3 The First Variation

Let $X : M \hookrightarrow \mathbf{R}^3$ be a regular surface and $(U, (x, y))$ be a coordinate neighbourhood. Let $X_1 = X_x$, $X_2 = X_y$, $g_{ij} = X_i \bullet X_j$, and $g = \det(g_{ij})$. Then

$$dA := \sqrt{g}\, dx \wedge dy$$

is a well defined two form on M and $dA \neq 0$ everywhere.

Let $f : M \to \mathbf{R}$ be a continuous function of compact support, or suppose f does not change sign on M, then the integral of f on M is defined by

$$\int_M f := \int_M f\, dA.$$

When M is precompact and $f \equiv 1$, $\int_M dA$ is the area of the surface $X : M \hookrightarrow \mathbf{R}^3$.

The adjective "minimal" of minimal surfaces comes from the fact that at any point of the surface there exists a neighbourhood such that the surface in that neighbourhood has the least area among all surfaces with the same boundary.

To be precise, let $\Omega \subset M$ be a precompact domain and $X : \Omega \to \mathbf{R}^3$ be a surface. Let $X(t) : \Omega \to \mathbf{R}^3$, $-1 < t < 1$ and $X(0) = X$, such that $X(t)|_{\partial\Omega} = X|_{\partial\Omega}$, and $X(t,p) = X(t)(p)$ is C^2 on $\Omega \times (-1, 1)$. Such a family of surfaces is called a *variation of* X.

Consider the *area functional*

$$A(t) = \int_\Omega dA_t,$$

where dA_t is the area form induced by $X(t)$. The definition of minimal surface from the point view of the calculus of variations is that for any variation family $X(t)$,

$$\frac{dA(t)}{dt}\Big|_{t=0} = 0. \tag{3.2}$$

We will prove that this is another equivalent definition of minimal surface.

Without loss of generality, we may assume that X is conformal. Let $p \in \Omega$ and $p \in U \subset \Omega$ be an isothermal coordinate neighbourhood of p for X. On U, dA_t is expressed as

$$dA_t = \sqrt{\det[g_{ij}(t)]}\, dx \wedge dy,$$

where $z = x + iy$ is the isothermal coordinate and $g_{ij}(t) = X_i(t) \bullet X_j(t)$ (note that z may not be an isothermal coordinate for $X(t)$). Hence

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \int_U dA_t &= \int_U \frac{d}{dt}\Big|_{t=0} dA_t = \int_U \frac{d\sqrt{\det[g_{ij}(t)]}}{dt}\Big|_{t=0} dx \wedge dy \\ &= \frac{1}{2}\int_U \frac{d \det[g_{ij}(t)]}{dt}\Big|_{t=0} \{\det[g_{ij}(0)]\}^{-1/2} dx \wedge dy. \end{aligned}$$

We need the formula

$$\frac{d\det(g_{ij}(t))}{dt} = \det(g_{ij}(t))\,\mathbf{Trace}\left(\left(\frac{dg_{ij}(t)}{dt}\right)(g^{ij}(t))\right), \tag{3.3}$$

where $(g^{ij}(t)) = (g_{ij}(t))^{-1}$. To see this, let $(e_1, \ldots, e_n)$ be the standard orthonormal basis of $\mathbf{R}^n$. For any $n \times n$ matrix $A(t)$, we can write

$$A(t) = (A_1(t), \cdots, A_n(t)) = (A(t)e_1, \cdots, A(t)e_n),$$

where $A_i(t)$ is the i-th column of $A(t)$. If $\det A(t) \neq 0$, then

$$\begin{aligned}
\frac{d\det A(t)}{dt} &= \frac{d}{dt}\det(A(t)e_1, \cdots, A(t)e_n) \\
&= \sum_{i=1}^{n}\det\left(A(t)e_1, \cdots, \frac{dA(t)}{dt}e_i, \cdots, A(t)e_n\right) \\
&= \det A(t)\sum_{i=1}^{n}\det A^{-1}(t)\det\left(A(t)e_1, \cdots, \frac{dA(t)}{dt}e_i, \cdots, A(t)e_n\right) \\
&= \det A(t)\sum_{i=1}^{n}\det\left[A^{-1}(t)\left(A(t)e_1, \cdots, \frac{dA(t)}{dt}e_i, \cdots, A(t)e_n\right)\right] \\
&= \det A(t)\sum_{i=1}^{n}\det\left(e_1, \cdots, A^{-1}(t)\frac{dA(t)}{dt}e_i, \cdots, e_n\right) \\
&= \det A(t)\sum_{i=1}^{n}\det\left(e_1, \cdots, \sum_{j=1}^{n}\left(A^{-1}(t)\frac{dA(t)}{dt}\right)_{ji}e_j, \cdots, e_n\right) \\
&= \det A(t)\sum_{i=1}^{n}\det\left(e_1, \cdots, \left(A^{-1}(t)\frac{dA(t)}{dt}\right)_{ii}e_i, \cdots, e_n\right) \\
&= \det A(t)\sum_{i=1}^{n}\left(A^{-1}(t)\frac{dA(t)}{dt}\right)_{ii} = \det A(t)\,\mathbf{Trace}\left(A^{-1}(t)\frac{dA(t)}{dt}\right) \\
&= \det A(t)\,\mathbf{Trace}\left(\frac{dA(t)}{dt}A^{-1}(t)\right).
\end{aligned}$$

