23 Annular Ends Lying above Catenoid Ends

The Cone Lemma (Theorem 21.1) gives a criterion for a minimal surface to have finite total curvature by looking at the picture of its image. In this section we will give another such criterion due to Fang and Meeks [18].

Consider the family of catenoids

$$C_t = \{(x, y, z) \in \mathbf{R}^3 \mid t^2x^2 + t^2y^2 = \cosh^2(tz)\},$$

for $t > 0$. We will show that a properly immersed, complete minimal annulus with one compact boundary that lies above some C_t must have finite total curvature. More precisely:

Theorem 23.1 *Let*

$$W_t = \{(x, y, z) \in \mathbf{R}^3 \mid t^2x^2 + t^2y^2 \leq \cosh^2(tz),\ z \geq 0\}.$$

Suppose $X\colon M \to \mathbf{R}^3$ is a complete, proper minimal immersion of an annulus with smooth compact boundary such that the image is contained in W_t for some $t > 0$. Then M has finite total curvature.

We will break the proof of Theorem 23.1 into several lemmas. First let us fix the notation.

Let C be a catenoid in $\mathbf{R}^3$ with the z-axis as symmetry axis. Let W be the closure of the component of $\mathbf{R}^3 - C$ that contains the z-axis. Let $\mathbf{H} = \{(x, y, z) \in \mathbf{R}^3 \mid z > 0\}$ and $\overline{\mathbf{H}}$ be its closure.

Conformally we can write $M = \{\zeta \in \mathbf{C} \,|\, 0 < r_1 \leq |\zeta| < r_2\}$. The smooth compact boundary of X corresponding to $|\zeta| = r_1$. Complete means that $X \circ \gamma$ has infinite arc length as γ diverges to $|\zeta| = r_2$. Let $A = X(M)$.

After homothetically shrinking or expanding C and A, we can assume that C is the standard catenoid, i.e., C has the conformal structure of $\mathbf{C} - \{0\}$ and is embedded in $\mathbf{R}^3$ as follows:

$$F : \mathbf{C} - \{0\} \hookrightarrow \mathbf{R}^3$$

$$F(\zeta) = \Re\left(\int_1^\zeta \omega_1, \int_1^\zeta \omega_2, \int_1^\zeta \omega_3\right) + (-1, 0, 0),$$

where

$$\omega_1 = \frac{1}{2}\frac{(1-\zeta^2)}{\zeta^2}\, d\zeta,\quad \omega_2 = \frac{i}{2}\frac{(1+\zeta^2)}{\zeta^2}\, d\zeta,\quad \omega_3 = \frac{d\zeta}{\zeta}.$$

The Gauss map of C is

$$N^C(\zeta) = \frac{1}{1+|\zeta|^2}(2\Re\zeta, 2\Im\zeta, |\zeta|^2 - 1).$$

All lemmas in the following having the same assumptions as for Theorem 23.1.

The first lemma is the key point of the proof of Theorem 23.1.

Lemma 23.2 *Let $p \in \mathrm{Int}(M)$ and P be the tangent plane of A through $X(p)$ and suppose $P \cap \partial A = \emptyset$. Then the component of $P \cap A$ that contains $X(p)$ is noncompact.*

Proof. Since A is noncompact, we may assume that A is not part of a plane. If $\vec{n}$ is the normal vector of P, then $h = (X - X(p)) \bullet \vec{n}$ is a harmonic function on M and $X^{-1}(A \cap P) = h^{-1}(0)$. Since h is harmonic and $h^{-1}(0) \subset \mathrm{Int}(M)$, the maximum principle implies that every component of $h^{-1}(0)$ is a one-dimension analytic subvariety of M. Suppose that the component of $P \cap A$ containing $X(p)$ is compact. Let Δ denote the preimage of this component on M. Note that Δ is compact since X is proper. Furthermore, by Corollary 4.6, p is a critical point of the harmonic function h, thus Δ is a *singular* compact analytic one-dimensional variety in M. But the complement of any such singular variety in the annulus M disconnects M into at least three components. One of the components of $M - \Delta$ has $\{|\zeta| = r_2\}$ as a component of its boundary, another contains $\{|\zeta| = r_1\}$ and at least one, say Σ, has compact closure $\overline{\Sigma}$ and $h|\partial\overline{\Sigma} = 0$. By the maximum principle, $X(\Sigma) \subset P$, which forces A to be contained in the plane P. This contradiction proves the lemma. □

The second lemma clarifies the conformal type of M and gives a specific representation of the third coordinate function X_3.

