26 Isoperimetric Inequalities for Minimal Surfaces

It is well known that for a plane Jordan curve with length L, the area $\mathbf{A}$ enclosed by the curve is less than or equal to $L^2/4\pi$, with equality holding if and only if the curve is a circle. In this section we give such *isoperimetric inequalities* for simply or doubly connected minimal surfaces. For more general discussions and applications of the isoperimetric inequalities the reader can see [69].

The proof of the next theorem is from [68].

Theorem 26.1 *Let $M \subset \mathbf{R}^3$ be an immersed simply connected minimal surface with $C = \partial M$ a closed curve. Let L be the arclength of C, $\mathbf{A}$ the area of M, then*

$$L^2 - 4\pi\mathbf{A} \geq 0. \tag{26.118}$$

Proof. From (3.6) we have

$$2\mathbf{A} = \int_C (X - a)\bullet\vec{n}\, ds$$

for any $a \in \mathbf{R}^3$. Here X is the coordinate function of M, $\vec{n}$ is the outward unit conormal to C and ds is the line element of C. Select $a \in C$. We need prove that

$$2\pi \int_C (X - a)\bullet\vec{n}\, ds \leq L^2.$$

Let $x(s)$ be the parametrisation of C by arclength and $x(0) = x(L) = a$. We want to select suitable frames in each $T_{x(s)}\mathbf{R}^3$. For this purpose, let $B(s) : T_{x(s)}M \to T_{x(s)}M$ be the linear mapping which rotates $\vec{n}$ by $\pi/2$ and is zero in $T^{\perp}_{x(s)}$. If we let $(\vec{n}, B\vec{n}, N)$ be the orthonormal basis of $T_{x(s)}\mathbf{R}^3$, then B has the matrix form

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

From this it is clear that

1. $|Bv| \leq |v|$ for any $v \in \mathbf{R}^3$.

2. $u\bullet Bv = -v\bullet Bu$.

Let $(e_1, e_2, e_3)(s)$ be vector fields along C such that

$$e_i'(s) = \frac{\pi}{L} B e_i(s), \quad i = 1,\ 2,\ 3, \tag{26.119}$$

and $(e_1, e_2, e_3)(0)$ is an orthonormal basis of $\mathbf{R}^3$. Then property 2 guarantees that $(e_i \bullet e_j)(s)$ is a constant, thus $(e_1, e_2, e_3)(s)$ is an orthonormal basis of $\mathbf{R}^3$ for any $s \in [0, L]$. We can write

$$x(s) - a = \sum_{i=1}^{3} c_i(s) e_i(s).$$

Then

$$x'(s) = \sum_{i=1}^{3} c_i'(s) e_i(s) + \frac{\pi}{L} \sum_{i=1}^{3} c_i(s) B e_i(s) = \sum_{i=1}^{3} c_i'(s) e_i(s) + \frac{\pi}{L} B[x(s) - a].$$

Thus

$$\begin{aligned}
|x'(s)|^2 &= \frac{\pi}{L} x'(s) \bullet B[x(s) - a] + x'(s) \bullet \sum_{i=1}^{3} c_i'(s) e_i(s) \\
&= \frac{\pi}{L} x'(s) \bullet B[x(s) - a] + \left[\sum_{i=1}^{3} c_i'(s) e_i(s)\right] \bullet \left[\sum_{i=1}^{3} c_i'(s) e_i(s)\right] \\
&\quad + \frac{\pi}{L} B[x(s) - a] \bullet \left[\sum_{i=1}^{3} c_i'(s) e_i(s)\right] \\
&= \frac{\pi}{L} x'(s) \bullet B[x(s) - a] + \sum_{i=1}^{3} c_i'(s)^2 + \frac{\pi}{L} B[x(s) - a] \bullet \left\{x'(s) - \frac{\pi}{L} B[x(s) - a]\right\} \\
&= \frac{2\pi}{L} x'(s) \bullet B[x(s) - a] + \sum_{i=1}^{3} c_i'(s)^2 - \frac{\pi^2}{L^2} B[x(s) - a] \bullet B[x(s) - a] \\
&= \frac{2\pi}{L} x'(s) \bullet B[x(s) - a] + \sum_{i=1}^{3} c_i'(s)^2 - \frac{\pi^2}{L^2} |x(s) - a|^2 \\
&\quad + \frac{\pi^2}{L^2} \left(|x(s) - a|^2 - |B[x(s) - a]|^2\right) \\
&= \frac{2\pi}{L} x'(s) \bullet B[x(s) - a] + \sum_{i=1}^{3} \left[c_i'(s)^2 - \frac{\pi^2}{L^2} c_i^2(s)\right] \\
&\quad + \frac{\pi^2}{L^2} \left(|x(s) - a|^2 - |B[x(s) - a]|^2\right).
\end{aligned}$$