This establishes (3.3).

Thus we have

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\int_U dA_t &= \frac{1}{2}\int_U \frac{d\det(g_{ij}(t))}{dt}\Big|_{t=0}[\det(g_{ij}(0))]^{-1/2}dx \wedge dy \\
&= \frac{1}{2}\int_U \mathbf{Trace}\left[\left(\frac{dg_{ij}(t)}{dt}\right)(g^{ij}(t))\right]\Big|_{t=0}\sqrt{\det(g_{ij}(0))}dx \wedge dy.
\end{aligned}$$

Since X is conformal, we have $g^{ij}(0) = \Lambda^{-2}\delta_{ij}$. Thus

$$\mathbf{Trace}\left(\left(\frac{dg_{ij}(t)}{dt}\right)(g^{ij}(t))\right)\Big|_{t=0} = \sum_{ij}\frac{d\,g_{ij}(t)}{dt}g^{ij}(t)\Big|_{t=0} = \Lambda^{-2}\sum_{i=1}^{2}\frac{d\,g_{ii}(t)}{dt}\Big|_{t=0}.$$

Define the *variation field* E as

$$E(p) := \frac{dX(t)(p)}{dt}\Big|_{t=0}, \quad p \in \Omega.$$

Then

$$\frac{d\,g_{ii}(t)}{dt}\Big|_{t=0} = \frac{d(X_i \bullet X_i)}{dt}\Big|_{t=0} = 2E_i \bullet X_i.$$

Since (X_1, X_2, N) is a basis of $\mathbf{R}^3$, where N is the unit normal, we can write $E = \alpha X_1 + \beta X_2 + \gamma N$, where α, β, and γ are C^1 functions defined in Ω. Using $N \bullet X_i = 0$, $\gamma = E \bullet N$, and

$$\gamma \Lambda^{-2} \sum_{i=1}^{2} X_{ii} \bullet N = (E \bullet N)(\triangle_X X \bullet N) = 2(E \bullet N)(HN \bullet N) = 2H(E \bullet N),$$

we have

$$\begin{aligned}
\mathbf{Trace}\left(\left(\frac{dg_{ij}(t)}{dt}\right)(g^{ij}(t))\right)\Big|_{t=0} &= 2\Lambda^{-2} \sum_{i=1}^{2} E_i \bullet X_i \\
&= 2(\alpha_1 + \beta_2) + 2\Lambda^{-2}(\alpha\Lambda_1^2 + \beta\Lambda_2^2) - 2\gamma\Lambda^{-2} \sum_{i=1}^{2} X_{ii} \bullet N \\
&= 2(\alpha_1 + \beta_2) + 2\Lambda^{-2}(\alpha\Lambda_1^2 + \beta\Lambda_2^2) - 4H(E \bullet N).
\end{aligned}$$

Again since X is conformal, $\sqrt{\det(g_{ij}(0))} = |X_1|^2 = |X_2|^2 = \Lambda^2$, we have

$$\frac{d}{dt}\Big|_{t=0} \int_U dA_t = \frac{1}{2} \int_U \mathbf{Trace}\left(\left(\frac{dg_{ij}(t)}{dt}\right)(g^{ij})(t)\right)\Big|_{t=0} \Lambda^2 dx \wedge dy$$

$$= \int_U \mathbf{Div}(\Lambda^2(\alpha, \beta)) dx \wedge dy - 2\int_U H(E \bullet N) dA_0 = \int_{\partial U} \Lambda^2(\alpha, \beta) \bullet n \, ds - 2\int_U H(E \bullet N) dA_0,$$

where n and ds are the outward unit normal vector field and the line element of ∂U in the Euclidean metric respectively. Dividing Ω into a finite number of disjoint isothermal coordinate neighbourhoods U_i,

