INTRODUCTION

The central concept in these lectures is that of a formal system. This is quite a different notion from that of a postulate system, as naively conceived a half century ago. In the older conception a mathematical theory consisted of a set of postulates and their logical consequences. The trouble with that idea is that no one knows exactly what a logical consequence is, and the paradoxes have shown that our intuitive feelings on the subject are not reliable. In the modern conception this vague and subjective notion is replaced by the objective one of derivability according to explicitly stated rules.

A formal system is a body of propositions, which we shall call elementary propositions, concerning which we have a very precise and objective criterion of truth. This criterion has a recursive character. We start with a set of these propositions the axioms - which are stated to be true outright as part of the definition of the system; and to these we add explicit rules for deriving further true propositions from those already established It is then understood that an elementary proposition is a theorem - i.e., is true - if and only if it is an axiom or is derived from the axioms by the rules. It is further required that the specifications as to rules, axioms, etc., be definite, in the sense that there be a finite constructive process for deciding in any given case whether the concept applies or not. Thus the truth of an elementary proposition, although not necessarily itself a definite concept (the system has a relatively trivial character when it is), is nevertheless precise and objective in that the checking of evidence for it - i.e., of a proof - is a definite process.

This notion of formal system is fundamental, in one form or another, to many types of modern logical investigations. Frequently the systems studied have to do with symbols as subject matter. Thus in Hilbert's metamathematics the elementary propositions are of the form

"a is provable"

where a is a certain type of combination of the symbols of an exactly specified "language." But that is not essential. One can, if one likes, set up high-school algebra as a formal system, in which the elementary propositions are the equations

$$a = b.$$

the axioms are the instances of the "laws of algebra," and the rules are the customary ones for manipulating equations.[1]

Now we do not study a formal system simply by deriving elementary theorems one after the other. Once the system has been precisely defined, we can talk about it significantly by the devices of ordinary language. For such a study of a formal system, going beyond the step-by-step derivations of the elementary theorems, I shall use the prefix "epi-"[2] and I shall speak of *epi-propositions*, *epi-theorems*, and the *epi-theoretic* method in an obvious sense. Thus the consistency of Hilbert's mathematics, if established, would be an epitheorem of his metamathematics; and indeed practically all of the work on his system is epi-theoretic in character. These epi-propositions, since they are formed from elementary propositions by means of ordinary language, are subject to the dangers of such an origin; and it may be important to find systematic ways of clarifying them.

The simplest epi-propositions are those formed from elementary ones by one or more applications of the logical connectives "or," "and," "if..., then...," "not," and the quantifiers "all" and "some." Such epi-propositions will be called *compound propositions*, with the understanding that when compound propositions are combined by these connectives the result is again compound. The principal concern of these lectures is with the clarification of compound propositions. For the moment we confine attention to the first three of the connectives, which will be denoted by "$\wedge$", "$\vee$", and "$\supset$" respectively. In ordinary language it is customary to define them by truth tables, viz.:

Table 1

A	B	$A \wedge B$	$A \vee B$	$A \supset B$
1	1	1	1	1
1	0	0	1	0
0	1	0	1	1
0	0	0	0	1

Here "1" represents truth and "0" falsity. Truth tables for compound propositions of higher order are to be defined by compounding these tables in obvious fashion. But in the present connection there are two objections to such a procedure. In the first place, although it is clear when an elementary proposition is true, yet it is by no means clear when it is false or even what its falsity means. We shall see later that the negation of

1. This is done in Example 9 of [23].

2. Previously, e.g., in [16], I used the prefix "meta-" in this sense. This usage was criticized by Kleene [54]. It is true that the prefix "meta-" is preoccupied for a semiotical concept. (See Chapter I §4.) Although the two concepts are related I am convinced it is worth while to distinguish them notationally.

a proposition can be defined in different ways. In the second place, even in those cases where formal truth is definite, Table 1 does not represent the senses which we wish to give to these connectives (particularly implication). What then are these senses? Is it possible to define them objectively for a large class of formal systems so as to do justice to our intentions? We shall see that, under certain limitations, we can; and that we are led to a form of predicate calculus akin to that proposed by Heyting.[3]

This study of compound propositions is of interest in another connection. It has been noticed by several persons, that, even in ordinary language, the two-valued propositional algebra does not adequately represent the intuitive logical relationships of actual propositions, particularly as regards implication. As a result there have been proposed a variety of improved systems; e.g., the strict implication of C. I. Lewis.[4] Yet these systems have a subjective character. There has always been doubt as to which of them was most adequate. Lewis, in fact, considered five different systems, and declared himself unable to decide which of them "expresses the acceptable principles of deduction." But from the present point of view, where we replace the notion of logical consequence by that of derivability according to stated rules, we can give objective definitions of the logical connectives, and determine what sort of logical calculus follows from these definitions. That calculus will then evidently express the acceptable principles of deduction so far as mathematics is concerned.

This situation suggests that our objective may well give a mode of approach to the algebra of propositions and calculus of predicates which will be significant apart from its bearing on questions related to formal systems. Indeed the variables of an ordinary propositional algebra may be regarded as the elementary propositions of a degenerate, vacuous formal system. Whether or not this point of view is fruitful, it at least allows the logical calculus to be seen from a new angle.

