THE TRANSIENT BEHAVIOR OF A SINGLE SERVER QUEUEING PROCESS WITH A POISSON INPUT

LAJOS TAKÁCS
COLUMBIA UNIVERSITY

1. Introduction

There is a huge literature on the stationary behavior of different types of queueing processes but only a few papers deal with their transient behavior. An extensive bibliography of the theory of queues by A. Doig [9] contains about seven hundred papers, most of which deal with stationary queues. The theory of stationary queues is very important because most of the queueing processes are ergodic, that is, starting from any initial state the process tends toward equilibrium irrespective of the initial state. In the state of equilibrium the process shows only statistical fluctuation with no tendency to a certain state. Many queueing processes rapidly approach equilibrium and this explains why we can apply with success the stationary approximation. However, the investigation of the transient behavior of queueing processes is also important, not only from the point of view of the theory but also in the applications. For instance, if we apply the stationary solution instead of the transient solution we are interested in the error of this approximation, and further, even in the case of the stationary process the linear least squares prediction presupposes a knowledge of the transient behavior of the process.

The mechanism of queueing processes is very simple. Customers are arriving at a counter according to a certain probabilistic law (Poisson input, Erlang input, recurrent input, and so forth). The customers will be served by one or more servers following a certain principle (service in order of arrival, random service, priority service, last come first served, batch service, and so forth). The service times are random variables governed by a given probabilistic law. After service the customers depart.

We shall always use the above terminology. Every conceivable process can always be described in this terminology. For instance, in the case of a telephone traffic process the calls, lines, and holding times are replaced by customers, servers, and service times, respectively.

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Of course there are several variants of the above-mentioned queueing processes, for example, queueing with balking, queue with repeated arrivals, queueing with feedback, queue with batch arrivals, and so forth. Further, it is interesting to investigate the interaction of different queues, for instance, queues in parallel, queues in series, and so on.

The three most important characteristics in the theory of queues are as follows:

The *waiting time* of each customer.

The *busy period*, that is, the time interval during which one or more servers are busy.

The *queue size*, that is, the number of customers in the system.

The waiting time concerns the customers, the busy period concerns the servers, and the queue size is important from the point of view of the design of the system, for example, the size of the waiting room or the waiting facility in telephone exchanges.

The theory of queues, like the theory of probability, gives abstract models which are applied in many different fields. One of the most important models is as follows.

Customers arrive at a counter at the instants $\tau_1, \tau_2, \cdots, \tau_n, \cdots$. The customers will be served by m servers in the order of their arrival. Let us denote by χ_n the service time of the nth customer and write $\vartheta_n = \tau_{n+1} - \tau_n$. It is supposed that the interarrival times $\{\vartheta_n\}$ and the service times $\{\chi_n\}$ are independent sequences of identically distributed positive random variables with respective distribution functions $P\{\vartheta_n \leqq x\} = F(x)$ and $P\{\chi_n \leqq x\} = H(x)$. Such a queueing process can be described by the triplet $[F(x), H(x), m]$.

The simplest particular case of the above process is the following: $F(x) = 1 - \exp(-\lambda x)$ for $x \geqq 0$, $H(x) = 1 - \exp(-\mu x)$ for $x \geqq 0$, and $m = 1$, that is, the input is a Poisson process, the service times have an exponential distribution, and there is a single server. The transient behavior of this process was investigated by A. N. Kolmogorov [13], W. Lederman and G. E. H. Reuter [15], N. T. J. Bailey [1], D. G. Champernowne [5], A. B. Clarke [6], B. W. Conolly [7], and S. Karlin and J. McGregor [12]. In paper [12] the case of many servers is also investigated.

The transient behavior of the above-mentioned process in the case when $F(x) = 1 - \exp(-\lambda x)$ for $x \geqq 0$, $H(x)$ is arbitrary, and $m = 1$ was investigated by V. E. Beneš [2], F. Pollaczek [17], E. Reich [20], [21], J. T. Runnenburg [22], F. Spitzer [23], and the author [24].

The case when $F(x)$ is arbitrary, $H(x) = 1 - \exp(-\mu x)$ for $x \geqq 0$, and $m = 1$ was investigated by B. W. Conolly [8] and the author [25].

The transient behavior of the process $[F(x), H(x), 1]$, where either $F(x)$ or $H(x)$ has a Gamma distribution, is investigated by the author in papers [26] and [27].

In the present paper I shall consider a modification of the process $[F(x), H(x), 1]$, supposing that each customer arriving at a time when the server is not available leaves the queue without being served with probability q. I shall deal only with the particular case $F(x) = 1 - \exp(-\lambda x)$ for $x \geqq 0$, that is, when the input process is a Poisson process.

2. The process considered

Let us consider a counter with a single server, at which customers are arriving in the instants $\tau_1, \tau_2, \cdots, \tau_n, \cdots$. Suppose that the interarrival times $\tau_n - \tau_{n-1}$, where $n = 1, 2, \cdots; \tau_0 = 0$, are identically distributed independent positive random variables with distribution function $P\{\tau_n - \tau_{n-1} \leqq x\} = F(x)$, where

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geqq 0, \\ 0, & x < 0, \end{cases} \tag{1}$$

that is, the input process is a homogeneous Poisson process with density λ. If a customer arrives at the counter at an instant when the server is idle then his service starts immediately. If he arrives at an instant when the server is busy then he may or may not join the queue. Suppose that the event of joining the queue is independent of any other events. Let p be the probability that he joins the queue and $q = 1 - p$ that he does not. If a customer joins the queue then his service starts immediately after the departure of the preceding customer in the queue. Suppose that the durations of the successive service times are identically distributed independent positive random variables with distribution function $H(x)$ and further that they are also independent of $\{\tau_n\}$.

Denote by $\eta(t)$ the virtual waiting time at the instant t, that is, the time which a customer would wait if he joined the queue at the instant t. Denote by $\xi(t)$ the queue size at the instant t, that is, the number of customers waiting or being served at the instant t. We say that the system is in state E_k at the instant t if $\xi(t) = k$.

Let us denote by $\tau_1', \tau_2', \cdots, \tau_n', \cdots$ the instants of successive departures. Further define $\eta_n = \eta(\tau_n - 0)$, that is, η_n is the waiting time of the nth customer if he joins the queue at all and let $\xi_n = \xi(\tau_n' + 0)$, that is, ξ_n is the queue size immediately after the nth departure.

In the following we shall determine the transient behavior of the stochastic processes $\{\eta(t)\}$ and $\{\xi(t)\}$ and that of the stochastic sequences $\{\eta_n\}$ and $\{\xi_n\}$. Further we shall determine the asymptotic behavior of these processes and the stochastic law of the busy period.

In the particular case where $p = 1$ the transient behavior of the process $\{\eta(t)\}$ has been investigated earlier by the author [24], V. E. Beneš [2], E. Reich [20], [21] and J. T. Runnenburg [22] and the transient behavior of the sequence $\{\eta_n\}$ by F. Pollaczek [17], [18] and F. Spitzer [23]. Further, we mention the paper of D. P. Graver [11], in which the case $0 \leqq p \leqq 1$ is also mentioned.

The present generalization for $0 \leqq p \leqq 1$, has been motivated by the fact that in several queueing processes (for example, in telephone traffic) the arriving customers (callers), unaware of the size of the queue, know only whether the server is idle or busy and their decision to join the queue depends solely on this eventuality.

3. Notation

Let $\{\chi_n\}$ be a sequence of identically distributed independent random variables with distribution function $P\{\chi_n \leqq x\} = H(x)$ and suppose that $\{\chi_n\}$ is independent of $\{\tau_n\}$ also. Further define a sequence of random variables $\{\epsilon_n\}$ as follows: $\epsilon_n = 1$ if the nth customer joins the queue and $\epsilon_n = 0$ otherwise.

Let

$$\psi(s) = \int_0^\infty e^{-sx}\, dH(x), \tag{2}$$

which is convergent if $\mathrm{Re}(s) \geqq 0$, and

$$\alpha = \int_0^\infty x\, dH(x). \tag{3}$$

Define

$$\begin{aligned} W(t, x) &= P\{\eta(t) \leqq x\}, \\ W_n(x) &= P\{\eta_n \leqq x\}, \\ P_{ij}(t) &= P\{\xi(t) = j \mid \xi(0) = i\}, \\ P_j(t) &= P\{\xi(t) = j\}. \end{aligned} \tag{4}$$

Finally, introduce the transforms

$$\begin{aligned} \Omega(t, s) &= E\{e^{-s\eta(t)}\} = \int_0^\infty e^{-sx}\, d_x W(t, x), \\ \Omega_n(s) &= E\{e^{-s\eta_n}\} = \int_0^\infty e^{-sx}\, dW_n(x), \end{aligned} \tag{5}$$

whenever $\mathrm{Re}(s) \geqq 0$ and write

$$\hat{\Omega}(s) = E\{e^{-s\eta(0)}\} = \Omega(0, s). \tag{6}$$

4. An auxiliary theorem

Throughout this paper we use

LEMMA 1. *If* $\mathrm{Re}(s) \geqq 0$ *and* $|w| \leqq 1$ *then* $z = \gamma(s, w)$, *the root of the equation*

$$z = w\psi[s + \lambda p(1 - z)] \tag{7}$$

which has the smallest absolute value, is

$$\gamma(s, w) = w \sum_{j=1}^\infty \frac{(-\lambda p w)^{j-1}}{j!} \left(\frac{d^{j-1}[\psi(y)]^j}{dy^{j-1}}\right)_{y=\lambda p+s}. \tag{8}$$

This is a regular function of s and w if $\mathrm{Re}(s) \geqq 0$ *and* $|w| \leqq 1$ *and further* $z = \gamma(s, w)$ *is the only root of* (7) *in the unit circle* $|z| < 1$ *if* $\mathrm{Re}(s) \geqq 0$ *and* $|w| < 1$ *or* $\mathrm{Re}(s) > 0$ *and* $|w| \leqq 1$ *or* $\mathrm{Re}(s) \geqq 0$ *and* $|w| \leqq 1$ *and* $\lambda p\alpha > 1$. *Specifically,* $\omega = \gamma(0, 1)$ *is the smallest positive real root of the equation*

$$\omega = \psi[\lambda p(1 - \omega)]. \tag{9}$$

If $\lambda p\alpha > 1$ *then* $\omega < 1$ *and if* $\lambda p\alpha \leqq 1$ *then* $\omega = 1$.