Lemma 23.3 *If $A \subset W \cap \overline{\mathbf{H}}$ then A contains a proper subannulus A' that is conformally parametrized by $E = \{\zeta \in \mathbf{C} \mid |\zeta| \geq 1\}$. Moreover, in this parametrization $G : E \hookrightarrow \mathbf{R}^3$ of A', the third component of G is*

$$G_3(\zeta) = a \log|\zeta| + b$$

for some $a, b \in \mathbf{R}$, $a > 0$, $b \geq 0$.

Proof. Since $X = (X_1, X_2, X_3) : M \hookrightarrow \mathbf{R}^3$ is a proper minimal immersion and $A = X(M) \subset W \cap \overline{\mathbf{H}}$, $X_3 : M \to \mathbf{R}$ is a proper harmonic function.

We claim that X_3 is unbounded. In fact, if X_3 is bounded, then $A = X(M)$ is contained in a compact set, contradicting the fact that X is proper.

Then by properness and $A \subset W \cap \mathbf{H}$, $X_3(\zeta) \to \infty$ as $|\zeta| \to r_2$. If $r_2 < \infty$, letting $g_{ij} = e^{X_3}\delta_{ij}$, we get a complete flat metric on M. By Proposition 10.6 this is impossible. Thus $r_2 = \infty$.

We claim that if $X_3(\zeta) > c := \max_{\zeta \in \partial M}\{X_3(\zeta)\}$, then $DX_3(\zeta) \neq (0,0)$. In fact, if $DX_3(\zeta) = (0,0)$, then the tangent plane P of A at $X(\zeta)$ is horizontal, hence by Lemma 23.2 $A \cap P$ should have an uncompact component, which contradicts that $A \subset W$ and X is proper.

Now let $t > c_1 > c$. Then $\gamma = X_3^{-1}(c_1)$ and $\gamma_t = X_3^{-1}(t)$ are compact one-dimensional submanifolds of M and thus are Jordan curves. The annulus A_t bounded by γ and γ_t is conformally $M_{R(t)} := \{1 \leq |\zeta| \leq R(t)\}$ for some $R(t) > 1$. Let $f_t : A_t \to M_{R(t)}$ be the conformal diffeomorphism.

Solving a Dirichlet problem on $M_{R(t)}$ we have

$$X_3 \circ f_t^{-1}(\zeta) = c_1 + \frac{t - c_1}{\log R(t)} \log |\zeta|.$$

This shows that for any $t > s > c_1$, $f_t(\gamma_s)$ is the circle

$$|\zeta| = R(t)^{(s-c_1)/(t-c_1)},$$

hence f_t sends A_s to $M_{R(s)}$, where

$$R(s) = R(t)^{(s-c_1)/(t-c_1)}.$$

In particular,

$$\frac{t - c_1}{\log R(t)} = \frac{s - c_1}{\log R(s)}.$$

Since the modulus of A_s must be $R(s)$, we know that $f_t|_{A_s} = f_s$. Thus we can define a conformal diffeomorphism

$$f : \bigcup_{t \geq c_1} A_t \to E := \{\zeta \in \mathbf{C} \,|\, |\zeta| \geq 1\},$$

such that

$$X_3 \circ f^{-1}(\zeta) = c_1 + a \log |\zeta|, \quad a = \frac{t - c_1}{\log R(t)}, \quad \text{for any} \quad t > c_1.$$

Taking $b = c_1$ and $G = X \circ f^{-1}$, we have proved the lemma. □

Suppose A' is the subannulus of A described in Lemma 23.3. Since A and A' both have finite total curvature or both have infinite total curvature, we will assume, without loss of generality, that $A = A'$.

Suppose now that A has infinite total curvature. We will exhibit a family of tangent planes P_n of A at $G(p_n)$ such that the component of $P_n \cap A$ containing $G(p_n)$ is compact. Furthermore, for n large enough, $P_n \cap \partial A = \emptyset$. The existence of such tangent planes contradicts Lemma 23.2.

