AN APPLICATION OF ERGODIC THEOREMS IN THE THEORY OF QUEUES

DAVID M. G. WISHART
UNIVERSITY OF BIRMINGHAM

1. Introduction

We establish some notation: $\{Z_k, k \geqq 0\}$ is a homogeneous Markov chain taking its values in a locally compact Hausdorff space $\mathfrak{X}$; we denote by Σ the σ-algebra of subsets of $\mathfrak{X}$ generated by the open sets; $ca(\Sigma)$ is the Banach space of totally finite regular measures on Σ; and $C_0(\mathfrak{X})$ is the Banach space of real-valued, bounded, continuous functions on $\mathfrak{X}$ which vanish at infinity. If $\Phi_k \in ca(\Sigma)$ is the probability measure of Z_k then there exists a bounded linear operator T on $ca(\Sigma)$ into itself such that $\Phi_{k+1} = T\Phi_k$. If this operator can be represented by a real-valued function P on the product space $\mathfrak{X} \times \Sigma$ with the properties

(a) $0 \leqq P(x, F) \leqq P(x, \mathfrak{X}) = 1$ for all $x \in \mathfrak{X}$, $F \in \Sigma$;
(b) for each $x \in \mathfrak{X}$, $P(x, \cdot) \in ca(\Sigma)$;
(c) for each $F \in \Sigma$, $P(\cdot, F)$ is Σ-measurable;

then the mapping of $ca(\Sigma)$ into itself takes the form

$$\Phi_{k+1}(F) = \int_{\mathfrak{X}} \Phi_k(dx)P(x, F) \tag{1}$$

for each $F \in \Sigma$. We define inductively a sequence of real-valued functions $P_r(\cdot, \cdot)$ on $\mathfrak{X} \times \Sigma$ by the relations

$$\begin{aligned} P_{r+1}(x, F) &= \int_{\mathfrak{X}} P_r(x, dy)\, P_1(y, F) \\ P_1(x, F) &\equiv P(x, F). \end{aligned} \tag{2}$$

We may identify the conditional probability $P\{Z_{k+r} \in F | Z_k = x\}$ with the function $P_r(x, F)$, so that the rth iterate of the operator T may be written

$$(T^r\Phi)(F) = \int_{\mathfrak{X}} \Phi(dx)\, P_r(x, F). \tag{3}$$

A principal problem of ergodic theory has been to determine conditions under which the sequence of operators $n^{-1} \sum_{r=0}^{n-1} T^r$ converges in some sense. The underlying Banach space on which T is defined induces various topologies on the space of bounded linear operators. Since the norm of an arbitrary operator S is well defined by $||S|| = \sup_{\mu} ||S\mu||/||\mu||$ we can ask whether there exists a bounded linear operator T_1 such that $\lim_{n\to\infty} ||n^{-1} \sum_{r=0}^{n-1} T^r - T_1|| = 0$. A weaker convergence requires the existence of an operator T_1 such that $\lim_{n\to\infty} ||(n^{-1} \sum_{r=0}^{n-1} T^r)\mu - T_1\mu|| = 0$ for each $\mu \in ca(\Sigma)$. Yosida and Kakutani [12] call these the uniform ergodic theorem and the mean ergodic theorem respectively, and they have proved that

(I) the uniform ergodic theorem holds if T is a quasi-strongly compact operator, that is, if there exist a compact operator V and an integer p such that $||T^p - V|| < 1$;

(II) the mean ergodic theorem holds if T is a quasi-weakly compact operator, that is, if there exist a weakly compact operator V and an integer p such that $||T^p - V|| < 1$.

If an operator T is (weakly) compact then it satisfies condition (II) I automatically. The problem of identifying a quasi-weakly compact operator is still open. In this connection Kendall has shown [7] that if T is a bounded linear operator on $ca(\Sigma)$ into itself which sends positive elements into positive elements of equal norm, and if T is the adjoint of an operator on $C_0(\mathfrak{X})$, then T is not quasi-weakly compact.

In section 2 we shall illustrate this theorem by means of an example from the theory of queues. Yet, having exhibited an operator T to which the theorems of Yosida and Kakutani do not apply, we may still ask if there is not some other sense in which the sequence of operators $n^{-1} \sum_{r=0}^{n-1} T^r$ converges. In section 3 we shall examine further our operator and show that (if the system is subject to some small restraint) it possesses a unique stationary distribution Γ. Then the ergodic theorem for Markov chains holds without any further restrictions on the transition probability $P(\cdot, \cdot)$.

Theorem (*Doob* [2]-*Kakutani* [4]). *For each $B \in \Sigma$ there exists a set $E_B \in \Sigma$ such that $\Gamma(E_B) = 0$ and*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} P_r(x_0, B) = Q(x_0, B) \tag{4}$$

for $x_0 \notin E_B$, where Q is almost everywhere bounded, nonnegative, Σ-measurable, and invariant in the sense that $Q(y, B) = \int_{\mathfrak{X}} P(y, dz)\, Q(z, B)$ for Γ-almost all y.