PROOF. If $\operatorname{Re}(s) \geqq 0$ and $|w| < 1$ or $\operatorname{Re}(s) > 0$ and $|w| \leqq 1$, then by Rouché's theorem it follows that (7) has one and only one root in the unit circle $|z| < 1$. For in this case $|w\psi[s + \lambda p(1 - z)]| < 1$ if $|z| = 1$. Similarly, if $\operatorname{Re}(s) \geqq 0$, $|w| \leqq 1$, and $\lambda p\alpha > 1$, then by Rouché's theorem it follows that (7) has exactly one root in the circle $|z| < 1 - \epsilon$ where ϵ is a sufficiently small positive number. For in this case $|w\psi[s + \lambda p(1 - z)]| \leqq \psi(\epsilon) < 1 - \epsilon$ if $|z| = 1 - \epsilon$ and $\epsilon > 0$ is small enough. Let us denote this root by $z = \gamma(s, w)$. This can be obtained in the form of an infinite series by Lagrange's theorem. (See, for example, E. T. Whittaker and G. N. Watson [28], p. 132.) Clearly $z = \gamma(s, w)$ is that root of (7) which has the smallest absolute value and is a regular function of s and w in this domain.

On the other hand, if $z = \gamma(s, w)$ is that root of (7) which has the smallest absolute value, then $\gamma(s, w)$ is defined uniquely for $\operatorname{Re}(s) \geqq 0$ and $|w| \leqq 1$ as an inverse function of $s = \psi^{-1}(z/w) - \lambda p(1 - z)$ for fixed w, or as an inverse function of $w = z/\psi[s + \lambda p(1 - z)]$ for fixed s. It can be shown that the function $z = \gamma(s, w)$ is a regular function of s and w if $\operatorname{Re}(s) \geqq 0$ and $|w| \leqq 1$. Since this function agrees with the earlier one in the domain $\operatorname{Re}(s) \geqq 0$ and $|w| < 1$ or $\operatorname{Re}(s) > 0$ and $|w| \leqq 1$ or $\operatorname{Re}(s) \geqq 0$ and $|w| \leqq 1$ if $\lambda p\alpha > 1$; consequently the latter function is the analytical continuation of the former one to the domain $\operatorname{Re}(s) \geqq 0$ and $|w| \leqq 1$ and it is defined by (8).

We have always $|\gamma(s, w)| \leqq 1$ if $\operatorname{Re}(s) \geqq 0$ and $|w| \leqq 1$. Note also that (7) has at most one root (possibly double) on the unit circle $|z| = 1$, namely $z = 1$ is a root if $w\psi(s) = 1$.

It remains only to prove the second half of the lemma. Clearly for real x the function $\psi(x)$ is monotone decreasing if $0 \leqq x < \infty$ and $\psi'(0) = -\alpha$. Consequently, if $\lambda p\alpha > 1$ then (9) has only one real root in the interval $(0, 1)$ and if $\lambda p\alpha \leqq 1$ then $\omega = 1$ is the only real root of (9). Furthermore, the equation $z = \psi[\lambda p(1 - z)]$ has only one root in the unit circle $|z| < 1$ if $\lambda p\alpha > 1$ and has no root if $\lambda p\alpha \leqq 1$. This latter statement can be proved by probabilistic reasoning. (See remarks 6 and 8.) This completes the proof of the lemma.

We introduce the notation $\gamma(s) = \gamma(s, 1)$ and $g(w) = \gamma(0, w)$. Clearly $\omega = \gamma(0) = g(1)$.

Finally let us note that if $\omega = 1$, then by using (7) we get

$$\gamma'(0) = \begin{cases} \dfrac{-1}{1 - \lambda\alpha p}, & \lambda\alpha p < 1 \\ \infty, & \lambda\alpha p = 1 \end{cases} \tag{10}$$

and

$$g'(1) = \begin{cases} \dfrac{1}{1 - \lambda\alpha p}, & \lambda\alpha p < 1 \\ \infty, & \lambda\alpha p = 1. \end{cases} \tag{11}$$

REMARK 1. The function $\zeta(s, w) = s + \lambda p[1 - \gamma(s, w)]$ satisfies the equation $\zeta = (\lambda p + s) - \lambda p w \psi(\zeta)$ if $\text{Re}(s) \geqq 0$ and $|w| \leqq 1$. Conversely, if $\text{Re}(s) > 0$ and $|w| \leqq 1$ or $\text{Re}(s) \geqq 0$ and $|w| < 1$, then by Rouché's theorem it follows that this equation has only one root $\zeta = \zeta(s, w)$ in the domain $\text{Re}(\zeta) \geqq 0$. For $|\psi(\zeta)| < |(\lambda p + s - \zeta)/\lambda p w|$ if $\text{Re}(\zeta) = 0$ or $\text{Re}(\zeta) \geqq 0$ and $|\zeta|$ is sufficiently large.

5. The distribution of the virtual waiting time

The stochastic behavior of the process $\{\eta(t), 0 \leqq t < \infty\}$ can be described as follows: $\eta(0)$ is the initial occupation time of the server. If $\eta(0) = 0$ then the server is idle at time $t = 0$. If $\eta(0) \neq 0$ then $\eta(0)$ gives the instant when the server ceases to be busy for the first time if no new customer joins the queue. In the instants τ_n where $n = 1, 2, \cdots$ the value of $\eta(t)$ has a jump of magnitude χ_n, if the nth customer joins the queue. The value of $\eta(t)$ decreases linearly with slope -1 until it jumps or reaches 0. If at the instant t the $\eta(t)$ reaches 0 then it remains 0 until a customer arrives at the counter. (Compare figure 1.)

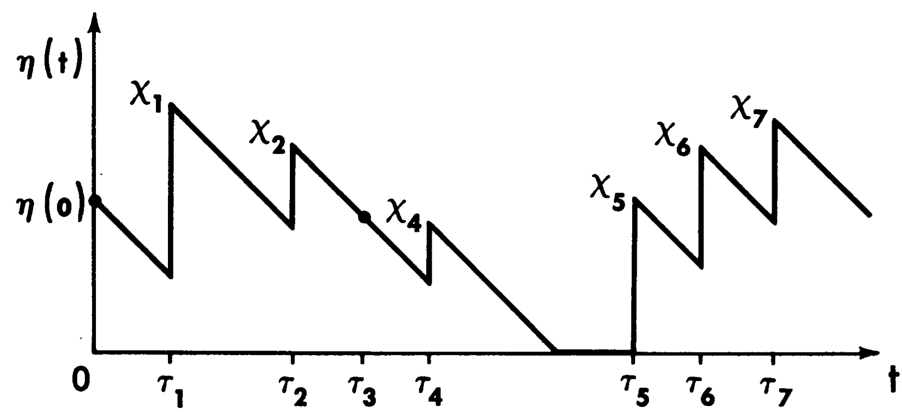

FIGURE 1

Graph of the stochastic behavior of the process $\{\eta(t), 0 \leqq t < \infty\}$ where $\eta(t)$ is the virtual waiting time at instant t.

The process described by the family of random variables $\{\eta(t), 0 \leqq t < \infty\}$ is a so-called continuous parameter Markov process of the mixed type. The change of state may happen continuously or by jumps. Processes of this type have been investigated by A. N. Kolmogorov [14] and W. Feller [10].

THEOREM 1. *The Laplace-Stieltjes transform of the distribution function of the virtual waiting time is given by*

$$\Omega(t, s) = \exp\{st - [1 - \psi(s)]\lambda p t\} \tag{12}$$

$$\left(\hat{\Omega}(s) - \{s + \lambda(1 - p)[1 - \psi(s)]\} \int_0^t \exp\{-su + [1 - \psi(s)]\lambda p u\} P_0(u) du\right),$$

where $P_0(t) = P\{\eta(t) = 0\}$ *can be obtained by the Laplace transform*

$$\int_0^\infty e^{-st} P_0(t)dt = \frac{\hat{\Omega}\{s + \lambda p[1 - \gamma(s)]\}}{s + \lambda - \lambda\gamma(s)} \tag{13}$$

if $\text{Re}(s) > 0$ *and* $z = \gamma(s)$ *is the root with the smallest absolute value of equation*

$$z = \psi[s + \lambda p(1 - z)]. \tag{14}$$

Proof. Denote by $\delta_{\Delta t}$ the number of customers arriving at the counter during the time interval $(t, t + \Delta t)$. By assumption

$$P\{\delta_{\Delta t} = j\} = e^{-\lambda\Delta t}\frac{(\lambda\Delta t)^j}{j!}, \qquad j = 0, 1, 2, \cdots. \tag{15}$$

Using the theorem of total probablilty we can write

$$\begin{aligned} \Omega(t + \Delta t, s) &= E\{e^{-s\eta(t+\Delta t)}\} \\ &= \sum_{j=0}^{\infty} P\{\delta_{\Delta t} = j\}E\{e^{-s\eta(t+\Delta t)}|\delta_{\Delta t} = j\}. \end{aligned} \tag{16}$$

Since $P\{\delta_{\Delta t} = 0\} = 1 - \lambda\Delta t + o(\Delta t)$, while $P\{\delta_{\Delta t} = 1\} = \lambda\Delta t + o(\Delta t)$ and $P\{\delta_{\Delta t} > 1\} = o(\Delta t)$, we have

$$\begin{aligned} \Omega(t + \Delta t, s) = (1 - \lambda\Delta t)E\{e^{-s\eta(t+\Delta t)}|\delta_{\Delta t} = 0\} \\ + \lambda\Delta t\, E\{e^{-s\eta(t+\Delta t)}|\delta_{\Delta t} = 1\} + o(\Delta t). \end{aligned} \tag{17}$$

Under the condition $\delta_{\Delta t} = 0$ we have

$$\eta(t + \Delta t) = \begin{cases} 0, & \eta(t) \leqq \Delta t, \\ \eta(t) - \Delta t, & \eta(t) > \Delta t, \end{cases} \tag{18}$$

whence

$$E\{e^{-s\eta(t+\Delta t)}|\delta_{\Delta t} = 0\} = W(t, \Delta t) + e^{s\Delta t}\int_{\Delta t}^\infty e^{-sx}\, d_x W(t, x). \tag{19}$$

Since $W(t, x)$ is right-continuous in x,

$$W(t, \Delta t) = W(t, 0) + O(\Delta t), \tag{20}$$

whence

$$0 \leqq \int_0^{\Delta t} x\, d_x W(t, x) \leqq \Delta t[W(t, \Delta t) - W(t, 0)] = o(\Delta t). \tag{21}$$

Thus

$$E\{e^{-s\eta(t+\Delta t)}|\delta_{\Delta t} = 0\} = (1 + s\Delta t)\Omega(t, s) - s\Delta t P_0(t) + o(\Delta t), \tag{22}$$

since $W(t, 0) = P_0(t)$.

