s_\mu(\rho)} \left(|X(0)|^2 + |X(\rho)|^2\right) - \frac{2}{s_\mu(\rho)} |X(0)|\cdot|X(\rho)| \ .$$

Proof Let

$$s(t) := \frac{1}{s_\mu(\rho)} \cdot \left(|X(0)|s_\mu(\rho-t) + |X(\rho)|s_\mu(t)\right) \ .$$

Then s solves

$$(2.2.10) \qquad s'' + \mu s = 0 \ , \qquad s(0) = |X(0)| \ , \qquad s(\rho) = |X(\rho)| \ ,$$

and

$$s \geq 0 \qquad \text{on} \quad [0,\rho]$$

and

$$(2.2.11) \qquad s'(0) = \frac{1}{s_\mu(\rho)} \left(|X(\rho)| - s_\mu'(\rho)|X(0)|\right)$$

$$s'(\rho) = \frac{1}{s_\mu(\rho)} \left(s_\mu'(\rho)|X(\rho)| - |X(0)|\right) \ .$$

Then the function

$$g := s|X|' - s'|X|$$

is differentiable where $|X| \neq 0$. (Note that the zeros of X are isolated, since X solves the Jacobi equation

$$X'' + R(c',X)c' = 0 , \qquad (2.2.12)$$

which is a linear second order equation.) Moreover

$$g' = s|X|'' - s''|X| = s\left(\frac{\langle X,X'\rangle}{|X|}\right)' + \mu s|X|$$

$$= s\,\frac{1}{|X|^3}\,(|X|^2|X'|^2 - \langle X,X'\rangle^2) - s\cdot\frac{1}{|X|}\langle X,R(c',X)c'\rangle + \mu s|X|$$

$$\geq 0 ,$$

since by assumption $\langle X,R(c',X)c'\rangle \leq \mu|X|^2$. Thus g is not decreasing on those intervals where it is differentiable. As was noted above, points τ where g' does not exist, i.e. $|X(\tau)| = 0$ are discrete, and moreover

$$g(\tau+0) - g(\tau-0) = 2s(\tau)|X'(\tau)| \geq 0 .$$

Thus, g is not decreasing on $[0,\rho]$, and defining

$$|X|'(\rho) = \lim_{\varepsilon\downarrow 0} |X|'(\rho-\varepsilon) , \qquad |X|'(0) = \lim_{\varepsilon\downarrow 0} |X|'(\varepsilon) ,$$

we conclude

$$0 \leq g(\rho) - g(0) = s(\rho)|X|'(\rho) - s'(\rho)|X(\rho)| - s(0)|X|'(0) + s'(0)|X(0)|$$

$$= \langle X,X'\rangle(\rho) - \langle X,X'\rangle(0) - \frac{s_\mu'(\rho)}{s_\mu(\rho)}(|X(0)|^2 + |X(\rho)|^2)$$

$$+ \frac{2}{s_\mu(\rho)}|X(0)|\cdot|X(\rho)| ,$$

by (2.2.11).

q.e.d.

We now turn to describe the effect of a lower curvature bound on Jacobi field estimates.

LEMMA 2.2.4 *Assume* $\lambda \le K \le \mu$ *, and either* $\lambda \le 0$ *or* $J^{\tan} \equiv 0$ *,* $|K| \le \Lambda^2$ *,* $|c'| \equiv 1$ *, and in addition that* $J(0)$ *and* $J'(0)$ *are linearly dependent. For a parameter* τ *, we define again* $f_\tau = |J(0)|c_\tau + |J|'(0)s_\tau$ *. If* $f_{\frac{1}{2}(\lambda+\mu)} > 0$ *on* $(0,\rho)$ *, then*

$$|J(s)| \le |J(0)|\ c_\lambda(s) + |J|'(0)\ s_\lambda(s)\ , \tag{2.2.13}$$

and in any case, if P_s *denotes parallel translation along* c

$$|J(s) - P_s(J(0) + sJ'(0))| \le |J(0)|(\cosh(\Lambda s) - 1) + |J|'(0)\left(\frac{1}{\Lambda}\sinh(\Lambda s) - s\right)\ . \tag{2.2.14}$$

Proof Let τ be a parameter, and $\eta = \max(\mu-\tau,\ \tau-\lambda)$. Let A be the vectorfield along c that satisfies

$$\frac{D}{ds}\frac{D}{ds}A + \tau A = 0\ ,\qquad A(0) = J(0)\ ,\qquad A'(0) = J'(0)\ .$$

Let a be the solution of

$$a'' + (\tau-\eta)a = \eta|A|\ ,\qquad a(0) = a'(0) = 0$$

and b the solution of

$$b'' + \tau b = \eta|J|\ ,\qquad b(0) = b'(0) = 0\ .$$

If P is a unit parallel field

$$|\langle J - A,P\rangle'' + \tau\langle J - A,P\rangle| = |\langle J'' - \tau J,P\rangle| \le \eta|J|\ .$$

Hence

$$d := \{\langle J - A,P\rangle - b\}''\ s_\tau - \{\langle J - A,P\rangle - b\}\ s_\tau'' \le 0$$

and

$$\left(\frac{1}{s_\tau}\{\langle J - A,P\rangle - b\}\right)'(s) = \frac{1}{s_\tau^{\,2}(s)}\int_0^s d \le 0 .$$

Thus $\frac{1}{s_\tau}\{\langle J - A,P\rangle - b\} \le 0$, since it vanishes at $s = 0$. If $s_\tau > 0$ on $(0,\rho)$, then this implies

$$|J - A| \le b \quad \text{on} \quad (0,\rho) \tag{2.2.15}$$

and

$$b'' + \tau b \le \eta b + \eta|A| .$$

In a similar way

$$\frac{1}{s_\tau}(b - a) \le 0 ,$$

(2.2.16) i.e. $$b \le a$$

(2.2.15) and (2.2.16) give

$$|J - A|(s) \le a(s) \quad \text{for} \quad s \in (0,\rho) . \tag{2.2.17}$$

Now

$$(\langle A',A'\rangle\langle A,A\rangle - \langle A,A'\rangle\langle A,A'\rangle)' = 0 \tag{2.2.18}$$

and thus

$$\langle A',A'\rangle\langle A,A\rangle - \langle A,A'\rangle\langle A,A'\rangle \equiv 0 ,$$

since it vanishes at $s = 0$, as $A(0)$ and $A'(0)$ are linearly dependent. This in turn implies

$$|A|'' + \tau\cdot|A| = 0 ,$$

i.e. $$|A| = f_\tau$$

and hence

$$a = f_{\tau-\eta} - f_\tau$$

and from (2.2.17)

$$|J| \leq f_{\tau-\eta} .$$

Choosing $\tau = \frac{1}{2}(\mu+\lambda)$, i.e. $\tau-\eta = \lambda$, then proves (2.2.13).

(2.2.18) also implies that $(A/|A|)' = 0$, i.e. $A/|A|$ is parallel, and choosing $\tau = 0$ then proves (2.2.14).

2.3 APPLICATIONS TO GEODESIC CONSTRUCTIONS

We let $c(s,t) = \exp_p(s\cdot(v+tw))$ be a family of geodesics radially emanating from the point p .

Then

(2.3.1) $$J(s) = \frac{\partial}{\partial t} c(s,t)\Big|_{t=0} = (d\exp_p)_{sv}\cdot sw$$

is a Jacobi field with

$$J(0) = 0 , \qquad J'(0) = w .$$

If we put $v = w$, then J is tangential to $c(s,0)$ and hence linear, i.e. $J(s) = sv$, which implies

$$|(d\exp_p)_v\cdot v| = |v|$$

or in other words, that $\exp_p : T_pM \to M$ is an isometry in the radial direction.

If w and v are orthogonal, then (2.2.6) and (2.2.13) imply

LEMMA 2.3.1 *If* $w \perp v$, $\lambda \leq K \leq \mu$, *then, if* $s \leq \frac{\pi}{\sqrt{\mu}}$ *in case* $\mu > 0$,

(2.3.2) $$|w| \cdot \frac{s_\mu(s)}{s} \leq |(d\exp_p)_{sv}\cdot w| \leq |w| \frac{s_\lambda(s)}{s} .$$

LEMMA 2.3.2 *Let* $B(m,\rho) := \{x \in M : d(m,x) \le \rho\}$ *be a ball in some manifold* M *which is disjoint to the cut locus of its centre* m *. We assume for the sectional curvatures* K *in* $B(m,\rho)$

$$-\omega^2 \le K \le \kappa^2 \qquad \textit{and} \qquad \rho < \frac{\pi}{2\kappa}$$

We define $r(x) := d(x,m)$ *and* $f(x) := \frac{1}{2} d(x,m)^2$. *Then* $f \in C^2(B(m,\rho),\mathbb{R})$ *and*

$$|\text{grad } f(x)| = r(x) \tag{2.3.3}$$

$$\kappa r(x) \text{ ctg}(\kappa r(x)) \cdot |v|^2 \le D^2 f(v,v) \tag{2.3.4}$$

$$\le \omega r(x) \coth(\omega r(x)) \cdot |v|^2$$

for $x \in B(m,\rho)$ *and* $v \in T_x M$.

Proof grad $f(x) = -\exp_x^{-1} m$ which implies (2.3.3).

Let $q(t)$ be a curve in M with $q(0) = x$ and $\dot{q}(0) = v$ and

$$c(s,t) = \exp_{q(t)}(s \exp_{q(t)}^{-1} m) .$$

Then grad $f(q(t)) = -\frac{\partial}{\partial s} c(s,t)\big|_{s=0}$, and hence

$$D_v \text{ grad } f(x) = -\frac{D}{\partial t}\frac{\partial}{\partial s} c(s,t)\big|_{s=0,t=0}$$

$$= -\frac{D}{\partial s}\frac{\partial}{\partial t} c(s,t) .$$

For fixed t , $J_t(s) = \frac{\partial}{\partial t} c(s,t)$ is the Jacobi field along the geodesic from m to $q(t)$ with $J_t(0) = \dot{q}(t)$ and $J_t(1) = 0 \in T_m M$. Hence D_v grad $f(x) = D_{J_o(0)}$ grad $f(x) = -J_o'(0)$. Since

$$D^2 f(v,v) = \langle D_v \text{ grad } f, v\rangle = -\langle J_o'(0), J_o(0)\rangle ,$$

(2.3.4) follows from (2.2.6) and (2.2.13) (since $J_t(1) = 0$, $J_t(1)$ and

$J_t'(1)$ are linearly dependent).

q.e.d.

2.4 CONVEXITY OF GEODESIC BALLS

The following convexity result was proved in [J2] and [BK], Prop. 6.4.6.