Thus we have

$$\frac{2\pi}{L} x'(s) \bullet B[x(s) - a] = |x'(s)|^2 - \sum_{i=1}^{3} \left[c_i'(s)^2 - \frac{\pi^2}{L^2} c_i^2(s)\right] - \frac{\pi^2}{L^2} \left(|x(s) - a|^2 - |B(x(s) - a)|^2\right).$$

Since $Bx'(s) = -\vec{n}$,

$$[x(s) - a] \bullet \vec{n} = -[x(s) - a] \bullet Bx'(s) = x'(s) \bullet B[x(s) - a],$$

we find that

$$2\pi \int_C (X - a) \bullet \vec{n} \, ds = 2\pi \int_0^L x'(s) \bullet B[x(s) - a] \, ds$$

$$= L^2 - L\int_0^L \sum_{i=1}^3 \left[c_i'(s)^2 - \frac{\pi^2}{L^2}c_i^2(s)\right] ds - \frac{\pi^2}{L}\int_0^L \left(|x(s)-a|^2 - |B[x(s)-a]|^2\right) ds.$$

The fact $x(0) = a$ and $x'(0)$ exists give that $c_i(0) = 0$, $c_i'(0) \in \mathbf{R}$, $i = 1,\ 2,\ 3$, thus the functions

$$b_i(s) = \frac{c_i(s)}{\sin\left(\frac{\pi s}{L}\right)}$$

are well defined for $i = 1,\ 2,\ 3$. Using the identities

$$\begin{aligned} c_i'(s)^2 - \frac{\pi^2}{L^2}c_i^2(s) &= b_i'(s)^2 \sin^2\frac{\pi}{L} + \frac{\pi}{2L}\frac{d}{ds}\left(b_i^2(s)\sin\frac{2\pi s}{L}\right) \\ &= b_i'(s)^2 \sin^2\frac{\pi}{L} + \frac{\pi}{L}\frac{d}{ds}\left(c_i^2(s)\cot\frac{\pi s}{L}\right), \end{aligned}$$

and $|B[x(s)-a]| \le |x(s)-a|$, we obtain

$$L^2 - 2\pi\int_C [x(s)-a]\bullet\vec{n}\, ds \ge L\sum_{i=1}^3 \int_0^L b_i'(s)^2 \sin^2\frac{\pi s}{L}\, ds \ge 0.$$

□

Remark 26.2 This isoperimetric inequality is also true for simply connected minimal surfaces in $\mathbf{R}^n$, $n \ge 3$. The proof is the same as above. See [68].

Next we study the doubly connected case, the proof is from [70]. We will use the notation in the last section.

Theorem 26.3 *Let $\mathbf{A}$ be the area of a minimal annulus $X : A \hookrightarrow \mathbf{R}^3$, L_1 and L_2 the length of its boundary curves C_1 and C_2, and let $L = L_1 + L_2$. If $\mathbf{Flux}(X) = 0$ or there are no planes separating the two boundary curves, then*

$$L_1^2 + L_2^2 \ge 4\pi\mathbf{A} \tag{26.120}$$

or, equivalently,

$$L^2 - 4\pi\mathbf{A} \ge 2L_1L_2. \tag{26.121}$$

For arbitrary minimal annulus, we have

$$L^2 - 4\pi\mathbf{A} \ge 2L_1L_2(1-\log 2). \tag{26.122}$$

Proof. From the area formula (3.6) we have

$$2\mathbf{A} = \int_{C_1} X\bullet\vec{n}\, ds + \int_{C_2} X\bullet\vec{n}\, ds.$$