In section 4 we shall show that for our queuing example the transition probability $P(x, \cdot)$ is absolutely continuous with respect to the stationary distribution $\Gamma(\cdot)$; then the limit (4) becomes universal in x_0 and the set function $Q(x_0, \cdot)$ is a stationary distribution for all x_0.

2. The transition operator

We shall discuss the queuing system usually denoted by $M/G/1$ whose primary characteristics are:

(i) the interarrival intervals, u_n, are assumed to be independent and identically distributed according to the law $dA(u) \equiv \exp(-\alpha u)\alpha\, du$, where $0 < u < \infty$;

(ii) the service times, v_n, are assumed to be independent of each other and of the u_n, and to be identically distributed according to the law $dB(v)$, where $0 < v < \infty$; $B(0+) = 0$. It will be found advisable to restrict the support D of $B(\cdot)$. If $B(V) = 1$ and $B(V - \epsilon) < 1$ for some finite V and all $\epsilon > 0$, then we define D as the closed interval $[0, V]$; if on the other hand $B(V) < 1$ for every finite V we take D as the half-closed real line $[0, \infty)$.

We shall assume that $0 < \alpha < \infty$, and $0 < b \equiv \int v dB(v) < \infty$. The *traffic intensity* of this system is then defined to be $\rho \equiv \alpha b$. (Recent descriptions of this system have been given by Cox [1], Gaver [3], Kendall [6], Takács [10], and Wishart [11].)

Our discussion will center on the stochastic process $\{[N(t), y(t)], t \geqq 0\}$ where $N(t)$ is the number of customers in the system at time t and $y(t)$ is the unexpended service time of the customer receiving service at time t. This process takes its values in the phase space

$$\mathfrak{X} = \{(r, x) : r = 0, 1, 2, \cdots ; x = 0 \text{ when } r = 0, \text{ and } x \in D \text{ otherwise}\}, \tag{5}$$

which was also introduced by Keilson and Kooharian [5]. We define certain special subsets $X_r \subset \mathfrak{X} : X_r \equiv \{(r, x) : x \in D\}$ for $r \geqq 1$; $X_0 \equiv \{(0, 0)\}$. The general subset of $\mathfrak{X}$ is then of the form $F = \bigcup F_r$ where $F_r \subset X_r$. We take as the open sets of $\mathfrak{X}$ those sets F whose components F_r are open in the usual topology for X_r. The compact sets in this topology are those sets F which are the union of a finite number of sets F_r compact in the usual topology for X_r (that is, F_r closed and bounded in X_r). With this topology $\mathfrak{X}$ is a locally compact Hausdorff space. We shall denote by Σ (Σ_r) the class of Borel subsets of $\mathfrak{X}$ (X_r): that is, the smallest σ-algebra of subsets of $\mathfrak{X}$ (X_r) containing the open sets.

We introduce now a class of bounded continuous functions on $\mathfrak{X}$ which vanish at infinity. For any real-valued function f on $\mathfrak{X}$ we define functions f_r on X_r by the relations $f[(0, 0)] = f_0, f[(r, x_r)] = f_r(x_r)$. If we require that $f_r \in C_0(X_r)$ with the usual supremum norm, and $\lim_{r\to\infty} ||f_r|| = 0$, and if we write $f \equiv [f_0, f_1, f_2, \cdots]$ with linear operations defined termwise, then the linear set thus defined is complete in the topology generated by the norm $||f|| = \max \{|f_0|, \sup_{r \geqq 1} ||f_r||\}$. It follows from the condition $\lim_{r\to\infty} ||f_r|| = 0$ that the functions of this set vanish at infinity, since if we are given $\epsilon > 0$ we can find N such that $||f_r|| < \epsilon$ for all $r > N$ and therefore the set $\{(r, x) : f_r(x) \geqq \epsilon\}$ is the union of at most N compact subsets $F_r \subset X_r$: that is, $\{(r, x) : f_r(x) \geqq \epsilon\}$ is a compact subset of $\mathfrak{X}$. We shall denote this normed linear space of bounded continuous functions on $\mathfrak{X}$ by $C_0(\mathfrak{X})$.

Because of the simplicity of the negative exponential input we shall make extensive use of the linear subset $\mathcal{E} \subset C_0(\mathfrak{X})$ spanned by functions of the form

$$e = [1, ze^{-sx_1}, z^2e^{-sx_2}, z^3e^{-sx_3}, \cdots], \tag{6}$$

where $(r, x_r) \in X_r$ and s, z are real numbers satisfying $0 < s < \infty$, $0 \leqq z < 1$. If

$$e' = [1, z_1 e^{-s' x_1}, z_1^2 e^{-s' x_2}, \cdots] \tag{7}$$

we define ring multiplication in $\mathcal{E}$ by the convention

$$ee' = [1, (zz_1)e^{-(s+s')x_1}, (zz_1)^2 e^{-(s+s')x_2}, \cdots] \tag{8}$$

and with this definition $\mathcal{E}$ is an algebra. The elements of $\mathcal{E}$ separate points in the sense that if $(n_1, y_1) \neq (n_2, y_2)$ we can find an $e \in \mathcal{E}$ such that $z^{n_1}e^{-sy_1} \neq z^{n_2}e^{-sy_2}$. Finally we observe that the elements of $\mathcal{E}$ have no common zero other than the point at infinity of $\mathfrak{X}$. Hence the elements of the closure of $\mathcal{E}$ (in the strong topology on $C_0(\mathfrak{X})$) have no other common zero, and so, by an appeal to the Stone-Weierstrass theorem [9], we can assert that $\mathcal{E}$ is dense in $C_0(\mathfrak{X})$ in the strong topology.