On the other hand, if $y > \Delta t$,

$$E\{e^{-s\eta(t+\Delta t)}|\delta_{\Delta t} = 1, \eta(t) = y\} = [(1 - p) + p\psi(s)]e^{-s(y-\Delta t)} \tag{23}$$

for the customer arriving during the time interval $(t, t + \Delta t]$ joins the queue with probability p or goes away with probability $1 - p$.

If $y \leqq \Delta t$, then

$$(24)\quad E\{e^{-s\eta(t+\Delta t)}|\delta_{\Delta t}=1,\eta(t)=y\}$$
$$=\frac{y}{\Delta t}\left[(1-p)+pH(\Delta t-y)+p\int_{\Delta t-y}^{\infty}e^{-s(x+y-\Delta t)}\,dH(x)\right]$$
$$+\frac{1}{\Delta t}\int_0^{\Delta t-y}\left[H(z)+\int_z^{\infty}e^{-s(x-z)}\,dH(x)\right]dz,$$

for the arrival instant of the customer is distributed uniformly in the interval $(t, t+\Delta t)$ and if it takes place in $(t, t+y]$, then the customer joins the queue with probability p, and if it takes place in $(t+y, t+\Delta t)$, then with probability 1. Thus

$$(25)\quad E\{e^{-s\eta(t+\Delta t)}|\delta_{\Delta t}=1,\eta(t)=y\}$$
$$=\begin{cases}[(1-p)+p\psi(s)]e^{-sy}(1+s\Delta t)+o(\Delta t), & y>\Delta t,\\ \psi(s)+\dfrac{y}{\Delta t}(1-p)[1-\psi(s)]+O(\Delta t), & y<\Delta t,\end{cases}$$

and dropping the condition $\eta(t)=y$ we get

$$(26)\quad E\{e^{-s\eta(t+\Delta t)}|\delta_{\Delta t}=1\}=\Omega(t,s)[(1-p)+p\psi(s)]+\psi(s)P_0(t)+O(\Delta t).$$

By (17), (22), and (26),

$$(27)\quad \Omega(t+\Delta t,s)=\Omega(t,s)+s\Delta t[\Omega(t,s)-P_0(t)]-\lambda\Delta t\Omega(t,s)$$
$$+\lambda\Delta t\Omega(t,s)[(1-p)+p\psi(s)]+o(\Delta t),$$

and letting $\Delta t\to 0$ we get

$$(28)\quad \frac{\partial\Omega(t,s)}{\partial t}=\{s-\lambda p[1-\psi(s)]\}\Omega(t,s)-\{s+\lambda(1-p)[1-\psi(s)]\}P_0(t).$$

The solution of this differential equation is (12).

It remains only to determine $P_0(t)$. This can be done by probabilistic methods (compare section 8) or by the theory of functions of a complex variable.

Let us write $s=\zeta$ in (12) and form the Laplace transform of $\Omega(t,\zeta)$ with respect to t, then we have

$$(29)\quad \int_0^{\infty}e^{-ts}\Omega(t,\zeta)dt=\frac{\hat{\Omega}(\zeta)-\{\zeta+\lambda(1-p)[1-\psi(s)]\}\int_0^{\infty}e^{-st}P_0(t)dt}{s-\zeta+\lambda p[1-\psi(\zeta)]}.$$

If $\mathrm{Re}(s)>0$ and $\mathrm{Re}(\zeta)\geqq 0$ then (29) is a regular function of ζ.

By Rouché's theorem it follows that the denominator of the right side of (29) has one and only one root $\zeta=\zeta(s)$ in this domain. By lemma 1 we have

$$(30)\quad \zeta(s)=s+\lambda p[1-\gamma(s)],$$

where $z=\gamma(s)$ is the root of the equation

$$(31)\quad z=\psi[s+\lambda p(1-z)]$$

that has the smallest absolute value. Accordingly $\zeta=\zeta(s)$ must be a root of the numerator of the right side of (29). Hence

$$\int_0^\infty e^{-st}P_0(t)dt = \frac{p\hat{\Omega}[\zeta(s)]}{\zeta(s) - (1-p)s}. \tag{32}$$

Putting (30) into (32) we obtain (13).

REMARK 2. The distribution function $W(t, x)$ satisfies the integro-differential equation

$$\begin{aligned} \frac{\partial W(t,x)}{\partial t} \\ = \frac{\partial W(t,x)}{\partial x} - \lambda p\left[W(t,x) - \int_0^x H(x-y)d_yW(t,y)\right] + \lambda(1-p)P_0(t)H(x) \end{aligned} \tag{33}$$

for almost all $x \geqq 0$ and $t \geqq 0$.

By the theorem of total probability we can write that

$$\begin{aligned} W(t+\Delta t, x) &= (1-\lambda p\Delta t)W(t, x+\Delta t) \\ &+ \lambda\Delta t\left\{p\int_0^x H(x-y)d_yW(t,y) - (1-p)P_0(t)[1-H(x)]\right\} + o(\Delta t). \end{aligned} \tag{34}$$

For the event $\eta(t+\Delta t) \leqq x$ may happen in several mutually exclusive ways.

(1) In the time interval $(t, t+\Delta t]$ no customer arrives at the counter and $\eta(t) \leqq x + \Delta t$, the probability of which is

$$(1-\lambda\Delta t)W(t, x+\Delta t) + o(\Delta t). \tag{35}$$

(2) In the time interval $(t, t+\Delta t]$ one customer arrives at the counter, he joins the queue and his service time is less than $x - \eta(t) + \vartheta_t\Delta t$, where $0 \leqq \vartheta_t \leqq 1$, the probability of which is

$$\lambda\Delta t\left[p\int_0^x H(x-y)d_yW(t,y) + (1-p)H(x)P_0(t)\right] + o(\Delta t). \tag{36}$$

(3) In the time interval $(t, t+\Delta t]$ one customer arrives at the counter, he does not join the queue and $\eta(t) \leqq x + \Delta t$, the probability of which is

$$\lambda(1-p)\Delta t[W(t, x+\Delta t) - P_0(t)] + o(\Delta t). \tag{37}$$

(4) In the time interval $(t, t+\Delta t]$ more than one customer arrives at the counter, the probability of which is $o(\Delta t)$.

REMARK 3. Let us denote by $a(t)$ the average waiting time at the instant t, that is, $a(t) = E\{\eta(t)\}$. We have

$$a(t) = a(0) + [1 - \lambda(1-p)\alpha]\int_0^t P_0(u)du - [1-\lambda\alpha]t. \tag{38}$$

Denote by $\nu(t)$ the number of customers joining the queue in the time interval $(0, t]$ and define $\zeta(t) = 1$ if the server is busy at the instant t and $\zeta(t) = 0$ if the server is idle at the instant t. We can write

$$\eta(t) = \eta(0) + \sum_{0 \leqq \tau_n \leqq t} \epsilon_n\chi_n - \int_0^t \zeta(u)du. \tag{39}$$

Now

$$E\{\sum_{0 \leqq \tau_n \leqq t} \epsilon_n \chi_n\} = \alpha E\{\sum_{0 \leqq \tau_n \leqq t} \epsilon_n\} = \alpha E\{\nu(t)\}$$

$$(40) \qquad = \alpha \int_0^t \{P_0(u)\lambda + [1 - P_0(u)]\lambda p\} du,$$

$$E\left\{\int_0^t \zeta(u) du\right\} = \int_0^t E\{\zeta(u)\} du = \int_0^t [1 - P_0(u)] du,$$

whence (38) follows.

EXAMPLE 1. Let us suppose that the distribution function of the service time is

$$(41) \qquad H(x) = \begin{cases} 1 - e^{-\mu x}, & x \geqq 0, \\ 0, & x < 0. \end{cases}$$

In this case $\psi(s) = \mu/(\mu + s)$. If we suppose that initially the queue size is i then

$$(42) \qquad \hat{\Omega}(s) = E\{e^{-s\eta(0)}\} = \left(\frac{\mu}{\mu + s}\right)^i.$$

