25 Minimal Annuli

The catenoid is topologically an annulus, and is the only embedded complete minimal annulus of finite total curvature by Theorem 18.1. Since any complete minimal surface has annular end, we want to study minimal surfaces of annular type, with or without boundary.

All the results in this section are due to Osserman and Schiffer [70].

First we fix $A := \{r_1 < |z| < r_2\} \subset \mathbf{C}$, $0 < r_1 \leq 1 \leq r_2 \leq \infty$ (by Lemma 9.1 and Proposition 9.2 we can always select such a representation of A). Let $X : A \to \mathbf{R}^3$ be a minimal annulus. Let g and $\eta = f(z)dz$ be the Weierstrass data for X and ϕ_i be as (6.15), $i = 1, 2, 3$. Let $\psi_i = z\phi_i$. Write $X = (X_1, X_2, X_3)$ and let

$$t = \log r = \log |z|.$$

We define

$$\overline{X}(r) := \frac{1}{2\pi}\int_0^{2\pi} X(re^{i\theta})d\theta.$$

Lemma 25.1

$$\frac{d^2\overline{X}(r)}{dt^2} = 0.$$

Proof.

$$\begin{aligned}\frac{d^2\overline{X}(r)}{dt^2} &= r\frac{d}{dr}\left(r\frac{d\overline{X}(r)}{dr}\right) = \frac{1}{2\pi}\int_0^{2\pi} r\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}X(r^{i\theta})\right]d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} r^2 \bigtriangleup X \, d\theta = 0,\end{aligned}$$

since

$$0 = \bigtriangleup X = r^{-1}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}X(re^{i\theta})\right] + r^{-2}\frac{\partial^2 X(re^{i\theta})}{\partial\theta^2}$$

and

$$\int_0^{2\pi}\frac{\partial^2 X(re^{i\theta})}{\partial\theta^2}d\theta = 0.$$

□

Let $C = \{|z| = 1\} \subset A$. Since C is the generator of the first homology group of A, by (17.72), we can define

$$\begin{aligned}\mathbf{Flux}(X) &= \mathbf{Flux}(C) = \Im\int_C \phi(z)dz = \Im\int_{|z|=r}\phi(z)dz = -i\int_{|z|=r}\phi(z)dz \\ &= \int_0^{2\pi}\phi(re^{i\theta})re^{i\theta}d\theta = \int_0^{2\pi}\psi(re^{i\theta})d\theta \qquad (25.91)\end{aligned}$$

where $\psi = z\phi$, for any $r_1 \leq r \leq r_2$. We have used the fact that since X is well defined,

$$\Re \int_{|z|=r} \phi \, dz = (0,0,0).$$

Lemma 25.2

$$\frac{d\overline{X}(r)}{dt} = r\frac{d\overline{X}(r)}{dr} = \frac{1}{2\pi}\mathbf{Flux}(X). \tag{25.92}$$

Proof.

$$\begin{aligned}
\frac{d\overline{X}(r)}{dt} &= r\frac{d\overline{X}(r)}{dr} = \frac{1}{2\pi}\int_0^{2\pi} r\frac{\partial X(re^{i\theta})}{\partial r}\, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial X(re^{i\theta})}{\partial x}\cos\theta + \frac{\partial X(re^{i\theta})}{\partial y}\sin\theta\right) r\, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \Re\left[re^{i\theta}\left(\frac{\partial X}{\partial x} - i\frac{\partial X}{\partial y}\right)(re^{i\theta})\right] d\theta \\
&= \Re\frac{1}{2\pi}\int_0^{2\pi}\left[re^{i\theta}\left(\frac{\partial X}{\partial x} - i\frac{\partial X}{\partial y}\right)(re^{i\theta})\right] d\theta \\
&= \Re\frac{1}{2\pi}\int_{|z|=r} -i\left(\frac{\partial X}{\partial x} - i\frac{\partial X}{\partial y}\right)(z)dz \\
&= \Im\frac{1}{2\pi}\int_{|z|=r}\phi(z)\, dz = \frac{1}{2\pi}\mathbf{Flux}(X).
\end{aligned}$$

□

Corollary 25.3 *Either*

$$\mathbf{Flux}(X) = (0,0,0),$$

or by a homothety, if necessary, we may assume that

$$\int_0^{2\pi}\psi(re^{i\theta})d\theta = 2\pi\frac{d\overline{X}(r)}{dt} = \mathbf{Flux}(X) = (0,0,2\pi). \tag{25.93}$$