PROP. 2.4.1 *Suppose the ball* $B(m,\rho)$ *is disjoint to the cut locus of* m *, and* $\rho < \frac{\pi}{2\kappa}$ *, where* κ^2 *is an upper bound for the sectional curvature of* $B(m,\rho)$ *. Then any two points in* $B(m,\rho)$ *can be joined in* $B(m,\rho)$ *by a unique geodesic arc. This arc is the shortest connection between its end points and thus in particular does not contain a pair of conjugate points.*

Proof Since the cut locus of a point m is closed, we can find some ρ' , $\rho < \rho' < \frac{\pi}{2\kappa}$, for which $B(m,\rho')$ is still disjoint to the cut locus of m . For any two points p and $q \in B(m,\rho)$, we can find a shortest connection $\gamma(t)$ in $B(m,\rho')$ by the standard Arzela-Ascoli argument. Let $\gamma(0) = p$, $\gamma(1) = q$, and let $c(\cdot,t)$ be the family of geodesics with $c(0,t) = m$, $c(1,t) = \gamma(t)$. The Jacobi fields $J_t(s) = \frac{\partial}{\partial t} c(s,t)$ are monotonically increasing in $s \in [0,1]$ by (2.2.5). Hence, in case γ leaves $B(m,\rho)$ somewhere between p and q , we can project it onto $B(m,\rho)$, i.e. take

$$\tilde{\gamma}(t) = \exp_m\left(\exp_m^{-1} \gamma(t) \cdot \min\left(1 , \frac{\rho}{d(\gamma(t),m)}\right)\right)$$

and obtain a shorter comparison curve in contradiction to the choice of γ . Hence γ is contained in $B(m,\rho)$ and hence in particular in the interior of $B(m,\rho')$ and is therefore geodesic. Furthermore, clearly length$(\gamma) \leq 2\rho$.

The exponential map has maximal rank along any geodesic in $B(m,\rho)$ of length $\leq 2\rho$ by Lemma 2.3.1. In particular, they do not contain pairs of conjugate points and are locally unique. Hence, the set of pairs $(p,q) \in B(m,\rho) \times B(m,\rho)$ with two geodesic connections is compact, since two geodesics cannot collapse in the limit into a single one with conjugate points. Thus, if this set were non empty, we could find such a pair (p,q) of minimal distance with two minimal geodesic connections γ_1 and γ_2. γ_1 and γ_2 then have to form a closed geodesic. Namely, otherwise, if they would form an angle $< \pi$ at p for example, then moving a little bit along the geodesic which bisects this angle, we could find a point $\tilde{p}$ which is closer to q and still has two different connections to q, in contradiction to the choice of p and q. (For more details on this argument, cf. [GKM]). On the other hand, by Lemma 2.3.2, $d^2(\cdot,m)$ is strictly convex on $B(m,\rho)$, and therefore the existence of a closed geodesic in $B(m,\rho)$ contradicts Cor. 1.7.1.

If now $p,q \in B(m,\rho)$ would have two geodesic connections, one of which, called γ, is longer than 2ρ, then γ ceases somewhere between p and q to be the shortest connection of its endpoints, and hence we could again find two minimal geodesics, in contradiction to what we already proved.

q.e.d.

This result can be somewhat improved in two dimensions. First of all, we have

LEMMA 2.4.1 *Let* S *be a compact surface, possibly with boundary. If the boundary* γ *is not empty, it is assumed to be convex, i.e. that through every point* $\tilde{q}$ *of* γ *there goes a geodesic arc which is disjoint to* S *in a neighbourhood of* $\tilde{q}$. *Let* $p,q \in S$. *Assume that there are two distinct homotopic geodesic arcs joining* p *and* q. *Then each of the points* p *and* q *has a conjugate point in* S, *and this point is conjugate to* p *or* q, *resp., with respect to a geodesic arc which is the shortest connection in its homotopy class.*

Proof We denote the two geodesic arcs by γ_1 and γ_2. We can assume w.l.o.g. that γ_1 and γ_2 are shortest connections in their homotopy class between p and q , since otherwise, starting e.g. from p and moving on γ_1 , we would find a point q_1 which would either be conjugate to p or would have a connection in S to p in the same homotopy class and of equal length as the segment of γ_1 between p and q_1 . (At this point, for the existence of such a connection, we have to use the convexity of γ). Since γ_1 and γ_2 are homotopic and distinct, because we could assume that they are shortest connections, they bound a set B of the topological type of the disc.

We now look at a geodesic line emanating from p into B . As γ_1 and γ_2 are shortest, this line has to cease somewhere in B to be shortest connection to p . Repeating the argument, if we have not yet found the desired conjugate point, we get a nested sequence of geodesic two-angles, i.e. configurations consisting of two homotopic geodesic arcs of equal length which furthermore are shortest possible in their homotopy class. In the limit, this construction has to yield a geodesic arc covered twice. The endpoint q_2 therefore is homotopic to p , and furthermore, the geodesic arc is the shortest connection in its homotopy class from p to q_2 .

q.e.d.

LEMMA 2.4.2 *Suppose* $B(p,R) := \{q \in \Sigma : d(p,q) \le R\}$ *, where* Σ *is a surface, is topologically a disc for some* $r < \frac{\pi}{\kappa}$ $(K \le \kappa^2)$. *Then* $\exp_p\{v : |v| = r\} = \partial B(p,r)$ *for all* $r \le R$ *, where* $\exp_p : T_p\Sigma \to \Sigma$ *is the exponential map. Furthermore,* $\partial B(p,r)$ *is convex, if* $r \le \frac{\pi}{2\kappa}$.

Proof Clearly, $\partial B(p,r) \subseteq \exp_p\{v : |v| = r\} \subseteq B(p,r)$. We assume now that

$$\exp_p\{v : |v| = r\} \cap \mathring{B}(p,r) \neq \phi \quad (2.4.1)$$

$\exp_p$ is a local diffeomorphism on $\{v : |v| < \frac{\pi}{\kappa}\}$ by Lemma 2.3.1, and therefore

$\exp_p\{v : |v| = r\}$ is an immersed smooth curve for $r < \frac{\pi}{\kappa}$. Since $\exp_p\{v : |v| = r\}$ is compact, we can find some $q \in \exp_p\{v : |v| = r\}$ with minimal distance to p. Consequently, the shortest geodesic γ from p to q is orthogonal to $\exp_p\{v : |v| = r\}$ at q and has length $< r$. On the other hand, $q = \exp_p w$, $|w| = r$, and the geodesic $\gamma' = \exp_p tw$, $t \in [0,1]$, is also orthogonal to $\exp_p\{v : |v| = r\}$ at q and different from γ, since its length is precisely r. Thus, γ and γ' have an angle of π at q and match together to a geodesic loop with corner at p. It is not difficult to see that every point inside this geodesic loop can be joined to p by a shortest geodesic, in spite of the fact that this loop might not be convex at p. Thus, we can carry over the argument of Lemma 2.4.1 to assert the existence of a point p' inside this loop which is conjugate to p w.r.t. a shortest geodesic γ''. Since $p' \in B(p,r)$ and $r < \frac{\pi}{\kappa}$, this is in contradiction to Lemma 2.3.1. This proves the first claim. Furthermore, since $\exp_p$ has maximal rank on $\{v \in T_p\Sigma : |v| < \frac{\pi}{\kappa}\}$, as noted above, we infer that every $v \in T_p\Sigma$ with $|v| = r$ has a neighbourhood V which is mapped under $\exp_p$ injectively onto its image (cf. [Kl], p.108f.). From this, we easily see that we may apply the estimate of Lemma 2.3.2. Therefore, if $r \leq \frac{\pi}{2\kappa}$, then h is a convex function on $B(p,r)$, and consequently, $\partial B(p,r) = \exp_p\{v : |v| = r\}$ is convex as a level set of a convex function.

PROP. 2.4.2 *Suppose now, that* $B(p,r)$ *is a geodesic disc on a surface, and* $r < \frac{\pi}{2\kappa}$ $(K \leq \kappa^2)$. *Then each pair of points* $q_1, q_2 \in B(p,r)$ *can be joined by a unique geodesic arc in* $B(p,r)$, *and this arc is free of conjugate points.*

Proof By virtue of Lemma 2.4.2, we could apply Lemma 2.4.1, if there would exist two geodesic arcs joining q_1 and q_2. Consequently, we would find a point q_3 conjugate to q_1 w.r.t. a shortest geodesic arc, i.e. an arc of length $\leq 2r < \frac{\pi}{\kappa}$. This would contradict Lemma 2.3.1.

q.e.d.

2.5 THE DISTANCE AS A FUNCTION OF TWO VARIABLES

We suppose again that the ball $B(p,M) \subset N$ is disjoint to the cut locus of p and that $M < \frac{\pi}{2\kappa}$, where $-\omega^2 \leq K \leq \kappa^2$ are curvature bounds. We define

$$q_\kappa(t) = \begin{cases} \frac{1}{\kappa^2}(1 - \cos \kappa t) & \text{if } \kappa > 0 \\ \frac{t^2}{2} & \text{if } \kappa = 0 \end{cases}$$

and note that

$$q_\kappa(t) = \int_0^t s_{\kappa^2} \, .$$

By assumption and 2.4, any two points $y_1, y_2 \in B(p,M)$ can be joined by a unique minimal geodesic in $B(p,M)$, and we can measure the distance between y_1 and y_2 by the length of the geodesic arc between them. We denote this (possibly modified) distance function again by $d(y_1,y_2)$. Then

$$Q_\kappa(y_1,y_2) := q_\kappa(d(y_1,y_2))$$

defines a C^2 function on $B(p,M) \times B(p,M)$, since $q_\kappa'(0) = 0$. Moreover, we note that

$$T_y(N\times N) = T_{y_1}N \oplus T_{y_2}N \quad \text{(isometrically)}$$

for $y = (y_1,y_2) \in N\times N$.

In the following lemma, we shall estimate the Hessian of Q_κ on $B(p,M) \times B(p,M)$, using the Jacobi field estimate of Lemma 2.2.3. This result is again due to Jäger-Kaul [JäK2].

LEMMA 2.5.1 *If* $y_1 \neq y_2$ *, then for all*

$$v \in T_y(N\times N) \ , \qquad y = (y_1,y_2) \ , \qquad y_1,y_2 \in B(p,M)$$

$$D^2Q_\kappa(v,v) \geq \frac{\langle \operatorname{grad} Q_\kappa(y), v\rangle^2}{2Q_\kappa(y)} - \kappa^2 Q_\kappa(y)\,|v|^2 \ . \tag{2.5.1}$$

If v *has the special form* $0 \oplus u$ *or* $u \oplus 0$ *, then*

$$D^2Q_\kappa(v,v) \geq (1 - \kappa^2 Q_\kappa(y))\,|u|^2 \ , \tag{2.5.2}$$

and this also holds for $y_1 = y_2$.

Proof First some definitions

$\rho := d(y_1,y_2)$

$v =: v_1 \oplus v_2 \in T_{y_1}N \oplus T_{y_2}N$,

$c : [0,\rho] \to B(p,M)$ is the unique geodesic arc from y_1 to y_2 with $|c'| = 1$,

$e_1(y) := -c'(0)$

$e_2(y) := c'(\rho)$

$v_i^{tan} := \langle v_i, e_i(y)\rangle\, e_i(y)$

$v_i^{nor} := v_i - v_i^{tan} \qquad (i = 1,2)$.