By theorem 1 we obtain

$$(43) \qquad \gamma(s) = \frac{(s + \lambda p + \mu) - [(s + \lambda p + \mu)^2 - 4\lambda p \mu]^{1/2}}{2\lambda p}$$

and

$$(44) \qquad \int_0^\infty e^{-st} P_0(t) dt = \frac{[\gamma(s)]^i}{\lambda + s - \gamma(s)} = \frac{[\gamma(s)]^{i+1}}{[1 - \gamma(s)][\mu + \lambda(1 - p)\gamma(s)]}$$

$$= \frac{1}{\mu + \lambda(1 - p)} \sum_{n=1}^\infty \left\{1 - \left[\frac{-\lambda(1 - p)}{\mu}\right]^n\right\} [\gamma(s)]^{i+n}$$

if $\mathrm{Re}(s) > 0$. If we use

$$(45) \qquad \int_0^\infty e^{-sx} g_n(x) dx = [\gamma(s)]^n$$

where

$$(46) \qquad g_n(x) = n\left(\frac{\mu}{\lambda p}\right)^{n/2} e^{-(\lambda p + \mu)x} \frac{I_n[2(\lambda p \mu)^{1/2} x]}{x}$$

$$= \mu\left(\frac{\mu}{\lambda p}\right)^{(n-1)/2} e^{-(\lambda p + \mu)x} \{I_{n-1}[2(\lambda p \mu)^{1/2} x] - I_{n+1}[2(\lambda p \mu)^{1/2} x]\}$$

and

$$(47) \qquad I_n(x) = \sum_{r=0}^\infty \frac{\left(\frac{x}{2}\right)^{n+2r}}{r!(n + r)!}, \qquad n = 0, 1, 2, \cdots,$$

is the modified Bessel function of order n, then we obtain from (44) by inversion

$$(48)\qquad P_0(t) = \frac{\mu e^{-(\lambda p+\mu)t}}{[\mu + \lambda(1-p)]}\sum_{n=1}^{\infty}\left\{1 - \left[\frac{-\lambda(1-p)}{\mu}\right]^n\right\}\left(\frac{\mu}{\lambda p}\right)^{(i+n-1)/2}$$
$$\{I_{n+i-1}[2(\lambda p\mu)^{1/2}t] - I_{n+i+1}[2(\lambda p\mu)^{1/2}t]\}$$
$$= e^{-(\lambda p+\mu)t}\left(\left(\frac{\mu}{\lambda p}\right)^{i} I_i[2(\lambda p\mu)^{1/2}t]\right.$$
$$+ \left(\frac{\mu}{\lambda p}\right)^{(i+1)/2}\left[1 + \frac{\lambda(1-p)}{\mu}\right] I_{i+1}[2(\lambda p\mu)^{1/2}t]$$
$$+ \frac{1}{1 + \frac{\lambda(1-p)}{\mu}}\sum_{r=i+2}^{\infty}\left\{\left(1 - \frac{\lambda p}{\mu}\right)\left(\frac{\mu}{\lambda p}\right)^{r/2}\right.$$
$$\left.\left.- \left[1 - \frac{\mu p}{\lambda(1-p)^2}\right]\left[\frac{\lambda(1-p)}{\mu}\right]^{r-i+1}\right\} I_r[2(\lambda p\mu)^{1/2}t]\right).$$

By (29) we have

$$(49)\qquad \int_0^{\infty} e^{-st}\Omega(t,\zeta)dt = \zeta\,\frac{\left(\frac{\zeta}{\mu+\zeta}\right)^{i-1} - \frac{[\mu+\zeta+\lambda(1-p)][\gamma(s)]^i}{s+\lambda[1-\gamma(s)]}}{s(\mu+\zeta) - \zeta(\mu+\zeta-\lambda p)}$$

and by inversion $W(t, x)$ can be expressed explicitly by Bessel functions.

6. The distribution function of the waiting time of the nth customer

The random variable η_n gives the waiting time of the nth customer if he joins the queue. Clearly

$$(50)\qquad p + (1-p)P\{\eta_n = 0\}$$

is the probability that the nth customer joins the queue, and

$$(51)\qquad (1-p)P\{\eta_n > 0\}$$

is the probability that he departs without being served. Now we can write

$$(52)\qquad \eta_{n+1} = [\eta_n + \epsilon_n\chi_n - \vartheta_n]^+,$$

where $\vartheta_n = \tau_{n+1} - \tau_n$ for $n = 1, 2, 3, \cdots$, and $\epsilon_n = 1$ if the nth customer joins the queue and $\epsilon_n = 0$ if not. The $\{\chi_n\}$ and $\{\vartheta_n\}$ are independent sequences of identically distributed independent random variables with distribution functions $P\{\chi_n \leqq x\} = H(x)$ and $P\{\vartheta_n \leqq x\} = F(x) = 1 - \exp(-\lambda x)$ if $x \geqq 0$.

We need the following

LEMMA 2. *Let ϑ and ξ be nonnegative independent random variables for which*

$$(53)\qquad P\{\vartheta \leqq x\} = \begin{cases} 1 - e^{-\lambda x}, & x \geqq 0 \\ 0, & x < 0, \end{cases}$$

then we have for $\mathrm{Re}(s) \geqq 0$,

$$E\{e^{-s[\xi-\vartheta]^+}\} = \begin{cases} \dfrac{\lambda E\{e^{-s\xi}\} - sE\{e^{-\lambda\xi}\}}{\lambda - s}, & s \neq \lambda, \\ \lambda E\{\xi e^{-\lambda\xi}\} + E\{e^{-\lambda\xi}\}, & s = \lambda, \end{cases} \tag{54}$$

and

$$P\{[\xi - \vartheta]^+ = 0\} = E\{e^{-\lambda\xi}\}. \tag{55}$$

PROOF. For nonnegative x values we have

$$E\{e^{-s[\xi-\vartheta]^+}|\xi = x\} = \begin{cases} \dfrac{\lambda e^{-sx} - se^{-\lambda x}}{\lambda - s}, & s \neq \lambda, \\ e^{-\lambda x} + \lambda x e^{-\lambda x}, & s = \lambda, \end{cases} \tag{56}$$

$$P\{[\xi - \vartheta]^+ = 0|\xi = x\} = e^{-\lambda x},$$

and unconditionally we obtain (54) and (55).

THEOREM 2. *The Laplace-Stieltjes transforms*

$$\Omega_n(s) = E\{e^{-s\eta_n}\}, \qquad n = 1, 2, 3, \cdots, \tag{57}$$

are given by the generating function

$$\sum_{n=1}^{\infty} \Omega_n(s) w^n \tag{58}$$
$$= \frac{(\lambda - s)w\Omega_1(s) + wsP\{\eta_1 = 0\} - \{s + w\lambda(1-p)[1-\psi(s)]\} \sum_{n=1}^{\infty} P\{\eta_n = 0\} w^n}{\lambda - s - w\lambda[(1 - p) + p\psi(s)]},$$

where

$$\sum_{n=1}^{\infty} P\{\eta_n = 0\} w^n = wp \frac{h(w)P\{\eta_1 = 0\} + w[\lambda - h(w)]\Omega_1[h(w)]}{h(w) - \lambda(1 - p)(1 - w)} \tag{59}$$

and $s = h(w)$ *is the only root of the equation*

$$\lambda - s = w\lambda[(1 - p) + p\psi(s)] \tag{60}$$

in the domain $\mathrm{Re}(s) \geqq 0$ *and* $|w| < 1$.

PROOF. It is easy to see that

$$E\{e^{-s(\eta_n + \epsilon_n \chi_n)}\} = [(1 - p) + p\psi(s)]\Omega_n(s) - (1 - p)[1 - \psi(s)]P\{\eta_n = 0\}. \tag{61}$$

By using (52) and lemma 2 we obtain

$$(\lambda - s)\Omega_{n+1}(s) \tag{62}$$
$$= \lambda[(1 - p) + p\psi(s)]\Omega_n(s) - \lambda(1 - p)[1 - \psi(s)]P\{\eta_n = 0\} - sP\{\eta_{n+1} = 0\}.$$

Forming the generating function of (62) we get (58). The left side of (58) is a regular function of s if $\mathrm{Re}(s) \geqq 0$ and $|w| < 1$. By Rouché's theorem it follows that the denominator of the right side of (58) has exactly one root in this domain. For,

$$|(1 - p) + p\psi(s)| < \left|\frac{\lambda - s}{w\lambda}\right| \tag{63}$$

if $\mathrm{Re}(s) = 0$ or $\mathrm{Re}(s) \geqq 0$ and $|s|$ is sufficiently large. Let us denote this root by $s = h(w)$. By lemma 1 this can also be expressed as

$$h(w) = \lambda(1-p)(1-w) + \lambda p\{1 - \gamma[\lambda(1-p)(1-w), w]\}, \tag{64}$$

where $\gamma(s, w)$ is defined by (8). Accordingly $s = h(w)$ must also be a root of the numerator of the right side of (58). So we obtain (59). This completes the proof of the theorem.

Now let us denote by η_n^* the waiting time of the nth customer joining the queue. We can see easily that

$$\eta_{n+1}^* = [\eta_n^* + \chi_n^* - \vartheta_n^*]^+, \tag{65}$$

where $\{\chi_n^*\}$ and $\{\vartheta_n^*\}$ are independent sequences of identically distributed independent random variables with distribution functions $P\{\chi_n^* \leqq x\} = H(x)$ and $P\{\vartheta_n^* \leqq x\} = 1 - \exp(-\lambda px)$ if $x \geqq 0$.

THEOREM 3. *The Laplace-Stieltjes transforms*

$$\Omega_n^*(s) = E\{e^{-s\eta_n^*}\}, \qquad n = 1, 2, \cdots, \tag{66}$$

are given by the generating function

$$\sum_{n=1}^{\infty} \Omega_n^*(s) w^n = w\,\frac{[1-g(w)](\lambda p - s)\Omega_1^*(s) - sg(w)\Omega_1^*\{\lambda p[1-g(w)]\}}{[1-g(w)][(\lambda p - s) - w\lambda p\psi(s)]}, \tag{67}$$

where $z = g(w)$ is the root with smallest absolute value of the equation

$$z = w\psi[\lambda p(1-z)]. \tag{68}$$

PROOF. By using (65) and lemma 2 we have

$$(\lambda p - s)\Omega_{n+1}^*(s) = \lambda p\Omega_n^*(s)\psi(s) - sP\{\eta_{n+1}^* = 0\} \tag{69}$$

and forming the generating function for $|w| < 1$ we get

$$\sum_{n=1}^{\infty} \Omega_n^*(s) w^n = \frac{(\lambda p - s)w\Omega_1^*(s) - s\sum_{n=2}^{\infty} P\{\eta_n^* = 0\}w^n}{(\lambda p - s) - w\lambda p\psi(s)}. \tag{70}$$

The left side of (70) is a regular function of s if $\mathrm{Re}(s) \geqq 0$ and $|w| < 1$. In this domain the denominator of (70) has exactly one root $s = \lambda p[1 - g(w)]$, where $z = g(w)$ is the root with the smallest absolute value of equation (68). This root must also be a root of the numerator of the right side of (70). So we obtain

$$\sum_{n=2}^{\infty} P\{\eta_n^* = 0\}w^n = \frac{wg(w)\Omega_1^*\{\lambda p[1-g(w)]\}}{1-g(w)}. \tag{71}$$

Putting (71) into (70) we get (67), which was to be proved.