Proof. Assume that $\mathbf{Flux}(X) \neq 0$. By Lemma 25.1, $\overline{X}(r) = \log r(c_1, c_2, c_3) + (d_1 + d_2 + d_3)$, where c_i and d_i are constants, $i = 1,\ 2,\ 3$. Thus the points $\overline{X}(r)$ lie on the straght line $t(c_1, c_2, c_3) + (d_1, d_2, d_3)$. After a linear homothety $H : \mathbf{R}^3 \to \mathbf{R}^3$, we may assume that $H[(c_1, c_2, c_3)] = (0,0,1)$. Thus by Lemma 25.2,

$$\mathbf{Flux}(H\circ X) = 2\pi\frac{d\overline{H\circ X}(r)}{dt} = 2\pi H\left[\frac{d\overline{X}(r)}{dt}\right] = 2\pi H[(c_1,c_2,c_3)] = (0,0,2\pi).$$

□

Remark 25.4 Thus we can always assume that X has vertical flux and that if $\mathbf{Flux}(X) \neq 0$ then

$$\mathbf{Flux}(X) = (0, 0, 2\pi)$$

after a suitable homothety. We will say that a minimal annulus with the above flux is *normalised.*

We are interested in the arclength of the closed curve $X|_{|z|=r}$. By (7.28)

$$\Lambda = \frac{1}{2}|f|(1+|g|^2) = \frac{1}{2}|\phi_3|\left(\frac{1}{|g|} + |g|\right).$$

The arclength $L(r)$ of the closed curve $X|_{|z|=r}$ is

$$L(r) = \int_{|z|=r} ds = \int_0^{2\pi} r\Lambda\, d\theta = \int_0^{2\pi} \frac{1}{2}\left(\left|\frac{\psi_3}{g}\right| + |\psi_3 g|\right) d\theta. \tag{25.94}$$

Theorem 25.5 *For any minimal annulus,*

$$\frac{d^2L}{dt^2} \geq L. \tag{25.95}$$

Equality holds if and only if the surface is the portion of a catenoid bounded by parallel coaxial circles, or an annulus in the plane.

Proof. The same calculation as in the proof of Lemma 25.1 leads to:

$$\frac{d^2L}{dt^2} = \int_{|z|=r} \frac{r^2}{2} \triangle \left(\left|\frac{\psi_3}{g}\right| + |\psi_3 g|\right) d\theta. \tag{25.96}$$

Now we have

$$\psi_1 = \frac{1}{2}\psi_3\left(\frac{1}{g} - g\right), \quad \psi_2 = \frac{i}{2}\psi_3\left(\frac{1}{g} + g\right)$$

and

$$\frac{\psi_3}{g} = \psi_1 - i\psi_2, \qquad \psi_3 g = -\psi_1 - i\psi_2.$$

Since both d^2L/dt^2 and L are continuous functions of r, it follows from (25.93), (25.94), (25.96) that in order to prove (25.95) it suffices to prove the following lemma. □

Lemma 25.6 *Let $F(z)$ be holomorphic in A, and satisfy*

$$\int_0^{2\pi} F(re^{i\theta})d\theta = 0. \tag{25.97}$$

Then on every circle $|z| = r$ where F has no zeros, the inequality

$$\int_0^{2\pi} r^2 \triangle |F| d\theta \geq \int_0^{2\pi} |F| d\theta \tag{25.98}$$

is valid. Equality holds if and if F is a constant multiple of z or $1/z$.

Proof. Let $G(z)$ be holomorphic in an annulus, and have the Laurent expansion

$$G(z) = \sum_{-\infty}^{\infty} a_n\, z^n. \tag{25.99}$$

Then

$$\int_0^{2\pi} |G(re^{i\theta})|^2 d\theta = 2\pi \sum_{-\infty}^{\infty} |a_n|\, r^{2n} \tag{25.100}$$

and

$$\int_0^{2\pi} |G'(re^{i\theta})|^2 d\theta = 2\pi \sum_{-\infty}^{\infty} n^2 |a_n|\, r^{2n-2}. \tag{25.101}$$