Then, since $\rho > 0$,

$\operatorname{grad} d(y) = e_1(y) \oplus e_2(y)$,

$\operatorname{grad} Q_\kappa(y) = s_{\kappa^2}(\rho)(e_1(y) \oplus e_2(y))$, and

$D^2Q_\kappa(y)(v,v) = \langle D_v \operatorname{grad} Q_\kappa, v\rangle$

$$= s'_{\kappa^2}(\rho)\langle e_1(y) \oplus e_2(y), v_1 \oplus v_2\rangle^2 + s_{\kappa^2}(\rho) D^2 d(v,v) \ . \tag{2.5.3}$$

If $c_t(s)$ is the geodesic arc with

$$c_t(0) = \exp_{y_1}(tv_1^{nor}) \ , \qquad c_t(\rho) = \exp_{y_2}(tv_2^{nor})$$

(note that c_t is unique, if $t \geq 0$ is small enough), then

$$J(s) := \frac{\partial}{\partial t} c_t(s)_{|s=0} \tag{2.5.4}$$

is a Jacobi field along c with

$$J(0) = v_1^{nor} \ , \qquad J(\rho) = v_2^{nor} \ .$$

By Synge's formula (cf. [GKM], §4.1),

$$\begin{aligned} D^2 d(v,v) &= \frac{\partial^2}{\partial t^2} \operatorname{length}(c_t)_{|t=0} \\ &= \int_0^\rho (|J'|^2 - \langle J, R(c',J)c'\rangle)\,ds \end{aligned} \tag{2.5.5}$$

(note that there is no boundary term, since

$$\langle J, c'\rangle = 0 \ )$$

We can apply Lemma 2.2.3 to obtain

$$\begin{aligned} D^2 d(v,v) &= \int_0^\rho (|J'|^2 + \langle J, J''\rangle)\,ds \\ &= \langle J, J'\rangle|_0^\rho \\ &\geq \frac{s'_{\kappa^2}(\rho)}{s_{\kappa^2}(\rho)} (|v_1^{nor}|^2 + |v_2^{nor}|^2) - \frac{2}{s_{\kappa^2}(\rho)} |v_1^{nor}| \cdot |v_2^{nor}| \ , \end{aligned}$$

and thus with (2.5.3)

$$\begin{aligned} D^2 Q_\kappa(v,v) \geq S'_{\kappa^2}(\rho) (&\langle e_1 \oplus e_2, v_1 \oplus v_2\rangle^2 + |v_1^{nor}|^2 + |v_2^{nor}|^2) \\ &- 2|v_1^{nor}||v_2^{nor}| \ . \end{aligned} \tag{2.5.6}$$

If $v = 0 \oplus u$, (2.5.6) implies

$$D^2 Q_\kappa(v,v) \geq s'_{\kappa^2}(\rho) \langle e_2(y), u\rangle^2 + s'_{\kappa^2}(\rho) \ |u^{nor}|^2$$

$$= s'_{\kappa^2}(\rho)\,|u|^2$$

$$= (1 - \kappa^2 Q\ (y))\,|u|^2 \ ,$$

while in the general case, we only have

$$\langle e_1 \oplus e_2,\ v_1 \oplus v_2\rangle^2 \le 2(|v_1^{tan}|^2 + |v_2^{tan}|^2) \ ,$$

and

$$|v_i|^2 = |v_i^{tan}|^2 + |v_i^{nor}|^2 \ ,$$

and therefore from (2.5.6),

$$\begin{aligned} D^2 Q_\kappa(v,v) &\ge s'_{\kappa^2}(\rho)\ \langle e_1 \oplus e_2,\ v_1 \oplus v_2\rangle^2 - (1 - s'_{\kappa^2}(\rho))(|v_1^{nor}|^2 + |v_2^{nor}|^2) \\ &\ge \tfrac{1}{2}(1 + s'_{\kappa^2}(\rho))\ \langle e_1 \oplus e_2,\ v_1 \oplus v_2\rangle^2 - (1 - s'_{\kappa^2}(\rho))(|v_1|^2 + |v_2|^2) \\ &= \frac{1}{2Q_\kappa(y)} \langle \mathrm{grad}\ Q_\kappa(y), v\rangle^2 - \kappa^2 Q\ (y)(|v_1|^2 + |v_2|^2) \ . \end{aligned}$$

q.e.d.

2.6 ALMOST LINEAR FUNCTIONS

We are now ready to introduce almost linear functions, one of the main tools of [JK1].

Let $B(m,\rho)$ be again a ball in some n-dimensional Riemannian manifold M which is disjoint to the cut locus of m, and assume curvature bounds

$$-\omega^2 \le K \le \kappa^2 \ , \qquad |K| \le \Lambda^2$$

and

$$\rho < \frac{\pi}{2\kappa} \ .$$

We put $r(x) = d(m,x)$, $f(x) = \frac{1}{2}d^2(m,x)$.

DEFINITION 2.6.1 Let $u \in T_m M$ be a unit vector, i.e. $|u| = 1$, and put $p(x) = \exp_m(r(x)u)$, $q(x) = \exp_m(-r(x)u)$. Then

$$\ell(x) := \frac{1}{4r(x)} \left(d(x,q(x))^2 - d(x,p(x))^2\right)$$

is called an *almost linear function.*

We observe that in the Euclidean case, this notion yields precisely the linear functions, because of Pythagoras' theorem. We furthermore note that

$$(2.6.1) \qquad -r(x) \le \ell(x) \le r(x) \ .$$

The estimates of [JK1] for almost linear functions are contained in

THEOREM 2.6.1 *Suppose* $B(m,\rho)$ *is disjoint to the cut locus of* m , $-\omega^2 \le K \le \kappa^2$, $|K| \le \Lambda^2$ *on* $B(m,\rho)$, *and* $\rho < \frac{\pi}{2\kappa}$. *Let* $u \in T_m M$, $|u| = 1$, $\ell(x)$ *the associated almost linear function, and* $u(x)$ *the radially parallel vector field on* $B(m,\rho)$ *with* $u(m) = u$. *Then*

$$(2.6.2) \qquad |\operatorname{grad} \ell(x) - u(x)| \le 2\kappa\Lambda \frac{\sinh(2\Lambda r)}{\sin(2\kappa r)} \cdot r^2(x)$$

$$(2.6.3) \qquad |D^2 \ell(x)| \le |9\kappa\Lambda \frac{\sinh(2\Lambda r)}{\sin(2\kappa r)} \omega r \operatorname{ctgh}(\omega r))\ r(x)$$

$$(2.6.4) \quad |\ell(x) - \langle \operatorname{grad} \ell(x), -\exp_x^{-1} m\rangle| \le \left(\frac{9}{2}\kappa\Lambda \frac{\sinh(2\Lambda r)}{\sin(2\kappa r)} \omega r \operatorname{ctgh}(\omega r)\right) r^3(x) \ .$$

Proof Let $\gamma(t)$ be a geodesic with $\gamma(0) = x$. We then look at the following families of geodesics, joining $\gamma(t)$ with $p(\gamma(t))$ or $q(\gamma(t))$, resp.,

$$c_1(s,t) = \exp_{\gamma(t)}(s \cdot \exp^{-1}_{\gamma(t)} p(\gamma(t)))$$

$$c_2(s,t) = \exp_{\gamma(t)}(s \cdot \exp^{-1}_{\gamma(t)} q(\gamma(t))) \ .$$

$J_i(\cdot,t) = \frac{\partial}{\partial t} c_i(\cdot,t)$ are Jacobi fields with

$$J_i(0,t) = \dot\gamma(t)$$

$$J_1(1,t) = \dot r u(t)$$

$$J_2(1,t) = -\dot r u(t)$$

where we have abbreviated $r(\gamma(t)) = r(t)$, $u(\gamma(t)) = u(t)$, etc. We also write again $c' = \frac{\partial}{\partial s} c$, $\dot{c} = \frac{\partial}{\partial t} c$. We note that

$$d^2(p(\gamma(t)), \gamma(t)) = c_1'(s,t)^2$$

$$d^2(q(\gamma(t)), \gamma(t)) = c_2'(s,t)^2 .$$

Now

$$\frac{d}{dt} \ell(\gamma(t)) = -\frac{c_2'^2 - c_1'^2}{4r^2} \dot{r} + \frac{1}{2r} \int_0^1 \{\langle c_2', \frac{D}{\partial t} c_2' \rangle - \langle c_1', \frac{D}{\partial t} c_1' \rangle\} ds$$

$$(2.6.5) \qquad = -\frac{c_2'^2 - c_1'^2}{4r^2} \dot{r} + \frac{1}{2r} \int_0^1 \{\langle c_2', J_2\rangle' - \langle c_1', J_1\rangle'\} ds$$

$$= -\frac{c_2'^2 - c_1'^2}{4r^2} \dot{r} - \frac{\dot{r}}{4r^2} \langle c_2' + c_1', 2ru\rangle_{s=1} - \frac{1}{2r} \langle c_2' - c_1', \dot{\gamma}\rangle_{s=0} .$$

In order to control $c_1' - c_2' - 2ru$ which vanishes in the Euclidean case, we need the following result which follows from [BK].

LEMMA 2.6.1 *Put* $\varepsilon(r) := \frac{2}{3} \kappa \Lambda r^3 \ \frac{\sinh(2\Lambda r)}{\sin(2\kappa r)}$

$$(2.6.6) \qquad |c_1' - (\exp_x^{-1} m + ru)| \ (x) \le \varepsilon(r)$$

$$(2.6.7) \qquad |c_2' - (\exp_x^{-1} m - ru)| \ (x) \le \varepsilon(r)$$

$$(2.6.8) \qquad |-c_1' - (\exp_m^{-1} x - ru)| \ (p(x)) \le \varepsilon(r)$$

$$(2.6.9) \qquad |-c_2^{-1} - (\exp_m^{-1} x + ru)| \ (q(x)) \le \varepsilon(r) .$$

Proof of Lemma 2.6.1 Let $v \in T_x M$, $c(t) = \exp tv$, $c(1) = q$, where q is some point in M . Let $w \in T_x M$ and $w(t)$ be the parallel vector field along $c(t)$.

We first want to estimate $d(F(w), G(w))$, where

$$F(w) = \exp_x(v + w)$$

$$G(w) = \exp_q(w(1)) .$$

We consider the family of geodesics

$$c(s,t) = \exp_{c(t)}(s \cdot (w(t) + (1-t)\,\dot{c}(t)))$$

and the corresponding Jacobi fields

$$J_t(x) = \dot{c}(s,t) \ .$$

The initial conditions are

$$(2.6.10) \qquad \begin{aligned} J_t(0) &= \dot{c}(t) \\ \frac{D}{\partial s} J_t(0) &= \frac{D}{\partial t}\frac{\partial}{\partial s} c(0,t) = -\dot{c}(t) \ . \end{aligned}$$

We let $J_t^{norm}(s)$ be the component of $J_t(s)$ which is orthogonal to $c'(s,t)$.

Since the curve $c(1,t)$ joins $F(w)$ and $G(w)$ and has tangent vector $J_t(1) = J_t^{norm}(1)$, because $J_t^{tan}(1) = 0$ (this follows from (2.6.10))

$$(2.6.11) \qquad d(F(w),\ G(w)) \le \int_0^1 |J_t^{norm}(1)|\,dt \ .$$

We now want to apply (2.2.14). Since $|c'|$ is not necessarily equal to 1, we have to rescale $c(\cdot,t)$, i.e. to look at the geodesics $\gamma(s,t) = c(\frac{s}{|c'|},t)$ and the Jacobi Fields $\tilde{J}(s,t) = J(\frac{s}{|c'|},t)$. This amounts to replacing Λ by $\Lambda|c'|$ in (2.2.14).