In the proof of Theorem 26.1, we have

$$M_1 := L_1^2 - 2\pi \int_{C_1} (X - p_1) \bullet \vec{n}\, ds \geq 0, \quad M_2 := L_2^2 - 2\pi \int_{C_2} (X - p_2) \bullet \vec{n}\, ds \geq 0,$$

where $p_i \in C_i$. (Note that we did not use that C_i encloses a simply connected minimal surface in the proof of the above inequalities.) Now remember that

$$-\int_{C_1} \vec{n}\, ds = \int_{C_2} \vec{n}\, ds = \mathbf{Flux}(X).$$

We have

$$L_1^2 + L_2^2 - 4\pi \mathbf{A} = M_1 + M_2 - 2\pi (p_2 - p_1) \bullet \mathbf{Flux}(X).$$

So if $\mathbf{Flux}(X) = 0$, then we have (26.120). If $\mathbf{Flux}(X) \neq 0$, then take a plane P_d defined by $x \bullet \mathbf{Flux}(X) = d$. All $d \in \mathbf{R}$ such that $P_d \cap C_i \neq \emptyset$ form two closed intervals in $\mathbf{R}$. If no planes separate C_1 and C_2, then these two intervals have common points, and thus we can find $p_i \in C_i$ such that $p_1 \bullet \mathbf{Flux}(X) = p_2 \bullet \mathbf{Flux}(X)$; again we get (26.120).

Now we consider the case that $\mathbf{Flux}(X) \neq 0$ and there is a plane separating C_1 and C_2. Note that after a homothety, both sides of (26.122) are multiplied by a positive constant, thus by Corollary 25.3 we can assume that

$$\mathbf{Flux}(X) = (0, 0, 2\pi).$$

So we have $\overline{X}_3(r) = \log r$. This implies that the planes $P_i := \{x_3 = \log r_i\}$ intersect C_i respectively. Thus selecting $p_i \in P_i \cap C_i$, we have

$$2\pi (p_2 - p_1) \bullet \mathbf{Flux}(X) = 4\pi^2 (\log r_2 - \log r_1) = 4\pi^2 \log \frac{r_2}{r_1},$$

and

$$L_1^2 + L_2^2 - 4\pi \mathbf{A} = M_1 + M_2 - 4\pi^2 \log \frac{r_2}{r_1}. \tag{26.123}$$

We now apply Theorem 25.10. Recall that $r_1 \leq 1 \leq r_2$ and that $L(r)$ is a minimum for $r = 1$. We let

$$K_i := \pi \left(r_i + \frac{1}{r_i} \right), \quad i = 1,\ 2,$$

be the lengths of the corresponding boundary circles on the standard catenoid. Then

$$\pi^2 \frac{r_2}{r_1} < K_1 K_2 < 4\pi^2 \frac{r_2}{r_1}.$$

By Theorem 25.10 and Lemma 25.8, $K_1 K_2 \leq L_1 L_2$. Finally, if we let $k_i = L_i/\pi$, we have

$$\begin{aligned} 2L_1L_2 - 4\pi^2 \log \frac{r_2}{r_1} &= 2\pi^2 \left(k_1k_2 - 2\log \frac{r_2}{r_1} \right) \\ &> 2\pi^2 \left(k_1k_2 - 2\log \frac{K_1K_2}{\pi^2} \right) \\ &\geq 2\pi^2 (k_1k_2 - 2\log k_1k_2) \\ &\geq 2\pi^2 k_1k_2 (1 - \log 2). \end{aligned}$$

The last inequality follows from the elementary fact that

$$2 \log x < x \log 2 \quad \text{for} \quad x > 4,$$

combined with $K_i \geq 2\pi$, $k_1 k_2 = L_1 L_2/\pi^2 \geq K_1 K_2/\pi^2 \geq 4$. Substituting in (26.123) gives (26.122), and the theorem is proved. □

Remark 26.4 The inequalities (26.120) and (26.121) are also true for minimal annuli in $\mathbf{R}^n$, $n \geq 3$, satisfying the corresponding conditions. The proof is similar, see [70]. The inequality (26.122) is true in $\mathbf{R}^3$ since we have Theorem 25.10, thus if Theorem 25.10 is true in $\mathbf{R}^n$ then (26.122) is also true in $\mathbf{R}^n$.