Now let us denote by ν_n the number of customers joining the queue among the first n customers. We shall prove

THEOREM 4. *The generating function $E\{z^{\nu_n}\}$ is given by*

$$\sum_{n=0}^{\infty} E\{z^{\nu_n}\}w^n = \frac{1 - (1-p)(1-z)\sum_{n=1}^{\infty} P\{\eta_n = 0\}E\{z^{\nu_{n-1}} \mid \eta_n = 0\}w^n}{1 - w[(1-p) + pz]}, \tag{72}$$

where

$$(73)\qquad \sum_{n=1}^{\infty} P\{\eta_n = 0\}E\{z^{\nu_{n-1}}|\eta_n = 0\}w^n = p\,\frac{w[\lambda - h(w, z)]\Omega_1[h(w, z)] + h(w, z)P\{\eta_1 = 0\}}{h(w, z) - \lambda(1 - p)(1 - w)}$$

and $s = h(w, z)$ *is the only root of the equation*

$$(74)\qquad \lambda - s = w\lambda[(1 - p) + pz\psi(s)]$$

in the domain $\mathrm{Re}(s) \geqq 0$, $|z| \leqq 1$, *and* $|w| < 1$.

PROOF. We have

$$(75)\qquad \nu_n = \nu_{n-1} + \epsilon_n, \qquad n = 1, 2, \cdots,$$

where $\nu_0 = 0$ and, by (52),

$$(76)\qquad \eta_{n+1} = [\eta_n + \epsilon_n\chi_n - \vartheta_n]^+, \qquad n = 1, 2, \cdots.$$

If $\mathrm{Re}(s) \geqq 0$ and $|z| \leqq 1$, then let us define

$$(77)\qquad \Omega_n(s, z) = E\{e^{-s\eta_n}z^{\nu_{n-1}}\}.$$

By using lemma 2 we get

$$(78)\qquad (\lambda - s)E\{e^{-s\eta_{n+1}}z^{\nu_n}\} = \lambda E\{e^{-s(\eta_n+\epsilon_n\chi_n)}z^{\nu_{n-1}+\epsilon_n}\} - sE\{e^{-\lambda(\eta_n+\epsilon_n\chi_n)}z^{\nu_{n-1}+\epsilon_n}\},$$

where

$$(79)\qquad \begin{aligned} E\{e^{-s(\eta_n+\epsilon_n\chi_n)}z^{\nu_{n-1}+\epsilon_n}\} &= z\psi(s)P\{\eta_n = 0\}E\{z^{\nu_{n-1}}|\eta_n = 0\} + \\ &\quad [(1 - p) + pz\psi(s)][\Omega_n(z, s) - P\{\eta_n = 0\}E\{z^{\nu_{n-1}}|\eta_n = 0\}], \\ E\{e^{-\lambda(\eta_n+\epsilon_n\chi_n)}z^{\nu_{n-1}+\epsilon_n}\} &= P\{\eta_{n+1} = 0\}E\{z^{\nu_n}|\eta_{n+1} = 0\}. \end{aligned}$$

Hence

$$(80)\qquad \begin{aligned} &(\lambda - s)\Omega_{n+1}(s, z) \\ &\quad = \lambda[(1 - p) + pz\psi(s)]\Omega_n(s, z) - \lambda(1 - p)[1 - z\psi(s)]P\{\eta_n = 0\}, \\ &E\{z^{\nu_{n-1}}|\eta_n = 0\} = sP\{\eta_{n+1} = 0\}E\{z^{\nu_n}|\eta_{n+1} = 0\}. \end{aligned}$$

Forming the generating function we get

$$(81)\qquad \begin{aligned} \sum_{n=1}^{\infty} \Omega_n(s, z)w^n &= \{(\lambda - s) - w\lambda\,[1 - p) + pz\psi(s)]\}^{-1} \\ &\quad \Big(w(\lambda - s)\Omega_1(s, z) + sP\{\eta_1 = 0\} \\ &\quad - \{s + \lambda w(1 - p)[1 - z\psi(s)]\}\sum_{n=1}^{\infty} P\{\eta_n = 0\}E\{z^{\nu_{n-1}}|\eta_n = 0\}w^n\Big). \end{aligned}$$

The left side of (81) is a regular function of s if $\mathrm{Re}(s) \geqq 0$, $|z| \leqq 1$, and $|w| < 1$. By the Rouché theorem it follows that the denominator of the right side of (81) has exactly one root in this domain. For,

$$(82)\qquad |(1 - p) + pz\psi(s)| < \left|\frac{\lambda - s}{\lambda w}\right|$$

if $\mathrm{Re}(s) = 0$ or $\mathrm{Re}(s) \geqq 0$ and $|s|$ is sufficiently large. Let us denote this root by $s = h(w, z)$. We can also write

$$h(z, w) = \lambda(1 - p)(1 - w) + \lambda p\{1 - \gamma[\lambda(1 - p)(1 - w), zw]\}, \tag{83}$$

where $\gamma(s, w)$ is defined by (8). Accordingly $s = h(w, z)$ must be a root of the numerator of the right side of (81) also. So we obtain

$$\sum_{n=1}^{\infty} P\{\eta_n = 0\}E\{z^{\nu_{n-1}}|\eta_n = 0\}w^n = p\,\frac{w[\lambda - h(w, z)]\Omega_1[h(w, z), z] + h(w, z)P\{\eta_1 = 0\}}{h(w, z) - \lambda(1 - p)(1 - w)}. \tag{84}$$

This proves (71). Finally, putting $s = 0$ in (81) we get (72). This completes the proof of the theorem.

7. The stochastic law of the busy period

It is clear that the time of the server is composed of alternating idle and busy periods. The durations of the successive idle periods and busy periods are independent random variables. The distribution function of the length of an idle period is clearly $F(x) = 1 - \exp(-\lambda x)$ if $x \geqq 0$. If $\eta(0) = 0$ then the process starts with an idle period and the length of every busy period has the same distribution function, say $G(x)$. If $\eta(0) \neq 0$ then the process starts with a busy period. In this case denote by $\hat{G}(x)$ the distribution function of the length of the initial busy period. The distribution function of the lengths of the other busy periods are $G(x)$. If $\eta(0) = 0$ then we agree to write $\hat{G}(x) = 1$ if $x \geqq 0$ and $\hat{G}(x) = 0$ if $x < 0$.

In order to determine $G(x)$ let us consider a customer who arrives at the counter when the server is idle. Denote by χ the duration of this service time. We have $P\{\chi \leqq x\} = H(x)$. Further denote by ν the number of new customers joining the queue during the service time of this customer. Clearly we have

$$P\{\nu = j|\chi = y\} = e^{-\lambda py}\frac{(\lambda py)^j}{j!}, \tag{85}$$

whence

$$P\{\nu = j\} = \int_0^{\infty} e^{-\lambda py}\frac{(\lambda py)^j}{j!}\,dH(y). \tag{86}$$

THEOREM 5. *Define $\{\theta_n\}$ as a sequence of identically distributed independent random variables having the distribution function $G(x)$ and suppose that $\{\theta_n\}$ is independent of ν and χ also. We have*

$$G(x) = P\{\chi + \theta_1 + \theta_2 + \cdots + \theta_\nu \leqq x\}, \tag{87}$$

where the empty sum is 0 when $\nu = 0$.

PROOF. First of all let us note that from the point of view of the server it is perfectly indifferent whether or not the customers are served in the order of their arrival. This affects the customers only. The distribution function of the waiting time of a customer is changed by this fact, but the distribution function of the busy period remains unaltered. We shall consider a special system of serving in which the busy period is composed of the serving of the first customer, the length of which is χ, and if during the service time of the first customer ν customers join the queue, then it contains ν further phases, the lengths of which are denoted by $\theta_1, \theta_2, \cdots, \theta_\nu$. Let us suppose that the first phase starts with the serving of one of the above-mentioned ν customers and continues with the serving of the new arrivals as long as they come. When there are no more new arrivals the second phase starts with the serving of one of the remaining $\nu - 1$ customers and this procedure is continued through all the ν phases. Thus the duration of the busy period is $\chi + \theta_1 + \theta_2 + \cdots + \theta_\nu$ where clearly $\theta_1, \theta_2, \cdots, \theta_n, \cdots$ are independent random variables with distribution function $P\{\theta_n \leqq x\} = G(x)$ and the sequence $\{\theta_n\}$ is independent of χ and ν also. This completes the proof of (87).

Now let us introduce the Laplace-Stieltjes transforms

$$\gamma(s) = \int_0^\infty e^{-sx} dG(x) \tag{88}$$

and

$$\hat{\gamma}(s) = \int_0^\infty e^{-sx} d\hat{G}(x) \tag{89}$$

for $\operatorname{Re}(s) \geqq 0$. We shall prove

THEOREM 6. *The Laplace-Stieltjes transform $\gamma(s)$ is the root with smallest absolute value in z of the equation*

$$z = \psi[s + \lambda p(1 - z)] \tag{90}$$

for $\operatorname{Re}(s) \geqq 0$. We have

$$\lim_{x \to \infty} G(x) = \omega, \tag{91}$$

where ω is the root with smallest absolute value in z of the equation

$$z = \psi[\lambda p(1 - z)]. \tag{92}$$

If $\lambda p\alpha \leqq 1$ then $\omega = 1$ and $G(x)$ is a proper distribution function, while if $\lambda p\alpha > 1$ then $\omega < 1$ and $G(x)$ is an improper distribution function, namely, in this case the busy period will be infinite with probability $1 - \omega$.

The Laplace-Stieltjes transform $\hat{\gamma}(s)$ can be obtained as follows,

$$\hat{\gamma}(s) = \hat{\Omega}[s + \lambda p - \lambda p \gamma(s)]. \tag{93}$$

If $\lambda p\alpha \leqq 1$ then $\hat{G}(\infty) = 1$ and if $\lambda p\alpha > 1$ and $\eta(0) \not\equiv 0$ then $\hat{G}(\infty) < 1$.