Since by (2.6.10) $J_t(0) + J_t'(0) = 0$, (2.2.14) yields, putting $\rho = \max(|w|,\ |v+w|)$, and using $\cosh x - \frac{\sinh x}{x} \le \frac{1}{3} x \sinh x$,

$$(2.6.12) \qquad |J_t^{norm}(1)| \le |J_t^{norm}(0)| \cdot |c'| \cdot \frac{1}{3}\Lambda \sinh(\Lambda\rho) \ .$$

Moreover,

$$\begin{aligned} |J_t(0)^{norm}|^2 \cdot \left|\frac{\partial c}{\partial s}\right|^2 &= \left|\frac{\partial c}{\partial t}\right|^2 \cdot \left|\frac{\partial c}{\partial s}\right|^2 - \left\langle \frac{\partial c}{\partial t}, \frac{\partial c}{\partial s}\right\rangle^2 \\ &= |v|^2\,|w + (1-t)\,v|^2 - \langle v, w + (1-t)\,v\rangle^2 \\ &= |v|^2\,|w|^2 - \langle v,w\rangle^2 \ . \end{aligned}$$

Therefore, (2.6.11) and (2.6.12) imply

$$(2.6.13) \quad d(F(w),\ G(w)) \le \frac{1}{3}\,|v|\cdot|w|\cdot\Lambda\,\sinh(\Lambda(|v|+|w|))\cdot\sin\measuredangle(v,w)\ .$$

In (2.6.13), we then put $v = \exp_x^{-1} m$, $w = \pm ru$.

Then

$$F(w) = \exp_x(\exp_x^{-1} m \pm ru)$$

$$G(w) = \exp_m(\pm ru) = p(x) \quad \text{or} \quad q(x) \quad \text{resp.}$$

$$= \exp_x c_1' \quad \text{or} \quad \exp_x c_2' \quad \text{resp.}$$

Therefore, (2.6.6) and (2.6.7) follow from (2.6.13) and (2.3.2). (2.6.8) and (2.6.9) follow in a similar manner.

q.e.d.

We now continue the proof of Thm. 2.6.1:

(2.6.6) and (2.6.7) yield

$$(2.6.14) \qquad |c_1' - c_2' - 2ru|(x) \le 2\varepsilon(r)\ ,$$

and similarly from (2.6.8) and (2.6.9), if p denotes parallel transport along radial geodesics

$$(2.6.15) \qquad |pc_1' - pc_2' - 2ru|(m) \le 2\varepsilon(r)\ .$$

(2.6.15) and $|c_1' + c_2'| \le 4r$ imply

$$(2.6.16) \qquad |c_2'^2 - c_1'^2 + \langle pc_2' + pc_1',\ 2ru\rangle| \le 8r\varepsilon(r)\ .$$

Since $|\dot r| \le |\dot\gamma|$, (2.6.5), (2.6.14), and (2.6.16) then yield

$$\langle \mathrm{grad}\ \ell - u,\ \dot\gamma\rangle\ | \le \frac{3}{r}\,\varepsilon(r)\,|\dot\gamma|$$

i.e. (2.6.2).

Differentiating (2.6.5), we get

$$(2.6.17)\quad \frac{d^2}{dt^2}\,\ell(\gamma(t)) = \langle c_2' + c_1',\ c_2' - c_1' + 2ru\rangle \left(\frac{-\ddot{r}}{4r^2} + \frac{\dot{r}^2}{2r^3}\right)$$

$$+ \frac{\dot{r}}{2r^2}\langle c_2' - c_1',\ \dot{\gamma}\rangle_{s=0} - \frac{\dot{r}}{4r^2}\langle c_2' + c_1',\ 2\dot{r}u\rangle_{s=1}$$

$$- \frac{1}{2r}\langle J_2' - J_1',\ \dot{\gamma}\rangle_{s=0} - \frac{\dot{r}}{4r^2}\langle J_2' + J_1',\ 2ru\rangle_{s=1}$$

$$- \frac{\dot{r}}{4r^2}\frac{d}{dt}\left(c_2'^2 - c_1'^2\right) .$$

In the course of (2.6.5), we obtained

$$\frac{d}{dt}\left(c_2'^2 - c_1'^2\right) = -\langle c_2' + c_1',\ 2\dot{r}u\rangle_{s=1} - \langle c_2' - c_1',\ 2\dot{\gamma}\rangle_{s=0} .$$

Hence

$$(2.6.18)\qquad \frac{d^2}{dt^2}\,\ell(\gamma(t)) = \langle c_2' + c_1',\ c_2' - c_1' + 2ru\rangle \left(\frac{-\ddot{r}}{4r^2} + \frac{\dot{r}^2}{2r^3}\right)$$

$$+ \frac{1}{2r}\Bigg(\frac{2\dot{r}}{r}\langle c_2' - c_1',\ \dot{\gamma}\rangle_{s=0}$$

$$+ \langle J_1', J_1\rangle(0) - \langle J_2', J_2\rangle(0) - \langle J_1', J_1\rangle(1) + \langle J_2', J_2\rangle(1)\Bigg)$$

Since $\ddot{f} = r\ddot{r} + \dot{r}^2$, with (2.3.4)

$$\left(\frac{-\ddot{r}}{4r^2} + \frac{\dot{r}^2}{2r^3}\right) \le \frac{|\dot{\gamma}|^2}{4r^3}\,(3 + \omega r\ \mathrm{ctgh}(\omega r)) .$$

(2.6.14) then gives

$$(2.6.19)\quad \langle c_2' + c_1',\ c_2' - c_1' + 2ru\rangle \left(\frac{-\ddot{r}}{4r^2} + \frac{\dot{r}^2}{2r^3}\right) \le \frac{2\varepsilon(r)}{r^2}\,(3 + \omega r\ \mathrm{ctgh}(\omega r))\ |\dot{\gamma}|^2$$

Furthermore, since

$$(J(s) - p\cdot J(0) - sJ'(s))' = sR(c',J)\ c' ,$$

$$|J(s) - p\ J(0) - sJ'(s)| \le \Lambda^2\ |c'|^2 \int_0^s \sigma|J(\sigma)| .$$

Using

$$|J(s)| \le |J(1)|\ \frac{\sin(\kappa|c'|s)}{\sin(\kappa|c'|)} + |J(0)|\ \frac{\sin(\kappa|c'|(1-s))}{\sin(\kappa|c'|)}$$

$$\le 2\ \max(|J(0)|,\ |J(1)|)\cdot\frac{\sin(\kappa r)}{\sin(2\kappa r)} ,$$

which follows from (2.2.8), we get

$$(2.6.20) \qquad |J(s) - p\,J(0) - s\,J'(s)| \le \frac{\Lambda^2 r^2 s^2 \sin(\kappa r)}{\sin(2\kappa r)} \max(|J(0)|,\ |J(1)|)$$

and similarly

$$|J(1-s) - p\,J(1) + (1-s)\,J'(1-s)|$$

is estimated by the same quantity.

We are now ready to control the second term of (2.6.18). First

$$(2.6.21) \qquad \left|\frac{2\dot r}{r} \langle c_2' - c_1',\ \dot\gamma\rangle + \langle 4\dot r u,\ \dot\gamma\rangle\right| \le \frac{4|\dot r|}{r}\,|\dot\gamma|\ \varepsilon(r)\ .$$

Next

$$(2.6.22) \qquad \langle p\,J_1(1) - J_1(0),\ J_1(0)\rangle - \langle p\,J_2(1) - J_2(0),\ J_2(0)\rangle$$
$$- \langle J_1(1) - p\,J_1(0),\ J_1(1)\rangle + \langle J_2(1) - p\,J_2(0),\ J_2(1)\rangle$$
$$- 4\,\langle \dot r u,\ \dot\gamma\rangle = 0\ ,$$

since $J_i(0) = \dot\gamma$, $J_1(1) = \dot r u$, $J_2(1) = -\dot r u$.
Since $|\dot r| \le |\dot\gamma|$, (2.6.20), (2.6.21), and (2.6.22) then give

$$(2.6.23) \qquad |\langle J_1'(0),\ J_1(0)\rangle - \langle J_2'(0),\ J_2(0)\rangle - \langle J_1'(1),\ J_1(1)\rangle$$
$$+ \langle J_2'(1),\ J_2(1)\rangle + \frac{2\dot r}{r}\langle c_2' - c_1',\ \dot\gamma\rangle|$$
$$\le \left(\frac{4\varepsilon(r)}{r} + 4\Lambda^2\,\frac{r^2 \sin(\kappa r)}{\sin(2\kappa r)}\right)\ |\dot\gamma|^2$$

(2.6.18), (2.6.19), and (2.6.23) finally yield

$$\left|\frac{d^2}{dt^2}\,\ell(\gamma(t))\right| \le \left(\frac{8\varepsilon(r)}{r^2} + \frac{2\varepsilon(r)}{r^2}\,\omega r\ \mathrm{ctgh}(\omega r) + 2\Lambda^2 r\frac{\sin(\kappa r)}{\sin(2\kappa r)}\right)\ |\dot\gamma|^2$$
$$\le \left(9\kappa\Lambda\,\frac{\sinh(2\Lambda r)}{\sin(2\kappa r)}\ \omega r\ \mathrm{ctgh}(\omega r)\right)\cdot r\cdot|\dot\gamma|^2\ .$$

Thus, (2.6.3) is proved.

For any geodesic c

$$\frac{d}{dt}\left(\ell(c(t)) - t \langle \operatorname{grad} \ell, \dot{c}(t)\rangle\right) = -t\, D^2\ell(\dot{c},\dot{c}) \ .$$

Taking the radial geodesic from m to x, we then see that (2.6.4) follows from (2.6.3).

q.e.d.

For later purposes, we also need to investigate how almost linear functions depend on the base point m. To emphasize this dependence, we now use a subscript m, i.e. write $\ell_m(x)$ for the corresponding almost linear function. Let now $\gamma(t)$ be a geodesic arc, $u(t)$ a parallel unit vector field along $\gamma(t)$ and $\ell_{\gamma(t)}(x)$ the corresponding almost linear functions.

LEMMA 2.6.2 *For* $z \in B(\gamma(t), \rho)$, $\rho < \min(i(\gamma(t)), \pi/2\kappa)$

(2.6.24)
$$\left|\frac{d}{dt}\, \ell_{\gamma(t)}(z)\right| \le (5 + c\, \Lambda^2\rho^2) \ .$$

Proof Let

$$\rho(t) = d(\gamma(t), z)$$

$$p(t) = \exp_{\gamma(t)}(\rho(t)\ u(t))$$

$$q(t) = \exp_{\gamma(t)}(-\rho(t)\ u(t)) \ .$$

Then

(2.6.25)
$$\ell_{\gamma(t)}(z) = \frac{1}{4\rho(t)}\left(d^2(z,q(t)) - d^2(z,p(t))\right) \ .$$

We look at the family of geodesics

$$c(s,t) = \exp_{\gamma(t)}(s\rho(t)\ u(t)) \ .$$

The corresponding Jacobi field $J_t(s) = \frac{\partial}{\partial t} c(s,t)$ then satisfies

$$J_t(0) = \dot{\gamma}(t)$$

$$\frac{D}{\partial s} J_t(0) = \dot{\rho}(t)\ u(t) \ , \qquad \text{since } u(t) \text{ is parallel along } \gamma$$

$$J_t(1) = \dot{p}(t) \ .$$

In particular, $\frac{D}{\partial s} J_t(0)$ is tangential to the geodesic $c(\cdot,t)$. Thus, $J_t^{norm}(0)$ and $J_t^{norm'}(0)$ are linearly dependent, and (2.2.13) implies

$$|\dot{p}| \leq |\dot{\rho}| + \cosh(\Lambda\rho)\ |\dot{\gamma}| \ , \tag{2.6.26}$$

and the same inequality holds for $|\dot{q}|$.

(2.6.24) then follows from (2.6.26), $|\dot{\rho}| \leq |\dot{\gamma}|$, and $d(z,q(t))$, $d(z,p(t)) \leq 2\rho(t)$.

q.e.d.