For the part of C in $\overline{\mathbf{H}}$ we have the following non-parametric expression: $x^2 + y^2 = \cosh^2 z$, $z \geq 0$. Hence, at any point $p = (x, y, z) \in C \cap \overline{\mathbf{H}}$, the normal vector is

$$N^C(p) = \frac{1}{\sqrt{1 + z_x^2 + z_y^2}}(-z_x, -z_y, 1),$$

where $z_x = 2x/\sinh 2z$, $z_y = 2y/\sinh 2z$, and

$$1 + z_x^2 + z_y^2 = (\sinh^2 2z + 4\cosh^2 z)/\sinh^2 2z = [4\cosh^2 z(\sinh^2 z + 1)]/\sinh^2 2z$$

$$= 4\cosh^4 z/(4\cosh^2 z \sinh^2 z) = \cosh^2 z/\sinh^2 z.$$

Suppose $p = (x, y, z) \in C \cap \overline{\mathbf{H}}$. Let $\theta(p)$ be the angle such that

$$\cos\theta(p) = N^C(p)\bullet(0,0,1) = \frac{1}{\sqrt{1+z_x^2+z_y^2}} = \frac{\sinh z}{\cosh z}.$$

Then

$$\sin\theta(p) = \sqrt{1-\cos^2\theta(p)} = \frac{1}{\cosh z}.$$

Thus $\sin\theta(p)$ is independent of x and y. We denote it by $\sin\theta(z)$. For $p_0 = (x_0, y_0, z_0) \in A \cap W \cap \overline{\mathbf{H}}$, $z_0 \geq 1$, consider the solid cylinders

$$L^{z_0} = \{(x,y,z) \in \mathbf{R}^3 \mid x^2 + y^2 \leq \cosh^2(z_0+1)\},$$

$$L_1^{z_0} = \{(x,y,z) \in L^{z_0} \mid z_0 - 1 \leq z \leq z_0 + 1\}.$$

If P is a plane passing through $p_0 = (x_0, y_0, z_0)$ and ν_P is the normal vector of P, define $-\pi/2 \leq \Psi_P \leq \pi/2$ by the formula $\cos\Psi_P = |\nu_P\bullet(0,0,1)|$.

Lemma 23.4 *If z_0 is large and*

$$|\Psi_P| < \frac{1}{16\cosh z_0} = \frac{\sin\theta(z_0)}{16},$$

then the component of $P \cap A$ that contains p_0 is compact and $P \cap \partial A = \emptyset$.

Proof. Since $p_0 = (x_0, y_0, z_0) \in L_1^{z_0}$, for any $(x,y,z) \in P \cap \partial L^{z_0}$ we have

$$|z - z_0| \leq 2\cosh(z_0+1)\tan|\Psi_P| = 2\cosh(z_0+1)\frac{\sin|\Psi_P|}{\cos|\Psi_P|}.$$

Since $\cos|\Psi_P| > \frac{1}{2}$ and $|\Psi_P| < \frac{1}{16\cosh z_0}$,

$$|z - z_0| < 4\frac{\cosh(z_0+1)}{16\cosh z_0}.$$

Note that $\cosh(z_0+1) = \cosh z_0 \cosh 1 + \sinh z_0 \sinh 1$, $\sinh 1 < \cosh 1 < 2$, and $\sinh z_0 < \cosh z_0$. Hence, $\cosh(z_0+1) < 4\cosh z_0$, and so $|z - z_0| < 1$. Hence, $P \cap \partial L^{z_0} = P \cap \partial L_1^{z_0}$ and $P \cap L^{z_0} = P \cap L_1^{z_0}$. This implies that the component γ of $A \cap P$ that contains p_0 must be compact (since $\gamma \subset P \cap L_1^{z_0}$ and $L_1^{z_0}$ is compact).

Let $z_0 - 1 > \max_{x\in\partial A}\{|x|\}$, then clearly $P \cap \partial A = \emptyset$. □

Now we prove Theorem 23.1.

Proof of Theorem 23.1. Assume A has infinite total curvature. Let $g : E \to \mathbf{C}\cup\{\infty\}$ be the Gauss map of A composed with stereographic projection. Similarly define $\tilde{g} : \mathbf{C} - \{0\} \to \mathbf{C}\cup\{\infty\}$ to be the Gauss map of C composed with stereographic projection. Recall, in fact, that in our original parametrization F of C, $\tilde{g}(\zeta) = \zeta$ for $\zeta \in \mathbf{C} - \{0\}$.