Thus

$$\int_0^{2\pi} r^2 |G'(re^{i\theta})|^2 d\theta \geq \int_0^{2\pi} |G(re^{i\theta})|^2 d\theta - 2\pi |a_0|^2 \tag{25.102}$$

and since $\triangle |G|^2 = 4(\partial^2/\partial z \partial \overline{z})(G\overline{G}) = 4|G'|^2$,

$$\int_0^{2\pi} r^2 \triangle |G(re^{i\theta})|^2 d\theta \geq 4 \int_0^{2\pi} |G(re^{i\theta})|^2 d\theta - 8\pi |a_0|^2. \tag{25.103}$$

Since $F \neq 0$ on $|z| = r$, we may choose an annulus (by "thickening" this circle) in which $F \neq 0$. Since F is holomorphic in this annulus,

$$\int_0^{2\pi} \frac{\partial}{\partial \theta} \arg F(re^{i\theta}) d\theta = 2\pi k$$

for some integer k. Corresponding to k is even or odd, there are two possibilities:

either **Case** 1. $F = G^2$ or **Case** 2. $F = zG^2$,

where G is holomorphic in the annulus.

Case 1. If G has the expansion (25.99), then the constant term in the expansion of $F = G^2$ is

$$a_0^2 + 2 \sum_{n=1}^{\infty} a_n\, a_{-n}.$$

But condition (25.97) is equivalent to the vanishing of the constant term in the Laurent expansion of F. Thus

$$a_0^2 = -2 \sum_{n=1}^{\infty} a_n\, a_{-n}$$

and

$$|a_0^2| = 2 \Big| \sum_{n=1}^{\infty} a_n r^n\, a_{-n} r^{-n} \Big| \leq \sum_{n=1}^{\infty} (|a_n|^2 r^{2n} + |a_{-n}|^2 r^{-2n}) = \frac{1}{2\pi} \int_0^{2\pi} |G(re^{i\theta})|^2\, d\theta - |a_0|^2,$$

or

$$4\pi |a_0^2| \leq \int_0^{2\pi} |G(re^{i\theta})|^2\, d\theta.$$

Substituting this in (25.103) and using $|F| = |G|^2$, yields

$$\int_0^{2\pi} r^2 \triangle |F| d\theta \geq 2 \int_0^{2\pi} |F| d\theta. \tag{25.104}$$

Thus in Case 1, not only does (25.98) hold but in fact a stronger form is valid, implying in particular that the inequality in (25.98) is strict.

Case 2. Here $F = zG^2$ and $|F| = r|G|^2$. We have

$$\begin{aligned}
\triangle|F| &= |G|^2 \triangle r + r \triangle |G|^2 + 2D\, r \bullet D|G|^2 \\
&= r^{-1}|G|^2 + 4r|G'|^2 + 4r^{-1}(x,y)\bullet\left(\Re(\overline{G}G'), -\Im(\overline{G}G')\right) \\
&= r^{-1}|G|^2 + 4r|G'|^2 + 4r^{-1}\Re(z\overline{G}G').
\end{aligned}$$

Note that

$$\int_0^{2\pi} re^{i\theta}G'(re^{i\theta})d\theta = 0$$

and

$$\begin{aligned}
4\Re \int_0^{2\pi} [re^{i\theta}\overline{(G-a_0)}G']d\theta &\geq -4\int_0^{2\pi} r|(G-a_0)G'|\, d\theta & (25.105)\\
&\geq -2\int_0^{2\pi} |G-a_0|^2\, d\theta - 2\int_0^{2\pi} r^2|G'|^2\, d\theta. & (25.106)
\end{aligned}$$

We have

$$\begin{aligned}
\int_0^{2\pi} r^2 \triangle |F|\, d\theta &= \int_0^{2\pi} r|G|^2\, d\theta + 4\int_0^{2\pi} r^3|G'|^2\, d\theta + 4\int_0^{2\pi} r\Re(z\overline{G}G')d\theta \\
&= \int_0^{2\pi} |F|\, d\theta + 4\int_0^{2\pi} r^3|G'|^2\, d\theta + 4\Re\int_0^{2\pi} r[re^{i\theta}\overline{(G-a_0)}G']d\theta \\
&\geq \int_0^{2\pi} |F|\, d\theta + 4\int_0^{2\pi} r^3|G'|^2\, d\theta \\
&\quad -2\int_0^{2\pi} r|(G-a_0)|^2 d\theta - 2\int_0^{2\pi} r^3|G'|^2 d\theta \\
&= \int_0^{2\pi} |F|\, d\theta + 2\int_0^{2\pi} r^3|(G-a_0)'|^2\, d\theta - 2\int_0^{2\pi} r|(G-a_0)|^2 d\theta \\
&\geq \int_0^{2\pi} |F|\, d\theta.
\end{aligned}$$