Actually, one can even show the stronger estimate

$$\left|\frac{d}{dt}\, \ell_{\gamma(t)}(z) - \langle u(t),\, \dot{\gamma}\rangle\right| \leq c\ \Lambda^2\rho^2 \ . \tag{2.6.27}$$

The proof is rather tedious, however, and hence left out, since we do not need (2.6.27) in the sequel.

2.7 APPROXIMATE FUNDAMENTAL SOLUTIONS AND REPRESENTATION FORMULAE

We first apply Lemma 2.3.2 to construct approximate fundamental solutions of the Laplace and the heat equation on manifolds.

LEMMA 2.7.1 *Let* $B(m,\rho)$ *be as in Lemma 2.3.2.* $\Lambda^2 := \max(\kappa^2,\, \omega^2)$ *, and let* Δ *be the Laplace-Beltrami operator on* M *, and* $n = \dim M$ *,* $h(x) := d(x,m)^2$.

$$|\Delta \log r(x)| \leq 2\Lambda^2 \qquad \textit{for}\ \ x \neq m\ \ \textit{if}\ n = 2 \tag{2.7.1}$$

$$|\Delta\, r(x)^{2-n}| \leq \frac{n-2}{2}\,\Lambda^2\ \ r^{2-n}(x) \qquad \textit{for}\ \ x \neq m\ \ \textit{if}\ n \geq 3 \tag{2.7.2}$$

and

$$\left|\left(\Delta - \frac{\partial}{\partial t}\right)\left(t^{-n/2}\exp\left(-\frac{h(x)}{4t}\right)\right)\right| \leq 2\Lambda^2\,\frac{h(x)}{4t}\ t^{-n/2}\exp\left(-\frac{h(x)}{4t}\right) \tag{2.7.3}$$

for $(x,t) \neq (m,0)$.

The proof follows through a straightforward computation from Lemma 2.3.2.

q.e.d.

We now derive approximate versions of Green's representation formula, first in the elliptic case.

LEMMA 2.7.2 *Let* $B(m,\rho)$ *be as above,* $h(x) = d(x,m)^2$. *Let* ω_n *denote the volume of the unit sphere in* $\mathbb{R}^n$. *If* $\phi \in C^2(B(m,\rho), \mathbb{R})$ *, then*

(2.7.4) *if* $n = 2$
$$\left|\omega_2\, \phi(m) + \int_{B(m,\rho)} \Delta\phi \cdot \log \frac{r(x)}{\rho} - \frac{1}{\rho} \int_{\partial B(m,\rho)} \phi \right| \le 2\Lambda^2 \int_{B(m,\rho)} |\phi|$$

(2.7.5) *if* $n \ge 3$
$$\left|(n-2)\omega_n\, \phi(m) + \int_{B(m,\rho)} \Delta\phi \left(\frac{1}{r(x)^{n-2}} - \frac{1}{\rho^{n-2}}\right) - \frac{(n-2)}{\rho^{n-1}} \int_{\partial B(m,\rho)} \phi \right| \le \frac{n-2}{2}\Lambda^2 \int_{B(m,\rho)} \frac{|\phi|}{r(x)^{n-2}} .$$

We note that the error term is of lower order than the other two terms which are the same as in the Euclidean version of the Green representation formula.

Proof We shall prove only (2.7.5) for simplicity. We put

$$g(x) = r(x)^{2-n} - \rho^{2-n} .$$

Then for $\varepsilon > 0$

$$\int_{B(m,\rho)\setminus B(m,\varepsilon)} (g\Delta\phi - \phi\Delta g) = \int_{\partial(B(m,\rho)\setminus B(m,\varepsilon))} \langle g \text{ grad } \phi - \phi \text{ grad } g, \vec{dO}\rangle .$$

Now

$$\left|\int_{B(m,\rho)\setminus B(m,\varepsilon)} \phi\Delta g\right| \le \frac{n-2}{2}\Lambda^2 \int_{B(m,\rho)} \frac{|\phi|}{r^{n-2}(x)} \qquad \text{by (2.7.2)}$$

$$g|\partial B(m,\rho) = 0$$

$$\int_{\partial B(m,\rho)} \phi \langle \text{grad } g, \vec{dO}\rangle = \frac{n-2}{\rho^{n-1}} \int_{\partial B(m,\rho)} \phi$$

$$\lim_{\varepsilon\to 0} \int_{\partial B(m,\varepsilon)} g \langle \text{grad } \phi, \vec{dO}\rangle = 0$$

$$\lim_{\varepsilon\to 0} \int_{B(m,\varepsilon)} \phi \langle \text{grad } g, \, d\vec{O}\rangle = (n-2)\omega_n \, \phi(m)$$

and (2.7.5) follows.

q.e.d.

In the parabolic case, the corresponding version is

LEMMA 2.7.3 *Let* $B(m,\rho)$ *be as above,*

$$B(m,\rho,t_0,t) := \{(x,\tau) \in B(m,\rho) \times [t_0,t]\},$$

$$\phi(\cdot,\tau) \in C^2(B(m,\rho), \mathbb{R}), \ \phi(x,\cdot) \in C^1([t_0,t], \mathbb{R}) \ .$$

Then

$$(2.7.6) \qquad \Big| (\sqrt{4\pi})^n \, \phi(m,t) + \int_{B(m,\rho,t_0,t)} \left(\Delta - \frac{\partial}{\partial t}\right) \phi(x,\tau)(t-\tau)^{-n/2} \left(\exp\left(-\frac{r^2(x)}{4(t-\tau)}\right) - \exp\left(\frac{-\rho^2}{4(t-\tau)}\right)\right) dx \, d\tau \Big|$$

$$\le \frac{c_n}{\rho^{n+2}} \int_{B(m,\rho,t_0,t)} |\phi| + \frac{c_n}{\rho^{n+1}} \int_{\substack{r(x)=\rho \\ t_0\le\tau\le t}} |\phi(x,\tau)|$$

$$+ (t-t_0)^{-n/2} \int_{B(m,\rho)} |\phi(x,t_0)| \, dx$$

$$+ 2\Lambda^2 \int_{B(m,\rho,t_0,t)} |\phi(x,\tau)| \frac{r^2(x)}{(t-\tau)} (t-\tau)^{-n/2} \exp\left(-\frac{r^2(x)}{4(t-\tau)}\right).$$

Here, c_n *is a constant depending only on* n .

Proof We put

$$g(x,\sigma) = \sigma^{-n/2}\left(\exp\left(-\frac{r^2(x)}{4\sigma}\right) - \exp\left(-\frac{\rho^2}{4\sigma}\right)\right) .$$

Let $\varepsilon > 0$. Then

$$\int_{B(m,\rho,t_0,t-\varepsilon)} \left\{ g(x,t-\tau)\left(\Delta - \frac{\partial}{\partial\tau}\right) \phi(x,\tau) - \phi(x,\tau)\left(\Delta + \frac{\partial}{\partial\tau}\right) g(x,t-\tau)\right\} dx \, d\tau$$

$$= \int_{\substack{r(x)=\rho \\ t_0\le\tau\le t-\varepsilon}} \langle g(x,t-\tau) \text{ grad } \phi(x,\tau) - \phi(x,\tau) \text{ grad } g(x,t-\tau), \, d\vec{O}\rangle \, d\tau$$

$$+ \int_{\substack{\tau=t-\varepsilon \\ r(x)\le\rho}} g(x,\varepsilon)\ \phi(x,t-\varepsilon)\ dx - \int_{\substack{\tau=t_0 \\ r(x)\le\rho}} \phi(x,t_0)(t-t_0)^{-n/2}$$

$$\left(\exp\left(-\frac{r^2(x)}{4(t-t_0)}\right) - \exp\left(-\frac{\rho^2}{4(t-t_0)}\right)\right) dx \ .$$

Now

$$\int_{B(m,\rho,t_0,t-\varepsilon)} \phi(x,\tau)\left(\Delta + \frac{\partial}{\partial\tau}\right) g(x,t-\tau)\ dx\ d\tau$$

$$\le 2\Lambda^2 \int_{B(m,\rho,t_0,t)} |\phi(x,\tau)|\ \frac{r^2(x)}{(t-\tau)}\ (t-\tau)^{-n/2}\ \exp\left(-\frac{r^2(x)}{4(t-\tau)}\right) dx\ d\tau$$

by (2.7.3)

$g(x,t-\tau) = 0$ if $r(x) = \rho$

$$\int_{\substack{r(x)=\rho \\ t_0\le\tau\le t}} \phi(x,\tau)(t-\tau)^{-n/2}\ \exp\left(-\frac{r^2(x)}{4(t-\tau)}\right) \frac{2r(x)}{4(t-\tau)} \langle \text{grad } r(x),\ d\vec{O}\rangle$$

$$\le \frac{c_n}{\rho^{n+1}} \int_{\substack{r(x)=\rho \\ t_0\le\tau\le t}} |\phi(x,\tau)|$$

since

$$\text{(2.7.7)} \qquad \exp(-y) \le c_\alpha\ y^{-\alpha} \quad \text{for } y > 0\ ,\ \alpha \ge 0\ .$$

$$\int_{B(m,\rho,t_0,t)} \phi(x,\tau)\ \frac{\partial}{\partial\tau}\left((t-\tau)^{-n/2}\ \exp\left(-\frac{\rho^2}{4(t-\tau)}\right)\right) dx\ d\tau$$

$$= \int_{B(m,\rho,t_0,t)} \phi(x,\tau)\left((t-\tau)^{-n/2-1}\ \exp\left(-\frac{\rho^2}{4(t-\tau)}\right)\right)\left(-\frac{n}{2}+\frac{\rho^2}{4(t-\tau)}\right) dx\ d\tau$$

$$\le \frac{c_n}{\rho^{n+2}} \int_{B(m,\rho,t_0,t)} |\phi(x,\tau)|\ dx\ d\tau \qquad \text{by (2.7.7) again}$$

$$\int_{r(x)\le\rho} \phi(x,t_0)(t-t_0)^{-n/2}\left(\exp\left(-\frac{r^2(x)}{4(t-t_0)}\right) - \exp\left(-\frac{\rho^2}{4(t-t_0)}\right)\right) dx$$

$$\le (t-t_0)^{-n/2} \int_{\substack{\tau=t_0 \\ r(x)\le\rho}} |\phi(x,t_0)|\ dx$$

$$\int_{r(x)\le\rho} \phi(x,t-\varepsilon)\ \varepsilon^{-n/2}\left(\exp\left(-\frac{r^2(x)}{4\varepsilon}\right)\left(-\exp\left(-\frac{\rho^2}{4\varepsilon}\right)\right)\right) dx$$

$$\to (\sqrt{4\pi})^n\ \phi(m,t) \qquad \text{as} \quad \varepsilon \to 0$$

and (2.7.6) follows.

q.e.d.

For a later purpose, we also note the following formula

$$(2.7.8)\qquad \Bigg| (\sqrt{4\pi})^n \phi(m,t) + \int_{B(m,\rho,t_0,t)} \left(\Delta - \frac{\partial}{\partial\tau}\right) \phi(x,\tau)(t-\tau)^{-n/2} \left(\exp\left(-\frac{r^2(x)}{4(t-\tau)}\right) - \exp\left(-\frac{\rho^2}{4(t-\tau)}\right)\right) dx\, d\tau - \int_{B(m,\rho)} \phi(x,t_0)(t-t_0)^{-n/2} \exp\left(-\frac{r^2(x)}{4(t-t_0)}\right) dx \Bigg|$$

$$\le \frac{c_n}{\rho^{n+2}} \int_{B(m,\rho,t_0,t)} |\phi| + \frac{c_n}{\rho^{n+1}} \int_{\substack{r(x)=\rho \\ t_0\le\tau\le t}} |\phi(x,\tau)| + c_n \int_{B(m,\rho)} |\phi(x,t_0)|\, dx + 2\Lambda^2 \int_{B(m,\rho,t_0,t)} |\phi(x,\tau)| \frac{r^2(x)}{(t-\tau)} (t-\tau)^{-n/2} \exp\left(-\frac{r^2(x)}{4(t-\tau)}\right) dx\, d\tau .$$

(2.7.8) also follows from the preceding proof by handling the boundary term at $t = t_0$ in a different way.