PROOF. Denote by $G_j(x)$ the distribution function of the random variable $\theta_1 + \theta_2 + \cdots + \theta_j$ for $j = 0, 1, 2, \cdots$, that is, $G_j(x)$ is the jth iterated convolution of $G(x)$ with itself. By (87) we have

$$\begin{aligned} &P\{\chi + \theta_1 + \cdots \theta_\nu \leqq x | \chi = y\} \\ &\quad = P\{\theta_1 + \cdots + \theta_\nu \leqq x - y | \chi = y\} = \sum_{j=0}^\infty e^{-\lambda p y} \frac{(\lambda p y)^j}{j!} G_j(x - y), \end{aligned} \tag{94}$$

where $\theta_1 + \cdots + \theta_\nu$ is a sum of a random number of random variables. Unconditionally we have

$$G(x) = \int_0^x \sum_{j=0}^{\infty} e^{-\lambda p y} \frac{(\lambda p y)^j}{j!} G_j(x - y) dH(y). \tag{95}$$

Passing from equation (95) to the Laplace-Stieltjes transform we obtain for $\mathrm{Re}(s) \geqq 0$ that

$$\begin{aligned} \gamma(s) &= \sum_{j=0}^{\infty} [\gamma(s)]^j \int_0^\infty e^{-(s+\lambda p)y} \frac{(\lambda p y)^j}{j!} dH(y) \\ &= \int_0^\infty e^{-[s+\lambda p - \lambda p \gamma(s)]y} dH(y) = \psi[s + \lambda p - \lambda p \gamma(s)]. \end{aligned} \tag{96}$$

that is, $z = \gamma(s)$ satisfies (90). If $\mathrm{Re}(s) > 0$ then $|\gamma(s)| < 1$ and (90) has exactly one root in this domain. Consequently, if $\mathrm{Re}(s) > 0$ then $z = \gamma(s)$ is the root with smallest absolute value of equation (90). The required result for $\mathrm{Re}(s) \geqq 0$ can be obtained by analytical continuation. Clearly $G(\infty) = \gamma(0) = \omega$ where ω is given by lemma 1.

If $P\{\eta(0) \leqq x\} = \hat{W}(x)$ then the distribution function of the length of the initial busy period is

$$\hat{G}(x) = \int_0^x \sum_{j=0}^{\infty} e^{-\lambda p y} \frac{(\lambda p y)^j}{j!} G_j(x - y) d\hat{W}(y). \tag{97}$$

For, to obtain $\hat{G}(x)$, we can apply word for word the proof of (87) except that χ is to be replaced by the initial occupation time $\eta(0)$ and ν is to be defined as the number of customers joining the queue in the time interval $(0, \eta(0)]$. Forming the Laplace-Stieltjes transform of (97) we obtain (93). Since

$$\lim_{x \to \infty} \hat{G}(x) = \lim_{s \to 0} \hat{\gamma}(s) = \hat{\Omega}[\lambda p(1 - \omega)] \tag{98}$$

we obtain that $\hat{G}(\infty) = 1$ if $\lambda p \alpha \leqq 1$ and $\hat{G}(\infty) < 1$ if $\lambda p \alpha > 1$ and $\eta(0) \neq 0$.

Now denote by $G_n^*(x)$ the probability that the busy period consists in servicing n customers and its length is at most x. Write

$$\Gamma_n(s) = \int_0^\infty e^{-sx} dG_n^*(x) \tag{99}$$

if $\mathrm{Re}(s) \geqq 0$. We shall prove

THEOREM 7. *If* $\mathrm{Re}(s) \geqq 0$ *and* $|w| \leqq 1$ *then*

$$\sum_{n=1}^{\infty} \Gamma_n(s) w^n = \gamma(s, w), \tag{100}$$

where $z = \gamma(s, w)$ *is the root with smallest absolute value in* z *of the equation*

$$z = w\psi[s + \lambda p(1 - z)]. \tag{101}$$

PROOF. A reasoning similar to the proof of (87) shows that

$$G_n^*(x) = P\{\chi + \theta_1 + \cdots + \theta_\nu \leqq x;\ \delta_1 + \cdots + \delta_\nu = n - 1\}, \tag{102}$$

where $\delta_1, \delta_2, \cdots, \delta_\nu$ denote the numbers of customers joining the queue during the 1st, 2nd, $\cdots$, νth phases of the busy period, respectively. By (102) we have

$$(103)\quad G_n^*(x) = \sum_{j+n_1+\cdots+n_j=n-1}^{\infty} \int_0^x e^{-\lambda py} \frac{(\lambda py)^j}{j!} G_{n_1}^*(x-y) * \cdots * G_{n_j}^*(x-y) dH(y),$$

whence

$$(104)\quad \Gamma_n(s) = \sum_{j+n_1+\cdots+n_j=n-1} \Gamma_{n_1}(s) \cdots \Gamma_{n_j}(s) \int_0^\infty e^{-(\lambda p+s)y} \frac{(\lambda py)^j}{j!} dH(y).$$

Multiplying both sides of this equation by w^n and summing over $n = 1, 2, \cdots$ we obtain that

$$(105)\quad \gamma(s, w) = \sum_{n=1}^{\infty} \Gamma_n(s) w^n$$

satisfies the equation $z = w\psi[s + \lambda p(1 - z)]$. Clearly $\gamma(s, w)$ is a regular function of s and w and $|\gamma(s, w)| \leqq 1$ whenever $\mathrm{Re}(s) \geqq 0$, and $|w| \leqq 1$. If $\mathrm{Re}(s) > 0$ then (101) has one and only one root in the unit circle and hence this is the required $\gamma(s, w)$. By analytical continuation we see that the theorem is also valid in the case $\mathrm{Re}(s) \geqq 0$.

REMARK 4. Now we shall give another proof of theorem 7. Denote by $G_{nk}(x)$ the probability that the busy period consists of at least n services, that at the end of the nth serving k customers are present in the queue, and that the total service time of the first n customers is at most x. Clearly

$$(106)\quad G_n^*(x) = G_{n0}(x).$$

$G_{nk}(x)$ can be determined by the recurrence formula

$$(107)\quad G_{nk}(x) = \sum_{r=1}^{k+1} \int_0^x G_{n-1,r}(x-y) e^{-\lambda py} \frac{(\lambda py)^{k-r+1}}{(k-r+1)!} dH(y), \qquad n = 2, 3, \cdots,$$

if we start from

$$(108)\quad G_{1k}(x) = \int_0^x e^{-\lambda py} \frac{(\lambda py)^k}{k!} dH(y).$$

Write

$$(109)\quad \Gamma_{nk}(s) = \int_0^\infty e^{-sx} dG_{nk}(x).$$

Then clearly

$$(110)\quad \Gamma_{n0}(s) = \Gamma_n(s).$$

Forming the Laplace-Stieltjes transforms of (107) and (108) we obtain

$$(111)\quad \begin{aligned} \Gamma_{nk}(s) &= \sum_{r=1}^{k+1} \Gamma_{n-1,r}(s) \int_0^\infty e^{-(\lambda p+s)y} \frac{(\lambda py)^{k-r+1}}{(k-r+1)!} dH(y), \\ \Gamma_{1k}(s) &= \int_0^\infty e^{-(\lambda p+s)y} \frac{(\lambda py)^k}{k!} dH(y). \end{aligned}$$

If we introduce the generating function

$$C_n(s, z) = \sum_{k=0}^{\infty} \Gamma_{nk}(s) z^k, \tag{112}$$

then we have

$$\begin{aligned} zC_n(s, z) &= \psi[s + \lambda p(1 - z)][C_{n-1}(s, z) - \Gamma_{n-1}(s)], \\ C_1(s, z) &= \psi[s + \lambda p(1 - z)], \end{aligned} \tag{113}$$

whence

$$\sum_{n=1}^{\infty} C_n(s, z) w^n = \frac{w\psi[s + \lambda p(1 - z)][z - \sum_{n=1}^{\infty} \Gamma_n(s) w^n]}{z - w\psi[s + \lambda p(1 - z)]}. \tag{114}$$

The left side of (114) is a regular function of z if $|z| \leqq 1$, $\mathrm{Re}(s) \geqq 0$, and $|w| < 1$. If $\mathrm{Re}(s) \geqq 0$ and $|w| < 1$ then the denominator of the right side has one root $z = \gamma(s, w)$ in the unit circle $|z| \leqq 1$. This must also be a root of the numerator. Therefore

$$\sum_{n=1}^{\infty} \Gamma_n(s) w^n = \gamma(s, w) \tag{115}$$

if $|w| < 1$ and it is also true for $|w| = 1$, which can be seen by analytical continuation.

THEOREM 8. *Denote by μ the expectation of the length of the busy period. We have*

$$\mu = \frac{\alpha}{1 - \lambda p \alpha} \tag{116}$$

if $\lambda p \alpha < 1$ and $\mu = \infty$ if $\lambda p \alpha \geqq 1$.

PROOF. Now

$$\mu = \int_0^{\infty} x dG(x) \tag{117}$$

and by (87) we have

$$\mu = E\{\chi + \theta_1 + \cdots + \theta_\nu\}. \tag{118}$$

Since $E\{\chi\} = \alpha$ and $E\{\nu\} = \lambda p \alpha$, therefore

$$\mu = \alpha + \lambda p \alpha \mu. \tag{119}$$

There are two possibilities: $\mu < \infty$ or $\mu = \infty$. If $\lambda p \alpha < 1$ then $\mu = \alpha/(1 - \lambda p \alpha)$ is finite and if $\lambda p \alpha \geqq 1$ then $\mu = \infty$.

We remark that if

$$\hat{\mu} = \int_0^{\infty} x d\hat{G}(x), \tag{120}$$

then by (97)

$$\hat{\mu} = a(0)[1 + \lambda p \mu] = \frac{a(0)}{\alpha} \mu. \tag{121}$$

REMARK 5. Theorem 8 can be proved directly as follows. Let us denote by ρ the expected number of services during a busy period (possibly $\rho = \infty$). Then we have $\mu = \rho\alpha$. Now we shall prove the relation

$$\frac{\rho}{\lambda p} = \rho\alpha + \frac{1}{\lambda p}. \tag{122}$$

If we suppose that the customers arrive at the counter according to a Poisson process of density λp and every customer joins the queue, then the stochastic law of the busy period remains unchanged. In this case consider the starting points of two consecutive busy periods. The expectation of the distance between them is on the one hand clearly $\rho/\lambda p$ and on the other hand it is equal to the sum of $\rho\alpha$, the expected length of the busy period and $1/\lambda p$, the expected length of the idle period. Thus we obtain (122). If $\lambda p\alpha < 1$ then $\rho = 1/(1 - \lambda p\alpha)$ is finite and if $\lambda p\alpha \geqq 1$ then ρ must be infinite.