This second point of view invites a slight broadening of the investigation. It is a curious fact that all the earlier attempts at "strict implication" were based, not on implication as basic idea, but on some notion of possibility or necessity. This suggests there is some interest in adding necessity and possibility to the operations for forming compound propositions. A method for dealing with these notions is to postulate, not a single formal system, but a whole family of them with some common

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3. This system was originally presented in [44]; cf. Glivenko [37]. A summary of the system is given in Feys [30] and Heyting [90]. For later development of this system see [62] and papers referred to therein.

4. See Lewis and Langford [57], especially Appendix 2, pp. 492-502. For recent developments concerning the Lewis system see McKinsey and Tarski [62] and papers referred to therein. Feys [30] gives a general survey of modal systems.

structure; then a proposition is necessary if it holds in all systems of the family, possible if it holds in at least one. This idea is not unrelated to recent analyses of modal logic by Carnap and McKinsey,[5] but it is totally independent of them, and indeed - at least so far as necessity is concerned - it antedates them.[6] The fact that such different approaches lead to similar results may well be interesting to those who are concerned about modal systems. Accordingly, the last chapter of these lectures is devoted to these notions.

The method of this investigation is that of Gentzen's thesis.[7] To him is due the idea of deriving the properties of the logical connectives from rules which, so to speak, flow directly from the associated meanings. This makes his approach to the logical calculus the most rapid and natural that has yet been made. Moreover, Gentzen anticipated some portions of a recent book by Carnap.[8] Indeed, Gentzen's introduction of "Sequenzen" with multiple antecedents and consequents is in effect equivalent to Carnap's involution;[9] thus Gentzen has a formulation of the classical predicate calculus which is a full formalization in the sense of Carnap.[10] Finally, Gentzen proved, for his L systems, a theorem of far-reaching importance to the effect that one of his rules - called "Schnitt"[11] - is superfluous. This theorem is here called the elimination theorem, because, from the present point of view, the rule in question does not flow as naturally from the meanings of the connectives as do the others; it represents not so much a rule as a property which it is desirable to establish. Indeed the proof of the elimination theorem in the epitheory of a formal system $\mathfrak{S}$ is tantamount to showing the consistency of the

5. For Carnap's theory see [10]. Carnap's book [9] was not available to me during the writing of these lectures. For McKinsey's theory see [60].

6. See the preface.

7. See Gentzen [35]. For expositions of Gentzen's methods see Bernays [3] and Feys [31] and [32]. Bernays considers only the "natural" systems for the classical calculus, here called the system TK. The papers by Feys give a summary not only of Gentzen's work but of other works of the same character by Hertz (see especially [89]; other references in [31] or [12]), Jaśkowski [48], and Ketonen [53]. The paper [33] shows that Gentzen was influenced by Hertz; in this paper he showed that Hertz's complex rule "Syllogismus" can be replaced by his own "Schnitt." The work of Jaśkowski, which he credited in part to Łukasiewicz, appears to have been independent of Gentzen. A recent paper [68] of Popper is of a character similar to the others mentioned, but shows no evidence of any contact with either Gentzen or Jaśkowski. This paper contains serious errors; see review to [67] in Mathematical Reviews (forthcoming). [It has since appeared, l.c. 9:321 (1948).]

8. [5].

9. L.c. § 32.

10. Carnap nowhere mentions Gentzen, so I conclude Carnap arrived at his analysis independently.

11. This rule appears to have been introduced in Gentzen [33].

system formed by adjoining a predicate calculus to $\mathfrak{S}$. Gentzen's proof of the consistency of arithmetic ([34], [36]) can be looked at in that light, and the same principle is back of some recent attempts at consistency proofs about which I have reported elsewhere [15].

No acquaintance, however, with Gentzen's technique will be assumed here. It is a part of the purpose of these lectures to give an exposition of that technique from the beginning. This will be done so that the clarification of the compound propositions of a formal system, an idea which does not occur in Gentzen, is still the main point of departure. Not only is that problem of some interest in its own right - it was the principal motivation for such research as these pages contain; but it sheds new light on the Gentzen methods and the logical calculus generally. From that standpoint, for example, the naturalness of the Heyting calculus (for the appropriate type of negation) is manifest; looked at from another angle we have a genuine interpretation of that calculus. The classical calculus emerges as a special case. Considerations relating to the classical calculus will be treated in parallel so as to have a complete exposition of Gentzen's main results.[12] Naturally there will be some difference from Gentzen's own treatment, both in the mode of approach and the details of the proofs.

On account of the emphasis on an underlying formal system, it will be necessary to precede the development of the theme proper by a discussion of the nature of formal systems which is more full than that in the opening paragraph. The connectives will then be introduced in the following order: 1) the finite positive connectives, $\supset$, $\wedge$, and $\vee$; 2) quantifiers; 3) negation; and 4) modalities. The treatment of the last will be less detailed and has a somewhat unfinished character.

It may be necessary to warn the reader about one detail. The relation between propositions A and B symbolized by the validity of $A \supset B$ is appropriately called formal deducibility, but it is not the most general situation under which the validity of B may be inferred from that of A. It is the case where there is a valid deduction according to certain rules leading from A to B. But it may happen that there is a valid process of transforming a proof of A into a proof of B without there being any deduction from A to B. An example of this, if we carry the formalization to a higher level, as indicated below, is the elimination theorem.

12. However, the more advanced aspects of Gentzen's theory as applied to the classical calculus - e.g., his strengthened Hauptsatz - are not considered here. For these cf. Ketonen [53].