The space $[C_0(\mathfrak{X})]^*$ adjoint to $C_0(\mathfrak{X})$ is the space $ca(\Sigma)$ of finite regular measures on Σ. We may represent an element $\Psi \in ca(\Sigma)$ in vector form $\Psi = [\Psi_0, \Psi_1, \Psi_2, \cdots]$, where $-\infty < \Psi_0 < \infty$ and $\Psi_r \in ca(\Sigma_r)$ with the usual norm $||\Psi_r||$ = total variation of Ψ_r over Σ_r. Then the norm of Ψ is given by $||\Psi|| = |\Psi_0| + \sum_{r=1}^{\infty} ||\Psi_r|| < \infty$. We shall use the notation (Ψ, f) for the value at the point $f \in C_0(\mathfrak{X})$ of the linear functional Ψ which is isomorphic to the measure $\Psi \in ca(\Sigma) : (\Psi, f) = \Psi_0 f_0 + \sum_{r=1}^{\infty} (\Psi_r, f_r)$ where $(\Psi_r, f_r) = \int_{X_r} f_r \, d\Psi_r$. The symbol $\hat{f}$ will denote the element of $[C_0(\mathfrak{X})]^{**}$ which is isomorphic to the element $f \in C_0(\mathfrak{X})$ under the natural mapping $(\Psi, f) = (\hat{f}, \Psi)$ for all Ψ, and $\hat{C}_0(\mathfrak{X})$ will denote the natural embedding of $C_0(\mathfrak{X})$ in $[C_0(\mathfrak{X})]^{**}$. If e is an element of $\mathcal{E}$ we shall write $\psi(s, z) = (\Psi, e)$.

Lastly we observe that if μ, ν are elements of $ca(\Sigma)$, then μ is absolutely continuous with respect to ν if and only if μ_r is absolutely continuous with respect to ν_r for all r. We shall use Halmos' notation and write $\mu \ll \nu$ if and only if $\mu_r \ll \nu_r$ for all r.

It was noted by D. G. Kendall [6] that the process $[N(t), y(t)]$ is Markovian for all $t \geqq 0$. We have investigated elsewhere its behavior in continuous time [11]. In this paper we shall confine our attention to the set of arrival epochs

$$\Pi = \{t_k : N(t_k) = N(t_k - 0) + 1, k = 0, 1, 2, \cdots \}, \tag{9}$$

taking $t_0 = 0$. Then, writing

$$[N_k, y_k] = [N(t_k - 0), y(t_k - 0)], \qquad t_k \in \Pi, \tag{10}$$

the Markov chain $\{[N_k, y_k], k = 0, 1, 2, \cdots \}$ is homogeneous with a transition operator on $ca(\Sigma)$ to itself which we shall denote by T: that is, if $\Phi_k \in ca(\Sigma)$ is the probability measure of $[N_k, y_k]$ then Φ_{k+1}, the probability measure of $[N_{k+1}, y_{k+1}]$, is given by $\Phi_{k+1} = T\Phi_k$. It is our purpose to show that T is the adjoint of an operator on $C_0(\mathfrak{X})$.

We may represent this operator by a real-valued function P on the product space $\mathfrak{X} \times \Sigma$ with the properties (a) to (c) enumerated in the introduction, so that the mapping of $ca(\Sigma)$ into itself which we have defined above takes the form (1).

The function P also determines an operator $\bar{T}$ on the linear space $\mathfrak{M} \subset [\mathrm{ca}(\Sigma)]^*$ of bounded measurable functions into itself by the relation

$$g \to \bar{T}g = \int_{\mathfrak{X}} P(\cdot, dx)g(x), \qquad g \in \mathfrak{M}, \tag{11}$$

and this mapping is the contraction to $\mathfrak{M}$ of the mapping T^* on $[ca(\Sigma)]^*$ to itself defined by $(h, T\mu) = (T^*h, \mu)$ for $h \in [ca(\Sigma)]^*$ and every $\mu \in ca(\Sigma)$. $\bar{T}$ is a positive operator of norm 1 leaving invariant the function which is constant everywhere on $\mathfrak{X}$.