EXAMPLE 2. Let us suppose that the service time has the distribution function $H(x) = 1 - \exp(-\mu x)$ for $x \geqq 0$ and that the initial queue size is i. In this case $\psi(s) = \mu/(\mu + s)$,

$$\hat{W}(x) = 1 - \sum_{j=0}^{i-1} \frac{e^{-\mu x}(\mu x)^j}{j!} \qquad \text{if } x \geqq 0, \tag{123}$$

and $\hat{\Omega}(s) = [\mu/(\mu + s)]^i$. Now by theorem 6,

$$\gamma(s) = \frac{(\lambda p + \mu + s) - [(\lambda p + \mu + s)^2 - 4\lambda p\mu]^{1/2}}{2\lambda p} \tag{124}$$

and by (93)

$$\hat{\gamma}(s) = [\gamma(s)]^i. \tag{125}$$

Using (45) and (46) we obtain

$$\frac{d\hat{G}(x)}{dx} = \mu\left(\frac{\mu}{\lambda p}\right)^{(i-1)/2} e^{-(\lambda p+\mu)x}\{I_{i-1}[2(\lambda p\mu)^{1/2}x] - I_{i+1}[2(\lambda p\mu)^{1/2}x]\}, \tag{126}$$

where $I_n(x)$ is the modified Bessel function of order n defined by (47).

If specifically $i = 1$, then $\hat{G}(x) = G(x)$.

8. The probability that the server is idle

THEOREM 9. *The probability $P_0(t)$ that the server is idle at time t satisfies the integral equation*

$$P_0(t) = \hat{G}(t) - \lambda \int_0^t [1 - G(t - u)]P_0(u)du. \tag{127}$$

The Laplace-Stieltjes transform of $P_0(t)$ is

$$\int_0^\infty e^{-st}P_0(t)dt = \frac{\hat{\gamma}(s)}{s + \lambda - \lambda\gamma(s)} = \frac{\hat{\Omega}[s + \lambda p - \lambda p\gamma(s)]}{s + \lambda - \lambda\gamma(s)}, \tag{128}$$

where $\gamma(s)$ is that root in z of the equation

$$z = \psi[s + \lambda p(1 - z)] \tag{129}$$

which has the smallest absolute value.

PROOF. We have

$$(130) \qquad 1 - P_0(t) = 1 - \hat{G}(t) + \lambda \int_0^t [1 - G(t-u)]P_0(u)du.$$

For the left side is the probability that the server is busy at the instant t. This event can occur in the following two exclusive ways: the length of the initial busy period is greater than t or at the instant u where $0 < u < t$ a busy period starts and its length is greater than $t - u$. At the instant u a busy period starts if and only if the server is idle and a customer arrives. This proves (127). The Laplace transform of $P_0(t)$ agrees with (13).

THEOREM 10. *The limit* $\lim_{t\to\infty} P_0(t) = P_0^*$ *always exists and is independent of the initial distribution of* $\eta(0)$. *We have*

$$(131) \qquad P_0^* = \frac{1 - \lambda p\alpha}{1 + \lambda(1-p)\alpha}$$

if $\lambda p\alpha < 1$ *and* $P_0^* = 0$ *if* $\lambda p\alpha \geqq 1$.

PROOF. Let us denote by $M_0(t)$ the expected number of transitions $E_0 \to E_1$ occurring in the time interval $(0, t]$ and denote by $N_0(t)$ the expected number of transitions $E_1 \to E_0$ occurring in the time interval $(0, t]$. Then we can write

$$(132) \qquad P_0(t) = \hat{G}(t) - \int_0^t [1 - G(t-u)]dM_0(u)$$

or

$$(133) \qquad P_0(t) = P_0(0)e^{-\lambda t} + \int_0^t e^{-\lambda(t-u)}dN_0(u).$$

The transitions $E_0 \to E_1$ form a recurrent process, that is, the distances between successive transitions $E_0 \to E_1$ are independent random variables with identical distribution function $G(t) * F(t)$. Clearly we have

$$(134) \qquad M_0(t) = \hat{G}(t) * F(t) + \hat{G}(t) * F(t) * G(t) + \cdots .$$

The distribution function $G(t) * F(t)$ is not a lattice distribution and its mean is evidently $\mu + (1/\lambda)$, where μ is defined by theorem 8. Thus, by using a theorem of D. Blackwell [1], we obtain for all $h > 0$

$$(135) \qquad \lim_{t\to\infty} \frac{M_0(t+h) - M_0(t)}{h} = \lim_{t\to\infty} \frac{M_0(t)}{t} = \frac{1}{\mu + \frac{1}{\lambda}}$$

and the limit is independent of $\hat{G}(t)$ and consequently also independent of the distribution of $\eta(0)$. If $\mu = \infty$ then (135) is taken to be zero.

If $\lambda p\alpha < 1$ then $\mu = \alpha/(1 - \lambda p\alpha)$ and $\hat{G}(\infty) = 1$, and by using (135) we obtain from (132) that

$$(136) \qquad \lim_{t\to\infty} P_0(t) = \frac{1}{1 + \lambda\mu} = \frac{1 - \lambda p\alpha}{1 + \lambda(1-p)\alpha}.$$

This proves the first half of theorem 8. On the other hand the representation (133) is suitable for the complete proof of the theorem. Similarly to $E_0 \to E_1$ the transitions $E_1 \to E_0$ also form a recurrent process in which the distances between successive transitions $E_1 \to E_0$ are independent random variables with identical distribution function $F(t) * G(t)$. Thus by the theorem of Blackwell [1] we have for all $h > 0$ that

$$\lim_{t\to\infty} \frac{N_0(t+h) - N_0(t)}{h} = \lim_{t\to\infty} \frac{N_0(t)}{t} = \frac{1}{\mu + \frac{1}{\lambda}} \tag{137}$$

irrespective of the initial distribution of $\eta(0)$. If we take into consideration that $|M_0(t) - N_0(t)| \leqq 1$ for all $t \geqq 0$, then we can conclude also from this fact that the limits (135) and (137) agree. Now by (133)

$$\lim_{t\to\infty} P_0(t) = \frac{1}{1 + \lambda\mu} \tag{138}$$

whether μ is infinite or finite. This proves the theorem.

REMARK 6. If $\lim_{t\to\infty} P_0(t) = P_0^*$ exists then obviously

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t P_0(u)du = P_0^* \tag{139}$$

also holds. By a well-known Tauberian theorem we can conclude that

$$\lim_{s\to 0} s \int_0^\infty e^{-st} P_0(t)dt = P_0^*. \tag{140}$$

Thus by (128)

$$P_0^* = \lim_{s\to 0} \frac{s}{s + \lambda[1 - \gamma(s)]}. \tag{141}$$

We have seen that $P_0^* > 0$ if $\lambda p\alpha < 1$. Consequently in this case $\gamma(0) = 1$ must hold. This proves that $\omega = 1$ if $\lambda p\alpha < 1$.

9. The limiting distribution of the waiting time

THEOREM 11. *If $\lambda p\alpha < 1$ then the limiting probability distribution $\lim_{t\to\infty} W(t, x) = W^*(x)$ exists and is independent of the initial distribution of $\eta(0)$. The Laplace-Stieltjes transform of $W^*(x)$ is given by*

$$\Omega^*(s) = \frac{(1 - \lambda p\alpha)\{s + \lambda(1-p)[1 - \psi(s)]\}}{[1 + \lambda(1-p)\alpha]\{s - \lambda p[1 - \psi(s)]\}}. \tag{142}$$

If $\lambda p\alpha \geqq 1$ then $\lim_{t\to\infty} W(t, x) = 0$ for all x.

PROOF. If $\lambda p\alpha < 1$ and we restrict ourselves to imaginary s then by (12) and (131) it can be proved that $\lim_{t\to\infty} \Omega(t, s) = \Omega^*(s)$ exists if $|s| < a$, where a is a sufficiently small positive number and further that $\Omega^*(s)$ is continuous at $s = 0$. Hence it follows by a theorem of A. Zygmund [29] that the limiting distribution $\lim_{t\to\infty} W(t, x) = W^*(x)$ exists and further that the Laplace-Stieltjes transform of $W^*(x)$ is $\lim_{t\to\infty} \Omega(t, s) = \Omega^*(s)$ defined for $\mathrm{Re}(s) \geqq 0$. Thus by (28) we can conclude that $\Omega^*(s)$ has to satisfy the equation

$$\{s - \lambda p[1 - \psi(s)]\}\Omega^*(s) = \{s + \lambda(1 - p)[1 - \psi(s)]\}P_0^*, \tag{143}$$

where P_0^* is defined by (131). This proves (142). If $\lambda p\alpha \geqq 1$ then $P_0^* = 0$ and therefore $\lim_{t\to\infty} \Omega(t, s) = 0$ for $\text{Re}(s) \geqq 0$. Hence it follows that $\lim_{t\to\infty} W(t, x) = 0$ for every x. This completes the proof of the theorem.

REMARK 7. If we suppose that $\lambda p\alpha < 1$ and $\hat{W}(x) = W^*(x)$, then we obtain a stationary process $\{\eta(t)\}$ for which

$$\begin{aligned} W(t, x) &= W^*(x), \\ P_0(t) &= P_0^* \end{aligned} \tag{144}$$

for all $t \geqq 0$. Conversely it is easy to see that the process $\{\eta(t)\}$ is stationary if and only if $\lambda p\alpha < 1$ and the initial distribution of $\eta(0)$ is $W^*(x)$.