The equality holds if and only if (25.105) and (25.106) are both equalities, and

$$\int_0^{2\pi} r^2|(G-a_0)'|^2\, d\theta = \int_0^{2\pi} |(G-a_0)|^2 d\theta. \tag{25.107}$$

In particular, by (25.100) and (25.101), (25.107) holds if and only if

$$a_n = 0 \quad \text{for} \quad |n| \neq 1 \text{ or } 0.$$

But if

$$G - a_0 = \frac{a_{-1}}{z} + a_1 z,$$

then

$$4\Re \int_0^{2\pi} re^{i\theta}\overline{(G-a_0)}G'\,d\theta = 8\pi(|a_1|^2r^2 - |a_{-1}|^2r^{-2}),$$

$$-2\int_0^{2\pi} |G-a_0|^2\,d\theta - 2\int_0^{2\pi} r^2|(G-a_0)'|^2\,d\theta = -8\pi(|a_1|^2r^2 + |a_{-1}|^2r^{-2}).$$

Comparing these two we have $a_1 = 0$.

Thus we have

$$G(z) = \frac{a_{-1}}{z} + a_0, \quad G^2(z) = \frac{a_{-1}^2}{z^2} + a_0^2 + 2\frac{a_0a_{-1}}{z},$$

and

$$F(z) = zG^2(z) = 2a_0a_{-1} + a_0^2z + \frac{a_{-1}^2}{z}.$$

Condition (25.97) then implies that $a_0a_{-1} = 0$, so that F is of the form stated. □

Remark 25.7 Note that in Case 2 the assumption (25.97) is not needed to deduce the inequality (25.98). Only in Case 1 did we use it, and there it is clearly necessary since, for example, (25.104) is false if F is a non-zero constant.

We now complete the proof of Theorem 25.5 by analysing when equality can hold in (25.95). Returning to (25.96) we see that for equality to hold in (25.95) we must have

$$\frac{\psi_3}{g} = c_1z \text{ or } \frac{c_2}{z}, \quad \psi_3 g = b_1 z \text{ or } \frac{b_2}{z}.$$

We therefore have four cases.

Case 1.

$$\frac{\psi_3}{g} = c_1z, \quad \psi_3g = b_1z.$$

Then g is a constant, and so is ϕ_3. It follows from (6.15), (6.18) and (6.26) that

$$\phi_1 = \frac{1}{2}c\left(\frac{1}{d} - d\right), \quad \phi_2 = \frac{i}{2}c\left(\frac{1}{d} + d\right), \quad \phi_3 = c.$$

The image surface is in a plane, and the map $X : A \hookrightarrow \mathbf{R}^3$ is a complex linear map into the plane.

Case 2.

$$\frac{\psi_3}{g} = \frac{c_2}{z}, \quad \psi_3g = \frac{b_2}{z}.$$

Again g is a constant, but this time

$$\phi_1 = \frac{1}{2}\frac{c}{z^2}\left(\frac{1}{d} - d\right), \quad \phi_2 = \frac{i}{2}\frac{c}{z^2}\left(\frac{1}{d} + d\right), \quad \phi_3 = \frac{c}{z^2}.$$

The image is again a plane, but the map is this time the composition of $1/z$ with a complex linear map.

Case 3.

$$\frac{\psi_3}{g} = c_1 z, \qquad \psi_3 g = \frac{b_2}{z}.$$

Then we have $\psi_3 = c$, $g = d/z$, and $\phi_3 = c/z$. Thus the Weierstrass data are $g = d/z$, $\eta = (\phi_3/g)dz = (c/d)dz$. Making change $z \to d/\zeta$, we see that $g(\zeta) = \zeta$ and $(c/d)dz = -c\,d\zeta/\zeta^2$. Thus c is real and the surface is part of a catenoid.

Case 4.

$$\frac{\psi_3}{g} = \frac{c_2}{z}, \qquad \psi_3 g = b_1 z.$$

Again these give $g = cz$ and $\eta = b\,dz/z^2$, and the surface is part of a catenoid.