We now use almost linear functions in order to also obtain an approximate version of the derivative of Green's function. This is important for obtaining derivative estimates for functions on manifolds.

LEMMA 2.7.4 *Let* $B(m,\rho)$ *be as before. For* $x \in B(m,\rho)$, $x \neq m$, *we define*

$$a(x) = \ell(x)(r(x)^{-n} - \rho^{-n}) ,$$

where $\ell(x)$ *is an almost linear function.*

Then

$$(2.7.9) \qquad |\Delta a| \le 9n^2 \kappa\Lambda \frac{\sinh(2\Lambda r)}{\sin(2\kappa r)} \omega r \operatorname{ctgh}(\omega r)\, r^{-n+1} \qquad \textit{for} \quad x \ne m .$$

Proof

$$(2.7.10) \quad \operatorname{grad} a = \operatorname{grad} \ell (r^{-n} - \rho^{-n}) - n \cdot \ell\, r^{-n-2} \operatorname{grad} f \qquad (f = \tfrac{1}{2} d(\cdot, m)^2)$$

and

$$\Delta a = -2nr^{-n-2} \langle \operatorname{grad} f, \operatorname{grad} \ell \rangle + \Delta \ell \cdot (r^{-n} - \rho^{-n})$$

$$- n\ell r^{-n-2} \Delta f + n(n+2)\ell\, r^{-n-4} |\operatorname{grad} f|^2$$

and hence

$$|\Delta a| \le |\Delta \ell| r^{-n} + 2n\, r^{-n-2} |\ell - \langle \operatorname{grad} f, \operatorname{grad} \ell \rangle| + n|\ell|\, r^{-n-2} |\Delta f - n|$$

since $\operatorname{grad} f = -\exp_x^{-1} m$ and $|\operatorname{grad} f| = r$, cf. (2.3.3).

(2.7.9) then follows from (2.6.3), (2.6.4), and (2.3.4).

q.e.d.

We now can prove that the gradient bound that is obtained in the Euclidean case by differentiating Green's representation formula, again holds on Riemannian manifolds up to a small error term.

LEMMA 2.7.5 *Suppose* $h \in C^2(B(m,\rho), \mathbb{R})$, *where* $B(m,\rho)$ *satisfies the same assumptions as before.*

Then

$$(2.7.11) \quad \omega_n |\operatorname{grad} h(m)| \le \frac{n}{\rho^n} \int_{\partial B(m,\rho)} |h(\cdot) - h(m)| + \int_{B(m,\rho)} \frac{|\Delta h|}{r^{n-1}}$$

$$+ c\,\Lambda^2 \int_{B(m,\rho)} \frac{|h(\cdot) - h(m)|}{r^{n-1}(\cdot)} .$$

Here c *is a constant which depends only on* n *and* $\Lambda\rho$.

Proof For simplicity, we assume $h(m) = 0$.

Let ℓ be an almost linear function with

$$(2.7.12) \qquad \langle \operatorname{grad} \ell(m), \operatorname{grad} h(m)\rangle = |\operatorname{grad} h(m)|$$

and let $a(x) = \ell(x)(r(x)^{-n} - \rho^{-n})$. Then for $\varepsilon > 0$

$$\int_{B(m,\rho)\setminus B(m,\varepsilon)} (a\cdot\Delta h - h\cdot\Delta a) = \int_{\partial(B(m,\rho)\setminus B(m,\varepsilon))} \langle a \operatorname{grad} h - h \operatorname{grad} a, \vec{dO}\rangle$$

Now

$$\int_{B(m,\rho)} |a\cdot\Delta h| \le \int_{B(m,\rho)} \frac{|\Delta h|}{r(x)^{n-1}} \qquad \text{since} \quad |\ell(x)| \le r(x)$$

$$\int_{B(m,\rho)} |h\cdot\Delta a| \le c\cdot\Lambda^2 \int_{B(m,\rho)} \frac{|h|}{r(x)^{n-1}} \qquad \text{by (2.7.9)}$$

$$a|\partial B(m,\rho) = 0$$

$$\int_{\partial B(m,\rho)} |\langle h \operatorname{grad} a, \vec{dO}\rangle| \le \frac{n}{\rho^n} \int_{\partial B(m,\rho)} |h| \qquad \text{by (2.7.10)}.$$

Furthermore by (2.6.4) and since $\vec{dO} = \frac{1}{r} \operatorname{grad} f\cdot|\vec{dO}|$

$$\left|\frac{1}{r^n}\langle \ell\cdot\operatorname{grad} h, \vec{dO}\rangle - \frac{1}{r}\langle \operatorname{grad} \ell, \operatorname{grad} f\rangle\cdot\frac{1}{r}\langle \operatorname{grad} h, \operatorname{grad} f\rangle\cdot\frac{|\vec{dO}|}{r^{n-1}}\right|$$
$$\le c_1\cdot r^3\cdot\frac{1}{r^n}\, |\operatorname{grad} h|\cdot|\vec{dO}|$$

and hence, using (2.7.12),

$$\lim_{\varepsilon\to 0} \int_{\partial B(m,\varepsilon)} \langle a \operatorname{grad} h, \vec{dO}\rangle = |\operatorname{grad} h(m)|\cdot\int_{S^{n-1}} \cos^2\theta \, d\omega^{n-1}$$
$$=: \alpha_n \, |\operatorname{grad} h(m)|.$$

Finally, since $h(x) = \langle \operatorname{grad} h, \operatorname{grad} f\rangle + O(r(x)^2)$, using (2.7.10)

$$\lim_{\varepsilon\to 0} \int_{\partial B(m,\varepsilon)} \langle h \operatorname{grad} a, \vec{dO}\rangle = \lim_{\varepsilon\to 0} \int_{\partial B(m,\varepsilon)} \langle \operatorname{grad} h, \operatorname{grad} f\rangle$$
$$\langle \operatorname{grad} \ell\cdot r^{-n} - n\cdot\ell\cdot r^{-n-2} \operatorname{grad} f, \vec{dO}\rangle$$
$$= \alpha_n(1-n)\, |\operatorname{grad} h(m)|, \text{ using (2.6.4) as before.}$$

The preceding estimates easily imply (2.7.11), noting $\omega_n = n\alpha_n$.

q.e.d.

2.8 REGULARITY PROPERTIES OF COORDINATES. HARMONIC COORDINATES

In this section, we are concerned with regularity properties of coordinates on manifolds. Eventually, we shall show that harmonic coordinates, i.e. ones for which the coordinate functions are harmonic, possess best possible regularity properties.

We start by noting that Riemannian normal coordinates have rather poor regularity properties. Namely, in [JK1] there was displayed the following example of a two-dimensional metric with Hölder continuous curvature which itself is only Hölder continuous in normal coordinates, but not better:

$$ds^2 = dr^2 + G^2(r,\phi)\, d\phi^2$$

with

$$G^2(r,\phi) = \begin{cases} r^2(1 + r^2 \sin^\alpha\phi)^2 & \text{for } 0 \le \phi \le \pi \qquad (0 < \alpha < 1) \\ r^2 & \text{for } \pi \le \phi \le 2\pi . \end{cases}$$

For this metric

$$K = -\frac{G_{rr}}{G} = \begin{cases} -\dfrac{6 \sin^\alpha\phi}{1 + r^2 \sin^\alpha\phi} & \text{for } 0 \le \phi \le \pi \\ 0 & \text{for } \pi \le \phi \le 2\pi . \end{cases}$$

The reason for this phenomenon is that the formula for K in normal coordinates does not involve any derivatives of G with respect to ϕ .

Our aim is to construct coordinates for which we can control - in contrast to normal coordinates - the Christoffel symbols in terms of curvature bounds.

Let us first derive some general identities for any coordinate map $H = (h^1,\dots,h^n) : (B,\langle\cdot,\cdot\rangle) \to \mathbb{R}^n$, where B is the coordinate domain and

$\langle\cdot,\cdot\rangle$ the Riemannian metric. If $v \in T_pB$, then its coordinates are $v^i = dh^i(p)v$. Thus $\langle v,w\rangle = g_{ij}\, v^i\, w^j$, and choosing $v = w = e_k$, where (e_k) is an orthonormal basis of T_pB , we get

$$g^{jk} = \langle \operatorname{grad} h^j, \operatorname{grad} h^k\rangle = dh^j \operatorname{grad} h^k . \tag{2.8.1}$$

Moreover

$$\begin{aligned} D^2_{v,w}H &= \langle D_v \operatorname{grad} H, w\rangle = v(dH\cdot w) - dH\cdot D_v w \\ &= v(dH\cdot w) - dH\, d_v w - dH\cdot\Gamma(v,w) \\ &= -\, dH\cdot\Gamma(v,w) \end{aligned} \tag{2.8.2}$$

since $dH = id$ is linear.

Hence we see that the Christoffel symbols Γ are given by the second derivatives of the coordinate functions. Thus, we have to control those second derivatives for suitable coordinates.

We first construct coordinates by almost linear functions. Let $U = \{u_1,\dots,u_n\}$ be an orthonormal basis of T_mM , and $\ell_1,\dots,\ell_n$ the corresponding almost linear functions.

We define $L : B(m,\rho) \to T_mM \cong \mathbb{R}^n$ via

$$L(x) = \ell_i(x)\cdot u_i(x) . \tag{2.8.3}$$

Then, if P denotes parallel transport along radial geodesics, from Thm. 2.6.1

$$|dL - P(u)| \le 2\sqrt{n}\,\kappa\Lambda\,\frac{\sinh(2\Lambda r)}{\sin(2\kappa r)}\, r^2(x) \tag{2.8.4}$$

$$|D^2L(x)| \le 9\sqrt{n}\,\kappa\Lambda\,\frac{\sinh(2\Lambda r)}{\sin(2\kappa r)}\,\omega r\,\operatorname{ctgh}(\omega r)\cdot r(x) \tag{2.8.5}$$

Note that the injectivity radius of p also enters, namely by restricting the size of the domain of definition of L . (2.8.4) implies that L is invertible on some ball $B(m,\delta)$, where δ depends on Λ , n, and the injectivity radius. Hence L defines coordinates on this ball, and the corresponding Christoffel symbols are bounded because of (2.8.2) and (2.8.5).

If we average this construction over all orthonormal bases U of T_mM, then the coordinates become canonical, since independent of a particular choice of U, while keeping the estimates (2.8.4) and (2.8.5).

We call these coordinates *almost linear coordinates*.

Let now $L : B(p,R) \to T_pM = \mathbb{R}^n$ be almost linear coordinates. We then take the harmonic map

$$H : B(p,R) \to \mathbb{R}^n$$

with

$$H|\partial B(p,R) = L|\partial B(p,R) .$$

We want to show that for some suitably chosen R, H is injective, i.e. a coordinate map.