Since A has infinite total curvature, g has an essential singularity at ∞. Recall that the Gauss map of C is

$$N^C(\zeta) = \frac{1}{1+|\zeta|^2}(2\Re\zeta, 2\Im\zeta, |\zeta|^2 - 1)$$

for $\zeta \in E$, and the Gauss map of A is

$$N^A(\zeta) = \frac{1}{1+|g(\zeta)|^2}(2\Re g(\zeta), 2\Im g(\zeta), |g(\zeta)|^2 - 1).$$

Also, recall that $\sin\theta(x,y,z) = \frac{1}{\cosh z}$. For any $(x,y,z) = F(\zeta)$, $\cos\theta(z) = N^C \bullet (0,0,1) = \frac{|\zeta|^2-1}{1+|\zeta|^2}$, so

$$\sin\theta(z) = \sqrt{1-\cos^2\theta(z)} = \frac{2|\zeta|}{1+|\zeta|^2}. \tag{23.86}$$

Similarly define the angle $-\pi/2 \leq \phi(\zeta) \leq \pi/2$ such that $\cos\phi(\zeta) = N^A \bullet (0,0,1) = \frac{|g(\zeta)|^2-1}{1+|g(\zeta)|^2}$. Then

$$\sin\phi(\zeta) = \sqrt{1-\cos^2\phi(\zeta)} = \frac{2|g(\zeta)|}{1+|g(\zeta)|^2}. \tag{23.87}$$

Since $z = G_3(\zeta) = a\log|\zeta| + b = F_3(\zeta^a \cdot \exp b)$, for some $a > 0$, $b \geq 0$,

$$\frac{\sin\phi(\zeta)}{\sin\theta(z)} = \frac{|\zeta^a \cdot \exp b|}{|g(\zeta)|}\left(\frac{1+1/|\zeta^a \cdot \exp b|^2}{1+1/|g(\zeta)|^2}\right). \tag{23.88}$$

Choose a positive integer $m > a$. Since $(\zeta^m \cdot \exp b)/g(\zeta)$ has an essential singularity at ∞, there is a divergent sequence $\{\zeta_n\}$ such that $|\zeta_n^m \cdot \exp b|/|g(\zeta_n)| \to 0$ as $n \to \infty$.

Delete a ray l in $\mathbf{C}$ such that l does not contain any ζ_n. Then on $\mathbf{C} - l$, ζ^a is well-defined and

$$\frac{|\zeta_n^a \cdot \exp b|}{|g(\zeta_n)|} < \frac{|\zeta_n^m \cdot \exp b|}{|g(\zeta_n)|} \to 0 \tag{23.89}$$

as $n \to \infty$. In particular, $g(\zeta_n) \to \infty$ as $n \to 0$. So $\theta(F_3(\zeta_n^a \cdot \exp b)) \to 0$, $\phi(\zeta_n) \to 0$ as $n \to \infty$. We see by (23.88) and (23.89) that

$$\frac{\phi(\zeta_n)}{\sin\theta(F(\zeta_n^a \cdot \exp b))} = \frac{\phi(\zeta_n)}{\sin\phi(\zeta_n)} \bullet \left(\frac{\sin\phi(\zeta_n)}{\sin\theta(F(\zeta_n^a \cdot \exp b))}\right) \to 0, \tag{23.90}$$

as $n \to \infty$. Here $\sin\theta(F_3(\zeta_n^a \exp b)) = \sin\theta(z_n) = 1/\cosh z_n$, and $z_n = F_3(\zeta_n^a \exp b) = G_3(\zeta_n) \to \infty$ as $n \to \infty$.

By Lemma 23.4, we can choose n so large that the tangent plane of A at $G(\zeta_n)$ does not intersect ∂A. By (23.90), we can also choose n so that

$$\frac{\phi(\xi_n)}{\sin\theta(F(\zeta n^a \cdot \exp b))} < 1/16.$$

It follows from Lemma 23.4 that the tangent plane of A at $G(\zeta_n)$ will have a compact component that contains $G(\zeta_n)$. The existence of such a tangent plane contradicts Lemma 23.2. This contradiction proves the theorem. $\square$

Remark 23.5 Rosenberg and Toubiana [73] have shown that there exist minimally immersed annuli in $\overline{\mathbf{H}}$ with proper third coordinate function which have infinite total curvature. Theorem 23.1 shows that such annuli do not lie above any catenoid.