To prove the isoperimetric inequality for minimal annuli in the next section, we need further study the function $L(r)$ for normalised surfaces.

Lemma 25.8 *For a non-zero flux normalised surface,*

$$L(r) \geq 2\pi \quad \text{for all} \quad r. \tag{25.108}$$

Equality can hold for at most one value of r. Moreover $L(r_0) = 2\pi$ if and only if the circle $|z| = r_0$ maps onto a horizontal plane $x_3 = c$ and each radial direction along the circle maps into a vertical direction in $\mathbf{R}^3$.

Proof. Since $\overline{X_3}(r) = \log r$,

$$0 = \frac{1}{2\pi}\int_0^{2\pi} [X_3(re^{i\theta}) - \log r]d\theta.$$

Thus there is a well defined harmonic function v conjugate to $X_3 - \log r$ in A such that $f := X_3 - \log r + iv$ is holomorphic in A. Then by the Cauchy-Riemann equations and $r^2 = z\bar{z}$, we have

$$f' = (X_3 - \log r)_x + iv_x = (X_3)_x - (\log r)_x - i(X_3)_y + i(\log r)_y = \phi_3 - \frac{x - iy}{r^2} = \phi_3 - \frac{1}{z}$$

and $\psi_3(z) = 1 + zf'(z)$. Since $\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$, we have

$$\Lambda^2 = \frac{1}{2}\sum_{i=1}^{3} |\phi_i|^2 \geq \frac{1}{2}(|\phi_1^2 + \phi_2^2| + |\phi_3|^3) = |\phi_3|^2.$$

Thus

$$\begin{aligned} L(r) &= \int_0^{2\pi} \Lambda(re^{i\theta})r\, d\theta \geq \int_0^{2\pi} |\psi_3| d\theta \geq \left| \int_0^{2\pi} \psi_3(re^{i\theta})\, d\theta \right| \\ &= \left| \int_0^{2\pi} (1 + re^{i\theta} f'(re^{i\theta})) d\theta \right| \qquad (25.109) \\ &= \left| 2\pi - i \int_{|z|=r} f'(z) dz \right| = 2\pi. \end{aligned}$$

The fact that $L(r)$ can attain the minimum value 2π for at most one value of r is an immediate consequence of Theorem 25.5, which says L is a strictly convex function of $\log r$.

$L(r_0) = 2\pi$ if and only both of the inequalities in (25.109) become equalities. This means

$$r_0 \Lambda(r_0 e^{i\theta}) = |\psi_3(r_0 e^{i\theta})|, \quad 0 \leq \theta \leq 2\pi, \qquad (25.110)$$

$$\int_0^{2\pi} |\psi_3(r_0 e^{i\theta})| d\theta = \left| \int_0^{2\pi} \psi_3(r_0 e^{i\theta}) d\theta \right|. \qquad (25.111)$$

Using the relation $\psi_3(z) = 1 + zf'(z)$ gives

$$2\pi = \int_0^{2\pi} \psi_3(r_0 e^{i\theta}) d\theta = \int_0^{2\pi} \Re[\psi_3(r_0 e^{i\theta})] d\theta + i \int_0^{2\pi} \Im[\psi_3(r_0 e^{i\theta})] d\theta.$$

Hence

$$\int_0^{2\pi} \Im[\psi_3(r_0 e^{i\theta})] d\theta = 0$$

and

$$\begin{aligned} \left| \int_0^{2\pi} \psi_3(r_0 e^{i\theta}) d\theta \right| &= \left| \int_0^{2\pi} \Re[\psi_3(r_0 e^{i\theta})] d\theta \right| \leq \int_0^{2\pi} |\Re[\psi_3(r_0 e^{i\theta})]| d\theta \qquad (25.112) \\ &\leq \int_0^{2\pi} |\psi_3(r_0 e^{\theta})| d\theta. \end{aligned}$$

For (25.111) to hold, we must have

$$\Im[\psi_3(r_0 e^{\theta})] = 0, \quad 0 \leq \theta \leq 2\pi. \qquad (25.113)$$

But

$$\begin{aligned} \psi_3 &= (x + iy)[(X_3)_x - i(X_3)_y] = x(X_3)_x + y(X_3)_y + i[y(X_3)_x - x(X_3)_y] \\ &= r(X_3)_r - i(X_3)_\theta. \qquad (25.114) \end{aligned}$$

Thus (25.113) holds if and only $X_3(r_0 e^{i\theta})$ is constant, and DX_3 is orthogonal to the circle $|z| = r_0$. From (25.110), at each point of $|z| = r_0$,

$$\Lambda = |DX_3| = |(X_3)_r|. \qquad (25.115)$$

But $\Lambda = |X_r|$, and so (25.115) holds if and only if $(X_i)_r = 0$ for $i = 1,\ 2$.