Since the Markov chain is homogeneous it will be sufficient to investigate a single relation, $\Phi_1 = T\Phi_0$, say. The suffixes may be dropped and we shall from now on write Φ and Ψ for Φ_0 and Φ_1 respectively, so that $\Psi = T\Phi$. In the notation established above we have $\varphi(s, z) = (\Phi, e)$ and $\psi(s, z) = (\Psi, e)$ for $e \in \mathcal{E}$. It is illuminating to cast these functions into the same form; we have

$$\varphi(s, z) = (\Phi, e) = (\hat{e}, \Phi) \tag{12}$$

and

$$\psi(s, z) = (\Psi, e) = (T\Phi, e) = (\hat{e}, T\Phi) = (T^*\hat{e}, \Phi). \tag{13}$$

Since $\hat{e} \in \mathfrak{M}$, we have $T^*\hat{e} = \bar{T}\hat{e}$ which, by equation (11), we can write as

$$\bar{T}\hat{e} = \mathbf{E}[z^{N_1}e^{-sy_1}|\,[N_0, y_0]]. \tag{14}$$

The last expression for $\psi(s, z)$ is still not very useful. However, $\bar{T}\hat{e}$ is a function which we can calculate and we shall show that $\bar{T}\hat{e} \in \hat{\mathcal{E}}$. Since $\mathcal{E}$ is dense in $C_0(\mathfrak{X})$ we conclude that T^* leaves $\hat{C}_0(\mathfrak{X})$ invariant in $[C_0(\mathfrak{X})]^{**}$. Consequently there is an operator S on $C_0(\mathfrak{X})$ to itself such that $S^* = T$, and the element $\bar{T}\hat{e}$ in $\hat{C}_0(\mathfrak{X})$ is isomorphic to Se in $C_0(\mathfrak{X})$. Hence the expression (13) for $\psi(s, z)$ may be rewritten as $\psi(s, z) = (T\Phi, e) = (\Phi, Se)$ which we now proceed to compute.

Underlying the present study is the product space

$$\mathfrak{X} \times \mathfrak{U}_1 \times \mathfrak{U}_2 \times \cdots \times \mathfrak{V}_1 \times \mathfrak{V}_2 \times \cdots, \tag{15}$$

where $\mathfrak{X}$ is the range space of $[N_0, y_0]$, $\mathfrak{U}_i$ is the range space of the random variable u_i and is a copy of the real line, and $\mathfrak{V}_j$ is the range space of the random variable v_j and is a copy of D. The distribution of $[N_0, y_0]$ is Φ so we may set up the product measure

$$\Phi \times A \times A \times \cdots \times B \times B \times \cdots, \tag{16}$$

since all the random variables are independent.

In order to evaluate

$$\mathbf{E}[z^{N_1}e^{-sy_1}|N_0 = m, y_0 = x], \tag{17}$$

we suppose first that $m \geqq 1$, and we write $V_j = v_1 + \cdots + v_j$. Then t_1 may occur in one of the following nonoverlapping intervals $(0, x)$, $(x, x + v_1)$, $(x + v_1, x + v_1 + v_2)$, $\cdots$, $(x + V_j, x + V_{j+1})$, $\cdots$, $(x + V_m, \infty)$; with $[N_1, y_1]$ taking the corresponding values $(m+1, x-u_1)$, $(m, x+v_1-u_1)$, $(m-1, x+v_1+v_2-u_1)$, $\cdots$, $(m-j, x+V_{j+1}-u_1)$, $\cdots$, $(0,0)$. We can therefore write down the expectation of $z^{N_1}\exp(-sy_1)$ with respect to the distribution of u_1 as a sum of terms

$$(18)\qquad \begin{aligned}&\int_0^x z^{m+1}e^{-s(x-u_1)}e^{-\alpha u_1}\alpha\,du_1 + \int_x^{x+v_1} z^m e^{-s(x+v_1-u_1)}e^{-\alpha u_1}\alpha\,du_1\\ &\qquad + \cdots + \int_{x+V_{m-1}}^{x+V_m} z^1 e^{-s(x+V_m-u_1)}e^{-\alpha u_1}\alpha\,du_1 + \int_{x+V_m}^{\infty} z^0 e^{-\alpha u_1}\alpha\,du_1\\ &= z^{m+1}\frac{\alpha}{\alpha-s}(e^{-sx}-e^{-\alpha x}) + z^m\frac{\alpha}{\alpha-s}e^{-\alpha x}(e^{-sv_1}-e^{-\alpha v_1})\\ &\qquad + \cdots + z\frac{\alpha}{\alpha-s}e^{-\alpha x-\alpha V_{m-1}}(e^{-sv_m}-e^{-\alpha v_m}) + z^0 e^{-\alpha x-\alpha V_m}.\end{aligned}$$

If we now take the expectation with respect to the product measure in the space $\mathfrak{V}_1\times\cdots\times\mathfrak{V}_m$ we obtain

$$(19)\qquad \begin{aligned}&\mathbf{E}[z^{N_1}e^{-sy_1}|N_0=m, y_0=x]\\ &= z^{m+1}\frac{\alpha}{\alpha-s}(e^{-sx}-e^{-\alpha x})\\ &\qquad + \frac{\alpha}{\alpha-s}e^{-\alpha x}[B^*(s)-B^*(\alpha)]\sum_{r=1}^{m} z^r[B^*(\alpha)]^{m-r} + [B^*(\alpha)]^m e^{-\alpha x},\\ &\qquad\qquad B^*(s) = \int_0^\infty e^{-sv}\,dB(v).\end{aligned}$$