THEOREM 2.8.1 *For each* $p \in M$ *there exists some* $R > 0$, *depending only on* $\Lambda^2 = \max(|K|)$ *(* K *is the sectional curvature of* M *),* $i(p)$ *(the injectivity radius of* p *), and* $n = \dim M$, *with the property that on* $B(p,R)$ *there exist harmonic coordinates.*

Proof Let ℓ be almost linear on some ball $B(p,R)$. We solve

$$\Delta h = 0 \quad \text{in} \quad B(p,R)$$

$$h|\partial B(p,R) = \ell|\partial B(p,R) .$$

Assuming $R < \frac{\pi}{2\Lambda}$ and putting $k = h-\ell$, (2.6.3) implies

$$(2.8.6) \qquad |\Delta k| \le 9n\Lambda^2 \cdot \Lambda d(x,p) \ \mathrm{ctgh}(\Lambda d(x,p)) \cdot \frac{\sinh(\Lambda d(x,p))}{\sin(\Lambda d(x,p))} \cdot d(x,p) .$$

On the other hand, for

$$\phi(x) = c_0\Lambda^2(d^3(x,p) - R^3)$$

by Lemma 2.3.2

$$\Delta\phi(x) \geq c_0\Lambda^2(3d^2(x,p)(n-1)\Lambda\, \mathrm{ctg}(\Lambda d(x,p)) + 6d) .$$

For given $R \leq R_0 < \frac{\pi}{2\Lambda}$, we can calculate $c_0 = c_0(\Lambda\cdot R_0,n)$ for which $k \pm \phi$ is sub- or superharmonic, resp. Since $k\pm\phi|\partial B(p,R) = 0$, the maximum principle implies

$$|k(x)| \leq |\phi(x)| \leq c_0\Lambda^2 R^3 \tag{2.8.7}$$

and for $x_1 \in \partial B(p,R)$, $x_2 \in B(p,R)$

$$\frac{|k(x_1) - k(x_2)|}{|x_1 - x_2|} \leq \frac{c_0|\phi(x_2)|}{|x_1 - x_2|} \leq 3c_0\Lambda^2 R^2 \tag{2.8.8}$$

or

$$|k(x_2)| \leq 3c_0\Lambda^2 R^2 d(x_2,\partial B(p,R)) . \tag{2.8.9}$$

Let $x \in B(p,R)$, $\rho := d(x,\partial B(p,R))$. Lemma 2.7.5, applied to $B(x,\rho)$ yields

$$\omega_n|\mathrm{grad}\ k(x)| \leq \frac{n}{\rho^n}\int_{\partial B(x,\rho)} |k(y) - k(x)|\, dy + \int_{B(x,\rho)} \frac{|\Delta k(y)|}{d(x,y)^{n-1}}\, dy$$
$$+ c_1(\Lambda\rho,n) \int_{B(x,\rho)} \frac{|k(y) - k(x)|}{d(x,y)^{n-1}}\, dy$$

and hence with (2.8.6) and (2.8.9)

$$|\mathrm{grad}\ k(x)| \leq c_2\Lambda^2 R^2 .$$

Here $c_2 = c_2(\Lambda R_0,n)$ remains bounded for fixed n and $R_0 \to 0$.

(2.6.2) then implies

$$|\mathrm{grad}\ h(x) - u(x)| \leq c_3\Lambda^2 R^2 , \tag{2.8.10}$$

$c_3 = c_3(\Lambda\cdot R_0,n)$.

Let $\{e_i\}$ be an orthonormal basis of T_pM, ℓ^i corresponding almost linear functions and h^i harmonic functions with $h^i|\partial B(p,R) = \ell^i|\partial B(p,R)$.

Putting $H(x) = h^i(x) e_i$, (2.8.10) implies

$$|dH - id| \le c_3 \sqrt{n}\, \Lambda^2 R^2 \quad \text{on} \quad B(p,R) . \tag{2.8.11}$$

We then average again over orthonormal bases of T_pM .

As for almost linear coordinates, we see that harmonic coordinates exist on fixed balls, the radius of which depends only on $i(p)$ (since $R < i(p)$ is necessary for the above constructions), Λ^2 , and n .

q.e.d.

If (g_{ik}) is the metric tensor for the harmonic coordinates constructed above, then from (2.8.1) and (2.8.10)

$$\begin{aligned} |g^{ik} - \delta^{ik}| &= |\langle \operatorname{grad} h^i - u^i, \operatorname{grad} h^k\rangle - \langle u^i, \operatorname{grad} h^k - u^k\rangle| \\ &\le (2 + c_3\Lambda^2 R^2)\, c_3\Lambda^2 R^2 = c_4\Lambda^2 R^2 . \end{aligned} \tag{2.8.12}$$

(2.8.12) implies

$$\|g_{ik}\|_\infty \le \frac{1}{1 - c_4 n\Lambda^2 R^2}$$

and hence

$$|g_{ik} - \delta_{ik}| \le c_4\Lambda^2 R^2 \, \|g_{ik}\|_\infty \le \frac{c_4\Lambda^2 R^2}{1 - c_4 n \Lambda^2 R^2} . \tag{2.8.13}$$

We now want to estimate the Christoffel symbols for harmonic coordinates.

LEMMA 2.8.1 *Let* $H = (h^1,\dots,h^n)$ *be harmonic coordinates. Then, if* (e_i) *is an orthonormal frame, satisfying* $\nabla_{e^i}(e_j) = 0$ *at* x

$$\begin{aligned} \Delta g^{ik} &= \Delta\langle \operatorname{grad} h^i, \operatorname{grad} h^k\rangle \\ &= 2\, R_{mn}\, h^i_{e^m}\, h^k_{e^n} + 2\, h^i_{e^j e^\ell}\, h^k_{e^j e^\ell} , \end{aligned} \tag{2.8.14}$$

where R_{mn} *is the Ricci tensor.*

The proof uses the calculations presented in 1.6.

LEMMA 2.8.2 *There exists some* $R_0 > 0$ *, depending only on* n *,* Λ^2 *,* $i(p)$ *, with the property that for all* $R \le R_0$ *on* $B(p,R)$ *there exist harmonic coordinates the metric tensor* g *of which satisfies*

$$(2.8.15) \qquad |dg(x)| \le \frac{c_5 \Lambda^2 R^2}{d(x,\partial B(p,R))} \quad \text{for } x \in B(p,R) \ ,$$

where $c_5 = c_5(n,\Lambda R_0)$.

Proof Since

$$(2.8.16) \qquad e_\ell \langle \operatorname{grad} h^i, \operatorname{grad} h^k\rangle = h^i_{e^j e^\ell} h^k_{e^j} + h^i_{e^j} h^k_{e^j e^\ell}$$

in normal coordinates, (2.8.10) and (2.8.14) imply

$$(2.8.17) \qquad |\Delta g| \le 2\|\mathrm{Ric}\| \ (1 + c_3\Lambda^2R^2)^2 + \frac{9}{2}\ (1 + c_3\Lambda^2R^2)\ |dg|^2 \ .$$

We now use a method of Heinz [Hz1] to obtain (2.8.15).

Let $\mu := \max\limits_{x \in B(p,R_0)} d(x,\partial B(p,R_0))\ |dg(x)|$.

Then there is some $x_1 \in B(p,R_0)$ with

$$(2.8.18) \qquad \mu = d(x_1,\partial B(p,R_0))\ |dg(x_1)| \ ,$$

and

$$(2.8.19) \qquad |dg(p)| \le \frac{\mu}{R_0} \ .$$

Let $d := d(x_1,\partial B(p,R_0))$, i.e. $\frac{\mu}{d} = |dg(x_1)|$.

By Lemma 2.7.5, applied to $B(x_1,d\theta)$, $0 < \theta < 1$

$$(2.8.20) \quad \frac{\mu}{d} \le \frac{c_5}{d^n\theta^n} \int_{d(x,x_1)=d\theta} |g(x) - g(x_1)| + c_6 \int_{B(x_1,d\theta)} \frac{|\Delta g(x)|}{d(x,x_1)^{n-1}}$$
$$+ c_7\Lambda^2 \int_{B(x_1,d\theta)} \frac{|g(x) - g(x_1)|}{d(x,x_1)^{n-1}}$$
$$=: I + II + III \ .$$

By (2.8.12)

$$I \leq \frac{c_8 \Lambda^2 R^2}{d\theta} ,$$

by (2.8.17)

$$II \leq c_9 \ d\theta (\|Ric\| + |dg|^2) \leq c_9 \|Ric\| \ d\theta + 2c_9 \ d\theta \ \frac{\mu^2}{d^2} ,$$

if we choose $\theta \leq \frac{1}{2}$, since then for $x \in B(x_1, d\theta)$ $d(x, \partial B(p, R_0)) \geq d(1-\theta) \geq \frac{d}{2}$ and by (2.8.12) again

$$III \leq c_{10} \Lambda^4 \ R^2 \ d\theta .$$

Hence

$$\text{(2.8.21)} \quad \mu \leq \frac{1}{\theta} (c_8 \Lambda^2 R^2 + c_9 \|Ric\| \ d^2 \theta^2 + c_{10} \Lambda^4 R^2 \ d^2 \theta^2) + 2c_9 \theta \mu^2$$

$$=: \frac{1}{2\theta} \ a\Lambda^2 R^2 + b\theta \ \frac{\mu^2}{2} .$$

a and b depend only on n and ΛR_0 (for $R \leq R_0$) .

We now choose R_0 so small that

$$\text{(2.8.22)} \qquad ab\Lambda^2 R_0^2 < 1 .$$

Then (2.8.21) implies that for each $\theta \leq \frac{1}{2}$ either

$$\mu \leq \frac{1 - \sqrt{1 - ab\Lambda^2 R^2}}{b\theta}$$

or

$$\mu \geq \frac{1 + \sqrt{1 - ab\Lambda^2 R^2}}{b\theta} \geq 2 \ \frac{1 + \sqrt{1 - ab\Lambda^2 R^2}}{b}$$

$$=: \mu_0 .$$

On the other hand, for each $\mu_1 > \mu_0$ there is some $\theta_1 < \frac{1}{2}$ with

$$\frac{1 - \sqrt{1 - ab\Lambda^2 R^2}}{b\theta_1} < \mu_1 < \frac{1 + \sqrt{1 - ab\Lambda^2 R^2}}{b\theta_1} .$$

Hence the second possibility cannot hold for any $\theta \leq \frac{1}{2}$, and the first one therefore is valid for each $\theta \leq \frac{1}{2}$, in particular for $\theta = \frac{1}{2}$, and

$$\mu \le 2a\Lambda^2 R^2 .$$

(2.8.15) then follows from the definition of μ .

q.e.d.

Lemmata 2.8.1 and 2.8.2 now imply in conjunction with linear elliptic theory, that dg^{ij} is Hölder continuous on balls $B(p,R)$, $R < R_0$ with any exponent $\alpha \in (0,1)$. We only have to observe that the Laplace-Beltrami operator, written in harmonic (or almost linear) coordinates, now is a divergence type elliptic operator with C^1-coefficients while the right-hand side of (2.8.14) is bounded since the Christoffel symbols can be expressed in terms of dg^{ik} . The corresponding estimates for the Green's functions of Δ can be found in [GW]. The important point is that even the Hölder norm of dg^{ik} for harmonic coordinates depends only on the dimension, the injectivity radius, and curvature bounds, but does not involve any curvature derivatives.

We want to present a simple proof of this result for $\alpha = \frac{2}{3}$, using almost linear functions.