$$\sum_i \int_{\partial U_i \cap \Omega} \Lambda^2(\alpha, \beta) \bullet n_i \, ds_i = 0$$

since each arc in $\partial U_i \cap \Omega$ appears twice in the summation and with opposite unit normal. Moreover, because $\alpha = \beta = 0$ on $\partial\Omega$, we have

$$\sum_i \int_{\partial U_i} \Lambda^2(\alpha, \beta) \bullet n_i \, ds_i = \sum_i \int_{\partial U_i \cap \Omega} \Lambda^2(\alpha, \beta) \bullet n_i \, ds_i + \int_{\partial\Omega} \Lambda^2(\alpha, \beta) \bullet n \, ds = 0,$$

where in the last integral n and ds are the outward unit normal vector field and the line element of $\partial\Omega$ in the Euclidean metric. Thus we finally have the *first variational formula* for the surface area functional:

$$\frac{dA}{dt}\Big|_{t=0} = -2\int_\Omega H(E\bullet N)dA_0. \tag{3.4}$$

If X is minimal, then $H=0$, so $\frac{dA}{dt}\Big|_{t=0} = 0$. On the other hand, if X is a stationary point for the area functional $A(t)$ (for example, if X has minimal area among all surfaces with the same boundary), then $\frac{dA}{dt}\Big|_{t=0} = 0$ for any variation of X. Since E can be any vector field, $\frac{dA}{dt}\Big|_{t=0} = 0$ forces that $H \equiv 0$, that is, X is a minimal surface.

Finally we will give an area formula for surfaces in $\mathbf{R}^3$. Suppose $X : \Omega \hookrightarrow \mathbf{R}^3$ is an immersion; without loss of generality, we may assume that X is conformal. Let $\vec{n}$ be the unit *conormal* on $X(\partial\Omega)$, i.e., $\vec{n}$ is tangent to $X(\Omega)$ and is perpendicular to $X(\partial\Omega)$. Let ds be the line element of $X(\partial\Omega)$, (e_1, e_2) be the standard orthonormal basis on U_i in the Euclidean metric. Let $n_i = ae_1 + be_2$. The integral

$$\int_{\partial U_i \cap \partial\Omega} \Lambda^2(\alpha,\beta)\bullet n_i\, ds_i$$

can be rewritten as

$$\begin{aligned}
&\int_{\partial U_i\cap\partial\Omega} \Lambda^2(\alpha e_1 + \beta e_2)\bullet(ae_1 + be_2)\, ds_i \\
&= \int_{\partial U_i\cap\partial\Omega} \Lambda^2(a\alpha + b\beta)ds_i = \int_{\partial U_i\cap\partial\Omega} \Lambda^{-1}[E\bullet dX(n_i)]X^*(ds) \\
&= \int_{X(\partial U_i\cap\partial\Omega)} \Lambda^{-1}[E\bullet dX(n_i)]ds = \int_{X(\partial U_i\cap\partial\Omega)} (E\bullet\vec{n})ds,
\end{aligned}$$

since $E = \alpha X_1 + \beta X_2 + \gamma N$, $dX(n_i) = aX_1 + bX_2$, $X^*(ds) = \Lambda ds_i$, and $\vec{n} = \Lambda^{-1}dX(n_i)$. Thus if we do not assume that α and β vanish on $\partial\Omega$, we have the first variation formula

$$\frac{dA}{dt}\Big|_{t=0} = -2\int H(E\bullet N)dA_0 + \int_{X(\partial\Omega)} (E\bullet\vec{n})ds. \tag{3.5}$$

Now let $a \in \mathbf{R}^3$ be any fixed vector; then $X(t)(p) = t(X(p)-a)$ is a variation of X, not fixed on boundary. Clearly $E(t)(p) = X(p) - a$ is the variation vector field independent of t. An easy calculation shows that

$$g_{ij}(t) = t^2 g_{ij}, \quad g^{ij}(t) = t^{-2}g^{ij}, \quad h_{ij}(t) = t\, h_{ij}.$$

Hence

$$dA_t = t^2 dA_1 = t^2 dA, \quad H(t) = t^{-1}H,$$

where $H = H(1)$, etc. Note that

$$A := \textbf{Area of } X(\Omega) = \int_\Omega dA,$$

and

$$A(t) := \textbf{Area of } X(t)(\Omega) = \int_\Omega dA_t = t^2 A.$$

Since $E(t) = X - a$, by (3.5)

$$2A = -2\int H[(X-a)\bullet N]dA + \int_{X(\partial\Omega)} [(X-a)\bullet \vec{n}]ds. \tag{3.6}$$

This formula is useful when we derive the isoperimetric inequalities for minimal surfaces.