This is for $m \geqq 1$. If $m=0$, t_1 may occur in $(0, v_0)$ or in (v_0, ∞), so that $[N_1, y_1]$ may take the values $(1, v_0-u_1)$ or $(0,0)$ and we have therefore two terms only,

$$(20)\qquad z^1\int_0^{v_0} e^{-s(v_0-u_1)}e^{-\alpha u_1}\alpha\,du_1 + z^0\int_{v_0}^{\infty} e^{-\alpha u_1}\alpha\,du_1,$$

and when we integrate with respect to v_0 we obtain

$$(21)\qquad \mathbf{E}[z^{N_1}e^{-sy_1}|N_0=y_0=0] = z\frac{\alpha}{\alpha-s}[B^*(s)-B^*(\alpha)] + B^*(\alpha).$$

If we write $e_{s,z}$ for e and e_0 for the vector $[1, 0, 0, \cdots]$ then the equations (18) and (21) may be written together in the form

$$(22)\qquad \begin{aligned}T^*\hat{e}_{s,z} &= \frac{z\alpha}{\alpha-s}(\hat{e}_{s,z}-\hat{e}_{\alpha,z}) + \hat{e}_{\alpha,B^*(\alpha)}\\ &\qquad + \frac{z\alpha}{\alpha-s}\,\frac{B^*(s)-B^*(\alpha)}{z-B^*(\alpha)}(\hat{e}_{\alpha,z}-\hat{e}_{\alpha,B^*(\alpha)})\\ &\qquad + \left\{\frac{z\alpha}{\alpha-s}[B^*(s)-B^*(\alpha)] + B^*(\alpha) - 1\right\}\hat{e}_0.\end{aligned}$$

Hence $T^*\hat{e}$ is an element of $\hat{\mathcal{E}}$ and since the e are dense in $C_0(\mathfrak{X})$ it follows that T^* leaves $\hat{C}_0(\mathfrak{X})$ invariant in $[C_0(\mathfrak{X})]^{**}$. If we write S for the contraction of T^* to

$\hat{C}_0(\mathfrak{X})$ transfered to $C_0(\mathfrak{X})$, then S (being a contraction of T^*) is a bounded linear operator. Also, for all $f \in C_0(\mathfrak{X})$

$$(S^*\Phi, f) = (\Phi, Sf) = (\widehat{Sf}, \Phi) = (T^*\hat{f}, \Phi) = (\hat{f}, T\Phi) = (T\Phi, f) \tag{23}$$

holds. Therefore $S^* = T$, and the element $T^*\hat{e}$ which we have calculated is isomorphic to the element Se in $C_0(\mathfrak{X})$.

We have therefore shown that the transition operator T associated with the Markov chain $\{[N_k, y_k], k \geqq 0\}$ is not quasi-weakly compact and consequently that the ergodic theorems of Yosida and Kakutani cannot be applied to this system.

3. The stationary distribution

Since $T^*\hat{e}$ is isomorphic with the element Se in $C_0(\mathfrak{X})$ we have seen that we may write $\psi(s, z) = (\Phi, Se)$. Hence, regarding (16) as an operator equation in $\mathcal{E}$ and taking the inner product with respect to Φ, we obtain

$$\begin{aligned}\psi(s, z) = {} & \frac{z\alpha}{\alpha - s}[\varphi(s, z) - \varphi(\alpha, z)] + \varphi[\alpha, B^*(\alpha)] \\ & + \frac{z\alpha}{\alpha - s}\frac{B^*(s) - B^*(\alpha)}{z - B^*(\alpha)}\{\varphi(\alpha, z) - \varphi[\alpha, B^*(\alpha)]\} \\ & + \phi_0\left\{\frac{z\alpha}{\alpha - s}[B^*(s) - B^*(\alpha)] + B^*(\alpha) - 1\right\},\end{aligned} \tag{24}$$

where $\phi_0 = \Phi[(0, 0)] = (\Phi, e_0)$.

We seek solutions (if any) of the equation $\psi = \lambda\varphi$. The existence of a solution of this equation implies the existence of an element $\Phi \in ca(\Sigma)$ such that $(T\Phi - \lambda\Phi, e) = 0$ for all $e \in \mathcal{E}$, and since $\mathcal{E}$ is dense in $C_0(\mathfrak{X})$ we conclude that Φ satisfies $T\Phi = \lambda\Phi$. If, further, $\Phi(F) \geqq 0$ for all $F \in \Sigma$, and $0 < \Phi(\mathfrak{X}) < \infty$ then Φ is readily normalized to a probability measure.