Conversely, X_r is vertical means that (25.115) holds, and this implies $(X_3)_r \neq 0$. Thus by (25.114), $\Re\psi_3$ cannot change sign on the circle $|z| = r_0$. The condition that X_3 is constant on this circle implies (25.113), again using (25.114). These two facts yield equality in (25.112), hence in (25.111), while (25.115) gives equality in (25.110). This completes the proof of the lemma. □

To prove the next theorem we need a lemma.

Lemma 25.9 *Let $f(t)$ satisfy $f''(t) \geq f(t)$ in some interval I. Then for all t_0, t in I,*

$$f(t) \geq f(t_0)\cosh(t - t_0) + f'(t_0)\sinh(t - t_0). \tag{25.116}$$

Equality holds for some $t_1 \neq t_0$ if and only if it holds for all t between t_0 and t_1 if and only if $f''(t) = f(t)$ for all t between t_0 and t_1.

Proof. We have

$$\frac{d}{dt}\left[f'(t)\cosh t - f(t)\sinh t)\right] = (f'' - f)\cosh t \geq 0. \tag{25.117}$$

Hence $t > t_0$ implies

$$f'(t)\cosh t - f(t)\sinh t \geq f'(t_0)\cosh t_0 - f(t_0)\sinh t_0.$$

Thus

$$\frac{d}{dt}\left(\frac{f(t)}{\cosh t}\right) \geq [f'(t_0)\cosh t_0 - f(t_0)\sinh t_0]\frac{1}{\cosh^2 t}$$

and

$$\frac{f(t)}{\cosh t} - \frac{f(t_0)}{\cosh t_0} \geq [f'(t_0)\cosh t_0 - f(t_0)\sinh t_0](\tanh t - \tanh t_0).$$

Multiplying out and simplifying, we obtain (25.116).

An analogous argument holds for $t < t_0$.

For equality to hold, it must hold in (25.117), so that $f'' = f$. □

Theorem 25.10 *Let $X : A \hookrightarrow \mathbf{R}^3$ be a minimal annulus. Further assume (by a reparametrisation of the form $z = \zeta/c$ if necessary) that $L(r)$ attains a minimum L_0 for $r = 1$. Then the lengths of the boundary curves are greater than or equal to $L_0/2\pi$ times the lengths of the corresponding boundary circles of the standard catenoid (the Weierstrass data are $g = z$, $\eta = dz/z^2$) based on the same annulus. Equality can hold only if X is itself the standard catenoid.*

Proof. There are three cases, depending on whether $L(r)$ is increasing throughout, decreasing throughout, or has an interior minimum. Again using the notation $t = \log r$, the minimum occurs at $r = 1$ or $t = 0$. If we use primes to denote derivatives with respect to t, then by Theorem 25.5, $L''(t) \geq L(t)$, and by Lemma 25.9,

$$L(t) \geq L(0)\cosh t + L'(0)\sinh t.$$

In the case of an interior minimum, then $L'(0) = 0$, and

$$L(t_1) \geq L(0)\cosh t_1, \quad L(t_2) \geq L(0)\cosh t_2$$

for the values t_1, t_2 corresponding to the boundary curves. But we have seen that for the catenoid, the length function is $L(t) = 2\pi \cosh t$.

If L is decreasing, then the boundary values are $t = 0$ and $t = t_1 < 0$. Using $L'(0) \leq 0$ we obtain $L(t_1) \geq L(0)\cosh t_1$, and the result is again true. A similar procedure applies if L is increasing.

For equality to hold in any of these cases, it follows from Lemma 25.9 that $L''(t) = L(t)$. According to Theorem 25.5, X must be a standard catenoid or else a plane annulus. However, a direct computation shows that for the plane annulus one has either $L(t) = L(0)e^{t}$, $t \geq 0$, or $L(t) = L(0)e^{-t}$, $t \leq 0$, and which is strictly greater than $L(0)\cosh t$. $\square$