Let us first define the notion of Hölder continuity in a way which is invariant under renormalizations. A map $f : B(p,R) \to Y$ is called Hölder continuous with exponent α , if for all $x,y \in B(p,R)$

$$d(f(x), f(y) \le \text{const.}\ R^{1-\alpha}\, d(x,y)^{\alpha} .$$

Similarly, the k-th derivative of f is Hölder continuous, if

$$|D^k f(x) - D^k f(y)| \le \text{const.}\ R^{1-(k+\alpha)}\, d(x,y)^{\alpha} .$$

THEOREM 2.8.2 *Let* $p \in X$. *There exists* $R_0 > 0$, *depending solely on the injectivity radius of* p , *the dimension* n *of the considered manifold* X *and bounds for the sectional curvature on* $B(p,R_0)$ *with the property that for*

$R \le R_0$ *there exist harmonic coordinates on* $B(p,R)$ *the metric tensor* $g = (g_{ij})$ *of which satisfies on each ball* $B(p,(1-\delta)R)$

(2.8.23) $$|dg|_{C^{2/3}} \le \frac{c(\Lambda R_0, n)}{\delta^2} \Lambda^2 R^2 .$$

In particular, the Hölder norms of the corresponding Christoffel symbols are bounded in terms of ΛR_0 *and* n .

Proof Let x be a basepoint, $U = (u^1,\dots,u^n)$ be an orthonormal base of T_xX , and denote by $L_x(z) = (\ell_x^1(z),\dots,\ell_x^n(z))$ the corresponding vector valued almost linear function. Finally, put

$$b_x(z) = L_x(z) \cdot d(x,z)^{-n} .$$

We now want to estimate $|\text{grad } v(x) - \text{grad } v(y)|$ for $v(z) = g^{ij}(z)$. The claim then follows from (2.8.12) and Lemma 2.8.2.

Let $x,y \in B(p,R)$, m be the average of x,y , i.e. that point on the geodesic arc joining x and y with equal distance to both of them, and $\rho = C\cdot d(x,y)^{1/3} \cdot R^{2/3}$, where C will be chosen later.

As in the proof of Lemma 2.7.5, we obtain

(2.8.24) $$\begin{aligned} \omega_n |\text{grad } v(x) - \text{grad } v(y)| &\le \lim_{\varepsilon\to 0} \Big|\int_{B(m,\rho)\setminus B(m,\varepsilon)} \{(v(z) - v(x))\, \Delta b_x(z) \\ &\quad - (v(z) - v(y))\, \Delta b_y(z)\}\, dz\Big| + \Big|\int_{B(m,\rho)} (b_x(z) - b_y(z))\, \Delta v(z)\, dz\Big| \\ &\quad + \Big|\int_{\partial B(m,\rho)} (b_x(z) - b_y(z)) \langle \text{grad } v(z), d\vec{O}\rangle\Big| \\ &\quad + \Big|\int_{\partial B(m,\rho)} \{(v(z) - v(x)) \langle \text{grad } b_x(z), d\vec{O}\rangle - (v(z) - v(y)) \\ &\qquad \cdot \langle \text{grad } b_y(z), d\vec{O}\rangle\}\Big| \\ &=: I + II + III + IV . \end{aligned}$$

First of all, by Lemmata 2.7.4 and 2.8.2

$$I \le \frac{c_{11}\Lambda^2R^2}{\delta R}\Lambda^2\rho^2 . \tag{2.8.25}$$

(Note that we do not exploit the difference $\Delta b_x - \Delta b_y$ in I, since we control only the absolute value of Δb, as we do not want to admit dependence of the estimates on curvature derivatives.)

Choosing w.l.o.g. x and y close together and C suitably, we can assume

$$5d(x,y) \le \rho = C\cdot d(x,y)^{1/3} R^{2/3} \le \delta R . \tag{2.8.26}$$

We then split II into

$$\int_{B(m,\rho)} = \int_{B(m,5d(x,y))} + \int_{B(m,\rho)\setminus B(m,5d(x,y))} \tag{2.8.27}$$

$$= II_a + II_b .$$

(2.8.15), (2,8.17) and the definition of b give

$$II_a \le c_{11}\Lambda^2 \; d(x,y)\left(1 + \frac{\Lambda^2R^2}{\delta R}\right)^2 . \tag{2.8.28}$$

For II_b, we write

$$b_x(z) - b_y(z) = \frac{\ell_x(z) - \ell_y(z)}{d(x,z)^n} + \ell_y(z)\left(\frac{1}{d(x,z)^n} - \frac{1}{d(y,z)^n}\right) \tag{2.8.29}$$

and use Lemma 2.6.2 and (2.8.15), (2.8.17) to get

$$II_b \le \frac{c_{12}}{1-\alpha}\,\frac{\Lambda^2R^2}{(\delta R)^2}\, d(x,y)^\alpha \,\rho^{1-\alpha} \tag{2.8.30}$$

taking $d(x,z),\ d(y,z) \ge d(x,y)$ on $B(m,\rho)\setminus B(m,5d(x,y))$ into account.

Similarly, we get

$$III \le \frac{c_{13}\Lambda^2R^2}{\delta R}\, d(x,y)\cdot\rho^{-1} . \tag{2.8.31}$$

Finally, we write the integrand of IV as

$$(v(z) - v(x))(\operatorname{grad} b_x z - \operatorname{grad} b_y z) - (v(x) - v(y)) \operatorname{grad} b_y(z) .$$

If we use the splitting of (2.8.29), then the only nontrivial expression to estimate is

$$|\operatorname{grad} \ell_x(z) - \operatorname{grad} \ell_y(z)| \ .$$

For this purpose, let $\gamma(t)$ be the geodesic arc from x to y and let P_t be the parallel transport along geodesics emanating from $\gamma(t)$. Then from (2.6.2)

$$|d\ell_{\gamma(t)}(z) - P_t \circ u(t)(z)| \le c_{14}\ d(\gamma(t),z)^2 \ .$$

Moreover

$$|P_t \circ u(t)(z) - P_\tau \circ u(\tau)(z)| \le c_{15}\ d(\gamma(t),z) \cdot d(\gamma(t),\gamma(\tau)) \ .$$

Thus

$$|\operatorname{grad} \ell_x(z) - \operatorname{grad} \ell_y(z)| \le c_{16}\ \rho^2 \qquad \text{for} \quad z \in \partial B(m,\rho) \ .$$

Altogether, we get

$$\text{IV} \le \frac{c_{17}\Lambda^2 R^2}{\delta R} (\Lambda^2 \rho^2 + d(x,y) \cdot \rho^{-1}) \ . \tag{2.8.32}$$

Putting everything together, and using $\rho = Cd(x,y)^{1/3}\ R^{2/3}$

$$\text{I} + \text{II} + \text{III} + \text{IV} \le \frac{c_{18}\Lambda^2 R^2}{\delta^2} \left(\Lambda^2 R^2 C^2 + \frac{1}{C}\right) R^{-5/3}\ d(x,y)^{2/3} \ .$$

This is just the right power of R, since $\operatorname{grad} v$ contains the second derivatives of the coordinate functions h^i. This finishes the proof.

q.e.d.

Moreover, we note that once having proved Thm. 2.8.2 or Lemma 2.8.2, (2.8.14) in conjunction with linear elliptic theory implies

THEOREM 2.8.2 *Let* $R \le R_0$, *where* R_0 *is chosen as in Thm. 2.8.2, and let* $g = (g_{ij})$ *be the metric tensor of the corresponding harmonic coordinates on* $B(p,R)$. *If the Riemann curvature tensor on* $B(p,R)$ *is of class* C^k *or* $C^{k+\beta}$ $(k \in \mathbb{N},\ \beta \in (0,1)$, *then* $g \in C^{k+1+\alpha}$ *(for every* $\alpha \in (0,1))$ *or* $g \in C^{k+2+\beta}$, *resp., in the interior of* $B(p,R)$. *The corresponding estimates* *depend in addition to the quantities mentioned in Thm. 2.8.2 on the* C^k *or* $C^{k+\beta}$*-norm, resp., of the curvature tensor.*

That harmonic coordinates possess best possible regularity properties was first pointed out by de Turck-Kazdan [dTK]. The explicit construction implying the existence of harmonic coordinates on fixed (curvature controlled) balls and the explicit estimates of this section are due to Jost-Karcher [JK1].

Finally, for later purposes, we need still another construction of coordinates. We want to introduce coordinates with curvature controlled Christoffel symbols in a neighbourhood of a point $q \in B(p,M)$, without using any information of the geometry outside $B(p,M)$. We suppose again that $M < \frac{\pi}{2K}$, $M < i(p)$. In case $d(p,q) \leq \frac{1}{2}M$, we taken an arbitrary orthonormal base $e_1,\dots e_n$ of T_qY $(B(p,M) \subset Y, \dim Y = n)$. If $d(p,q) > \frac{1}{2}M$, we choose $e_1,\dots,e_n$ in such a way that $\Sigma\, e_i$ is tangent to the geodesic from q to p. We now want to show that the geodesics $\exp_p(t\cdot e_i)$ stay inside $B(p,M)$ for $t \leq t_0$, where $t_0 > 0$ can be estimated from below in terms of ω, M, and n. Indeed, by the Rauch-Toponogow Comparison Theorem (cf. [GKM], p.194f),

$$d(p, \exp_q t\cdot e_i) \leq d^{\omega}(\tilde{p}, \exp_{\tilde{q}} t\cdot \tilde{e}_i) ,$$

where the right hand side is the distance in the comparison triangle in the plane of constant curvature $-\omega^2$, with $d^{\omega}(\tilde{p},\tilde{q}) = d(p,q)$, $\tilde{e}_i$ having the same angle with the geodesic form $\tilde{q}$ to $\tilde{p}$ as e_i has with the geodesic from q to p. Consequently

$$\cosh(\omega d(p, \exp_q te_i)) \leq \cosh \omega t \cdot \cosh(\omega d(p,q)) - \frac{1}{n} \sinh \omega t \cdot \sinh(\omega d(p,q))$$

$$\leq \cosh \omega t \cdot \sinh \omega M - \frac{1}{n} \sinh \omega t \cdot \sinh \omega M ,$$

if $t \leq \frac{1}{2}M$

$$\leq \cosh \omega M ,$$

if $t \leq \bar{t}$, say.

Then, for $t \le t_0 = \min(\bar{t}, \frac{1}{2}M)$, $d(p, \exp_q te_i) \le M$, and consequently the geodesics $\exp_q te_i$ stay inside $B(p,M)$ for $t \le t_0$.

LEMMA 2.8.4 *In a neighbourhood* $B(q,\tau) \cap B(p,M)$ *of* $q \in B(p,M)$ *, we can define local coordinates for which the Christoffel symbols are bounded in absolute value and* $\tau > 0$ *is bounded from below, both in terms of* ω *,* κ *,* n *,* M *only, via*

$$k_i(s) := \frac{1}{2t_0} (d^2(s, \exp_q t_0 e_i) - d^2(s,q)) \ .$$

Proof By Lemma 2.3.2

$$(2.8.33) \qquad |D^2 k_i(s)| \le \frac{\omega M}{t_0} \coth \frac{\omega M}{2}$$

if $d(s,q) \le \frac{1}{2}M$, and

$$(2.8.34) \qquad dk|_q \quad \text{is an isometry ,}$$

where $k = (k_1, \ldots, k_n) : B(p,M) \to \mathbb{R}^n$.

This easily implies a lower bound τ for the radius of the set on which k is injective. Furthermore, the Christoffel symbols are given by $D^2 k$ (cf. (2.8.2)), and hence the bound on the Christoffel symbols follows from (2.8.33).

q.e.d.