We set aside for the moment the question of positivity and investigate the existence of solutions of the equation $\psi = \lambda\varphi$ which satisfy the condition $0 < \varphi(0, 1) < \infty$. Putting first $z = 0$ we obtain $\varphi(\alpha, B^*(\alpha)) = \phi_0[1 + \lambda - B^*(\alpha)]$, and inserting this in (24) we have

$$\begin{aligned}\left(\lambda - \frac{z\alpha}{\alpha - s}\right)\varphi(s, z) = {} & \phi_0\left\{\frac{z\alpha}{\alpha - s}[B^*(s) - B^*(\alpha)]\frac{z - 1 - \lambda}{z - B^*(\alpha)} + \lambda\right\} \\ & + \varphi(\alpha, z)\frac{z\alpha}{\alpha - s}\frac{B^*(s) - z}{z - B^*(\alpha)}.\end{aligned} \tag{25}$$

The function $\varphi(s, z)$ is a regular function of the complex variables s and z in the region defined by the inequalities $\operatorname{Re} s > 0$, and $|z| < 1$. Consequently the right side of (25) must be zero at points s_λ defined by $s_\lambda = \alpha(\lambda - z)/\lambda$; substitution in (25) enables us now to determine $\varphi(\alpha, z)$. We obtain

$$(26) \qquad 0 = \phi_0 \left\{ \lambda [B^*(s_\lambda) - B^*(\alpha)] \frac{z - 1 - \lambda}{z - B^*(\alpha)} + \lambda \right\} + \varphi(\alpha, z) \lambda \frac{B^*(s_\lambda) - z}{z - B^*(\alpha)},$$

and introducing this into (19) we derive finally

$$(27) \qquad \left(\lambda - \frac{z\alpha}{\alpha - s} \right) \frac{\varphi(s, z)}{\phi_0} = \lambda + \frac{z\alpha}{\alpha - s} \left\{ (z - \lambda) \frac{B^*(s) - B^*(s_\lambda)}{z - B^*(s_\lambda)} - 1 \right\}.$$

Let us now assume that $\lambda \neq 1$, and impose the condition $0 < \varphi(0, 1) < \infty$; equation (27) yields $(\lambda - 1)\varphi(0, 1)/\phi_0 = \lambda + (1 - \lambda - 1) = 0$, which contradicts our assumption. There exist therefore no characteristic values of T other than $\lambda = 1$ which can give probabilistically meaningful results. When $\lambda = 1$ we have

$$(28) \qquad \varphi(s, z) = \phi_0 \left\{ 1 + \alpha z \frac{1}{1 - \dfrac{1 - K(z)}{1 - z}} \frac{B^*(s_1) - B^*(s)}{s - s_1} \right\},$$

where we have written

$$(29) \qquad K(z) = B^*(s_1) = \int_0^\infty e^{-\alpha(1-z)v}\, dB(v).$$

We note that $K(1) = 1$, and $K'(1) = \alpha b = \rho$. We also have obtained the function $\varphi(s, z)$ in [11] as

$$(30) \qquad \lim_{t \to \infty} \mathbf{E}[z^{N(t)} e^{-sy(t)}].$$

The power series expansion of $K(z)$ has all its coefficients positive and $K(0) = B^*(\alpha) > 0$, so that for real z it is a monotonically increasing function passing through the point (1, 1). If $\rho > 1$, then $K'(1) = \rho > 1$, and there exists a real zero of the function $K(z) - z$ at a point $z_0 < 1$, say. Rouché's theorem shows that $z = z_0$ is the only zero of $K(z) - z$ properly inside the unit circle. The right side of (28) has therefore a pole at $z = z_0$, whereas the left side, by construction, is regular for all z inside the unit circle. It is not therefore possible that there exists a stationary distribution when $\rho > 1$.

We shall show that when $\rho \leqq 1$ the right side of (28) is a regular function of z everywhere inside the unit circle. For

$$(31) \qquad G(z) \equiv \frac{1 - K(z)}{1 - z} = \sum_{r=0}^{\infty} z^r (k_{r+1} + k_{r+2} + \cdots)$$

is a power series with positive coefficients and

$$(32) \qquad G(1) = \lim_{z \to 1} \frac{1 - K(z)}{1 - z} = \rho \leqq 1.$$

When $\rho < 1$ it follows immediately from Rouché's theorem that $1 - G(z)$ has no zeros inside the unit circle, since on $|z| = 1$, we have $|G(z)| \leqq G(1) < 1$. When $\rho = 1$ we observe that $G(1 - \epsilon) < 1$ for every $\epsilon > 0$, from which we can conclude that $1 - G(z)$ has no zero inside any circle $|z| = 1 - \epsilon < 1$. Hence $1 - G(z)$ is regular and nonzero inside the unit circle when $\rho \leqq 1$ and therefore

$[1 - G(z)]^{-1}$ is also regular inside the unit circle when $\rho \leqq 1$, and our assertion is proved.

We have thus obtained an invariant function φ corresponding to a finite measure which we will denote by $\Gamma \in ca(\Sigma)$. For this function to be probabilistically meaningful we must normalize it, and this condition enables us to evaluate ϕ_0 (which from now on we must call Γ_0). We require $\varphi(0, 1) = 1$: then we obtain from (22)

$$\Gamma_0 = 1 - \rho, \qquad \rho \leqq 1. \tag{33}$$

(Note. Although we have shown that (22) is a regular function of z for $\rho \leqq 1$, the function $\varphi(s, z)$ can only be said to exist when $\rho < 1$.)

We shall now establish the positivity of the set function Γ. We consider $\varphi(s, z)$ given by (28) in the region of the real (s, z)-plane defined by the inequalities $0 \leqq z < 1$, $\alpha < s < \infty$. Since $s_1 = \alpha(1 - z)$ we have also $0 < s_1 \leqq \alpha < s$. We saw above that $G(z) = [1 - K(z)]/(1 - z)$ is a power series with positive coefficients, and $G(1) = \rho < 1$. Then for $0 \leqq z < 1$ we have $G(z) \leqq \rho < 1$, so that

$$(1 - \rho)\left\{1 - \frac{1 - K(z)}{1 - z}\right\}^{-1} = P(z) = \sum_{r=0}^{\infty} p_r z^r \tag{34}$$

is absolutely convergent in the same half-open interval and has also positive coefficients. Further,

$$\begin{aligned}
\frac{B^*(s_1) - B^*(s)}{s - s_1} &= \int_0^\infty \frac{e^{-s_1 v} - e^{-sv}}{s - s_1}\, dB(v) = \int_0^\infty e^{-s_1 v} \frac{1 - e^{-(s-s_1)v}}{s - s_1}\, dB(v) \\
&= \int_0^\infty e^{-s_1 v}\left\{\int_0^v e^{-(s-s_1)t}\, dt\right\} dB(v) \\
&= \int_0^\infty e^{-st}\, dt \int_t^\infty e^{-s_1(v-t)}\, dB(v) \\
&= \int_0^\infty e^{-st}\, dt \int_0^\infty e^{-\alpha(1-z)w}\, dB(w + t) \\
&= \int_0^\infty e^{-st}\, dt \sum_{r=0}^\infty z^r \int_0^\infty \frac{(\alpha w)^r}{r!} e^{-\alpha w}\, dB(w + t) \\
&= \int_0^\infty e^{-st}\, dt \sum_{r=0}^\infty z^r \beta_r(t),
\end{aligned} \tag{35}$$

say, where

$$\beta_r(t) = \int_0^\infty \frac{(\alpha w)^r}{r!} e^{-\alpha w}\, dB(w + t). \tag{36}$$

Consequently, the rth component of Γ has a density which we can write in the form

$$d\Gamma_r(t) = \alpha \sum_{n=0}^{r-1} p_{r-n}\beta_n(t)\, dt, \qquad 0 \leqq t < \infty; r \geqq 1. \tag{37}$$

$\beta_n(t)$ is positive for $0 \leqq t < \infty$ and we have shown that $P(z)$ has a power series expansion with positive coefficients so that Γ_r has a positive density with respect to the Lebesgue measure on (X_r, Σ_r) for each $r \geqq 1$. If D is the finite closed interval $[0, V]$ then we define

$$\beta_r(t) = \int_t^V \frac{[\alpha(v-t)]^r}{r!} e^{-\alpha(v-t)}\, dB(v). \tag{38}$$

With this modification equation (38) remains correct for $0 \leqq t \leqq V$. (In this case it will be usual that $\beta_r(t) > 0$ for $0 \leqq t < V$ but $\beta_r(V) = 0$.) Also $\Gamma_0 = 1 - \rho$. Hence $\Gamma(F) \geqq 0$ for every $F \in \Sigma$, as was to be shown.

We have therefore shown in this section that

(i) $\lambda = 1$ is the unique characteristic value of T;

(ii) there exist no solutions of $\psi = \varphi$ for $\rho \geqq 1$;

(iii) when $\rho < 1$ there exists a unique solution of $\psi = \varphi$ given by (28) with $\Gamma_0 = 1 - \rho$, and the inverse function Γ satisfies $T\Gamma = \Gamma$;

(iv) $\Gamma(F) \geqq 0$ for every $F \in \Sigma$.

4. Absolute continuity

In this section we prove that $P(\xi, \cdot)$ is absolutely continuous with respect to $\Gamma(\cdot)$ for every $\xi \subset \mathfrak{X}$.

Let us denote the transition probability $P(\xi, \cdot)$ as an element of $ca(\Sigma)$ by the vector $\pi^\xi = [\pi_0^\xi, \pi_1^\xi, \pi_2^\xi, \cdots]$. Then our assertion is proved when we have shown that $\pi_r^\xi \ll \Gamma_r$ for each r and every ξ. With this notation the expression (19) may be written as (π^ξ, e) where $\xi = (m, x)$. Thus we must compare the coefficient of z^r in (19) with the coefficient of z^r in (28).

We have already seen that when $\rho < 1$ there exists for each $r \geqq 1$ a Lebesgue summable function g_r such that

$$d\Gamma_r = g_r\, dx_r, \qquad r \geqq 1, \tag{39}$$

where $g_r(u) > 0$ for $0 \leqq u < \infty$.

We treat the terms of (19) in the same way: the coefficient of z^r, where $1 \leqq r \leqq m$, is

$$\begin{aligned}\alpha e^{-\alpha x}[B^*(\alpha)]^{m-r} & \frac{B^*(\alpha) - B^*(s)}{s - \alpha} \\ &= \alpha e^{-\alpha x}[B^*(\alpha)]^{m-r} \int_0^\infty e^{-st}\, dt \int_0^\infty e^{-\alpha w}\, dB\,(w + t)\end{aligned} \tag{40}$$

so that

$$d\pi_r^\xi = f_r^\xi\, dx_r, \tag{41}$$

where

$$f_r^\xi(t) = \alpha e^{-\alpha x}[B^*(\alpha)]^{m-r} \int_0^\infty e^{-\alpha w}\, dB\,(w + t), \qquad 0 \leqq t < \infty;\ 1 \leqq r \leqq m, \tag{42}$$

is everywhere positive [the modifications necessary when D is bounded will be clear from the remarks following equation (37)]; the coefficient of z^{m+1} is

$$\frac{e^{-\alpha x} - e^{-sx}}{s - \alpha} = \int_0^x e^{-st} e^{-\alpha(x-t)}\, dt \tag{43}$$

so that

$$d\pi_{m+1}^{\xi} = e^{-\alpha(x - x_{m+1})}\, dx_{m+1}, \qquad 0 \leqq x_{m+1} \leqq x. \tag{44}$$

Also $\pi_r^{\xi} = 0$ for $r > m + 1$. Hence

$$d\pi_r^{\xi} = h_r^{\xi}\, d\Gamma_r, \qquad r \geqq 1, \tag{45}$$

where $h_r^{\xi} = f_r^{\xi}/g_r$. But h_r^{ξ} will be indeterminate at infinity (if the support of $B(\cdot)$ extends so far); it may be indeterminate at V if the support of $B(\cdot)$ is bounded. In either case $\lim_{u\to\infty} h_r^{\xi}(u)$ or $\lim_{u\to V} h_r^{\xi}(u)$ can be evaluated, so that for all ξ and each $r \geqq 1$ the function h_r^{ξ} is well defined, nonnegative, and Γ_r-summable on X_r.

There remains $r = 0$. We need to show that $(\pi^{\xi}, e_0) = 0$ whenever $\Gamma_0 = 0$. But $\Gamma_0 = 1 - \rho$ which is never zero since we are concerned here only with the case $\rho < 1$. Hence the value of (π^{ξ}, e_0) does not concern us and we can write $\pi^{\xi} \ll \Gamma$ or $P(\xi, \cdot) \ll \Gamma(\cdot)$ for all ξ when $\rho < 1$.

Therefore, when $\rho < 1$, the sequence of partial sums $Q_n(x, F) = n^{-1} \sum_{r=1}^{n} P_r(x, F)$ determined by (2) converges to a function $Q(x, F)$ which is a stationary distribution for each x. But the argument of the last section showed that there is a unique stationary distribution Γ which is independent of the starting point x. Since, also, it is clear that $Q(x, \mathfrak{X}) = 1$ for all x, we can aver that $Q(x, F) \equiv \Gamma(F)$ for all x: that is,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} P_r(x, F) = \Gamma(F) \tag{46}$$

for all x.

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APPENDIX

Another Markov chain associated with the system $M/G/1$ is the waiting time of the rth customer. In fact, if $w(t)$ is the time a customer would have to wait if he entered the system at time t, then $\{w(t), t \geqq 0\}$ is a Markov process (Takács [10]). If we write $w_r = w(t_r - 0)$, for $t_r \in \Pi$, then $\{w_r, r \geqq 0\}$ is a homogeneous Markov chain and Lindley has shown [8] that if $F_r(x) = P\{w_r \leqq x\}$ then

$$F_{r+1}(x) = \int_0^{\infty} G(x - y) F_r(dy), \tag{47}$$

where

$$G(t) = P\{v_n - u_n \leqq t\} = \begin{cases} B(t) + e^{\alpha t} \int_t^{\infty} e^{-\alpha v}\, dB(v), & t \geqq 0, \\ e^{\alpha t} B^*(\alpha), & t \leqq 0. \end{cases} \tag{48}$$

Our phase space in this case is the nonnegative real line $\mathcal{R}$, and (47) determines a transformation T on the space of functions of bounded variation to itself. As in section 2 the transition probability $G(x - y)$ determines an operator $\overline{T}$ on the space of bounded measurable functions to itself by the relation

$$(\overline{T}g)(y) = \int_0^\infty g(x)G(dx - y). \tag{49}$$

The functions of the form $e_s(\cdot) \equiv \exp(-s\cdot)$ are dense in $C_0(\mathcal{R})$ and pursuing our previous argument we can show that

$$\overline{T}\hat{e}_s = \frac{sB^*(\alpha)}{s - \alpha}\hat{e}_\alpha - \frac{\alpha B^*(s)}{s - \alpha}\hat{e}_s. \tag{50}$$

It follows as before that T is the adjoint of an operator on $C_0(\mathcal{R})$ and so is not quasi-weakly compact. We can show also that $\lambda = 1$ is the unique characteristic value of T and obtain the well-known stationary distribution

$$(\overline{F}, e_s) = (1 - \rho)\left\{1 - \rho\frac{1 - B^*(s)}{bs}\right\}^{-1} \tag{51}$$

which exists when $\rho < 1$. Lastly,

$$G(x - y) = P\{w_r \leqq x | w_{r-1} = y\} \tag{52}$$

is absolutely continuous with respect to the distribution function $\overline{F}(x)$, and so the argument of section 4 may be repeated.

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