THEOREM 12. *If $\lambda p\alpha < 1$ then the limiting probability distribution $\lim_{n\to\infty} P\{\eta_n \leqq x\} = W^*(x)$ exists and is independent of the initial distribution η_1. The Laplace-Stieltjes transform of $W^*(x)$ is given by (142). If $\lambda p\alpha \geqq 1$ then $\lim_{n\to\infty} P\{\eta_n \leqq x\} = 0$ for all x.*

PROOF. The statement concerning the existence of the limiting distribution is a consequence of a theorem of D. V. Lindley [16]. It remains only to find $\Omega^*(s)$, the Laplace-Stieltjes transform of $W^*(x)$ in the case $\lambda p\alpha < 1$. Here $\Omega^*(s)$ is independent of the initial distribution. If we suppose that $\eta_1 \equiv 0$ then by using Abel's theorem it follows from (58) that

$$\begin{aligned} \Omega^*(s) &= \lim_{w\to 1} (1 - w) \sum_{n=1}^{\infty} \Omega_n(s)w^{n-1} \\ &= \frac{(1 - \lambda p\alpha)\{s + \lambda(1 - p)[1 - \psi(s)]\}}{[1 + \lambda(1 - p)\alpha]\{s - \lambda p[1 - \psi(s)]\}} \end{aligned} \tag{145}$$

because $h'(1) = -\lambda/(1 - \lambda p\alpha)$. This agrees with (142).

THEOREM 13. *If $\lambda p\alpha < 1$ then the limiting probability distribution $\lim_{n\to\infty} P\{\eta_n^* \leqq x\} = \tilde{W}(x)$ exists and is independent of the initial distribution of η_1^*. The Laplace-Stieltjes transform of $\tilde{W}(x)$ is given by*

$$\tilde{\Omega}(s) = \frac{1 - \lambda p\alpha}{1 - \lambda p \dfrac{1 - \psi(s)}{s}}. \tag{146}$$

If $\lambda p\alpha \geqq 1$ then $\lim_{n\to\infty} P\{\eta_n^* \leqq x\} = 0$ for all x.

PROOF. The proof is similar to that of theorem 12. If $\lambda p\alpha < 1$ then by (67) we have

$$\tilde{\Omega}(s) = \lim_{w\to 1} (1 - w) \sum_{n=1}^{\infty} \Omega_n^*(s)w^n = \frac{s[1 - \lambda p\alpha]}{s - \lambda p[1 - \psi(s)]} \tag{147}$$

because $g'(1) = 1/(1 - \lambda p\alpha)$. If $\lambda p\alpha \geqq 1$ then $\Omega(s) \equiv 0$.

REMARK 8. If $\lambda p\alpha = 1$ then $\tilde{\Omega}(s) \equiv 0$ and therefore it is impossible that $|g(1)| < 1$. This proves that $\omega = g(1) = 1$ must hold if $\lambda p\alpha = 1$.

10. The distribution of the queue size

At this point let us suppose that there is a departure at time $t = -0$ and write $\xi_0 = \xi(0)$. It is easy to see that the sequence of random variables $\{\xi_n\}$ forms a homogeneous Markov chain. The transition probabilities

$$p_{ik} = P\{\xi_{n+1} = k | \xi_n = i\}, \qquad n = 1, 2, \cdots, \tag{148}$$

are given by

$$p_{ik} = \begin{cases} p_{k-i+1} & \text{if } k \geqq i - 1 \text{ and } i = 1, 2, \cdots, \\ p_k & \text{if } k \geqq 0 \quad \text{ and } i = 0, \\ 0 & \text{if } k < i - 1 \text{ and } i = 2, 3, \cdots, \end{cases} \tag{149}$$

where

$$p_j = \int_0^\infty e^{-\lambda p x} \frac{(\lambda p x)^j}{j!} \, dH(x), \qquad j = 0, 1, 2, \cdots, \tag{150}$$

is the probability that during a serving exactly j customers join the queue. Now the matrix of transition probabilities has the following form

$$\pi = \begin{Vmatrix} p_0 & p_1 & p_2 & \cdots \\ p_0 & p_1 & p_2 & \cdots \\ 0 & p_0 & p_1 & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{Vmatrix}. \tag{151}$$

THEOREM 14. *The higher transition probabilities*

$$p_{ik}^{(n)} = P\{\xi_n = k | \xi_0 = i\}, \qquad n = 1, 2, \cdots, \tag{152}$$

are given by the generating function

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} p_{ik}^{(n)} w^n z^k = \frac{z^{i+1}[1 - g(w)] - (1 - z) w \psi[\lambda p(1 - z)][g(w)]^i}{[1 - g(w)]\{z - w\psi[\lambda p(1 - z)]\}}, \tag{153}$$

where $z = g(w)$ is the root with the smallest absolute value of the equation

$$z = w\psi[\lambda p(1 - z)]. \tag{154}$$

PROOF. Now we can write

$$\xi_{n+1} = [\xi_n - 1]^+ + \nu_{n+1}, \tag{155}$$

where $\{\nu_n\}$ is a sequence of independent random variables with distribution $P\{\nu_n = j\} = p_j$, for $j = 0, 1, 2, \cdots$, defined by (150). Now let us suppose that $\xi_0 = i$ is fixed and write

$$U_n(z) = E\{z^{\xi_n}\}. \tag{156}$$

Then by (155) we have

$$U_{n+1}(z) = \psi[\lambda p(1 - z)] \left[\frac{U_n(z) - P\{\xi_n = 0\}}{z} + P\{\xi_n = 0\} \right]. \tag{157}$$

Taking into consideration that $U_0(z) = z^i$ and $P\{\xi_n = 0\} = p_{i0}^{(n)}$ and forming the generating function of (157) we obtain

$$(158)\qquad \sum_{n=0}^{\infty} U_n(z)w^n = \frac{z^{i+1} - w(1-z)\psi[\lambda p(1-z)] \sum_{n=0}^{\infty} p_{i0}^{(n)} w^n}{z - w\psi[\lambda p(1-z)]}.$$

The left side of (158) is a regular function of z if $|z| \leqq 1$ and $|w| < 1$. The denominator of the right side has exactly one root $z = g(w)$ in the unit circle $|z| < 1$. This must be a root of the numerator also. So we have

$$(159)\qquad \sum_{n=0}^{\infty} p_{i0}^{(n)} w^n = \frac{[g(w)]^i}{1 - g(w)}.$$

Putting (159) into (158) we get (153), which was to be proved.

REMARK 9. If we suppose that $\xi(0) = 0$ and the distribution function of the waiting time of the nth customer joining the queue is denoted by $W_n^*(x)$, then we can write the obvious relation

$$(160)\qquad P\{\xi_n = k\} = \int_0^\infty e^{-\lambda px} \frac{(\lambda px)^k}{k!} d[W_n^*(x) * H(x)],$$

whence

$$(161)\qquad E\{z^{\xi_n}\} = \Omega_n^*[\lambda p(1-z)]\psi[\lambda p(1-z)].$$

Now using (67) we can prove (153) in this way also. If the initial queue size is arbitrary then $E\{z^{\xi_n}\}$ can be obtained similarly.

The Markov chain $\{\xi_n\}$ is irreducible and aperiodic and we have

THEOREM 15. *If $\lambda p\alpha < 1$ then the Markov chain $\{\xi_n\}$ is ergodic and the limiting probability distribution $\lim_{n\to\infty} P\{\xi_n = k\} = P_k$ for $k = 0, 1, 2, \cdots$ exists and is independent of the initial distribution. We have*

$$(162)\qquad U(z) = \sum_{k=0}^{\infty} P_k z^k = \frac{(1-\lambda p\alpha)(1-z)\psi[\lambda p(1-z)]}{\psi[\lambda p(1-z)] - z}.$$

If $\lambda p\alpha \geqq 1$ then $\lim_{n\to\infty} P\{\xi_n = k\} = 0$ for every k.

PROOF. The limit $\lim_{n\to\infty} P\{\xi_n = k\} = P_k$ always exists and is independent of the initial distribution. Either every $P_k > 0$ and $\{P_k\}$ is a probability distribution or every $P_k = 0$. By using Abel's theorem we have

$$(163)\qquad \sum_{k=0}^{\infty} P_k z^k = \lim_{w\to 1} (1-w) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} p_{ik}^{(n)} z^k w^n$$

and the right side can be calculated by (153). If we use $g(1) = 1$ and $g'(1) = 1/(1-\lambda p\alpha)$ when $\lambda p\alpha \leqq 1$ then we obtain (162) and since $g(1) < 1$ when $\lambda p\alpha > 1$ we obtain that $P_k = 0$ for every k if $\lambda p\alpha > 1$.

Now we shall investigate the distribution of $\xi(t)$ for finite t. Let us suppose that there is a departure at time $t = -0$. We need the joint distribution of ξ_n and τ_n' for every n. In this case $\xi_0 = \xi(0)$ and $\tau_0' = 0$.

THEOREM 16. *Let us define*

$$(164)\qquad U_n(s, z) = E\{e^{-s\tau_n'} z^{\xi_n}\}, \qquad n = 0, 1, 2, \cdots$$

for $\operatorname{Re}(s) \geqq 0$ and $|z| \leqq 1$. We have for $|w| < 1$ that

$$(165) \qquad \sum_{n=0}^{\infty} U_n(s, z)w^n = \frac{zU_0(s, z) - \dfrac{w\psi[s + \lambda p(1 - z)][s + \lambda(1 - z)]U_0[s, \gamma(s, w)]}{s + \lambda[1 - \gamma(s, w)]}}{z - w\psi[s + \lambda p(1 - z)]},$$

where $\gamma(s, w)$ *is the root with smallest absolute value in* z *of the equation*

$$(166) \qquad z = w\psi[s + \lambda p(1 - z)].$$

PROOF. Now we can write

$$(167) \qquad \xi_{n+1} = [\xi_n - 1]^+ + \nu_{n+1}$$

and

$$(168) \qquad \tau'_{n+1} = \tau'_n + \chi^*_{n+1} + \begin{cases} 0, & \xi_n \geqq 1, \\ \vartheta^*_{n+1}, & \xi_n = 0, \end{cases}$$

where $\{\chi^*_n\}$ and $\{\vartheta^*_n\}$ are independent sequences of identically distributed independent random variables with distribution functions $P\{\chi^*_n \leqq x\} = H(x)$ and $P\{\vartheta^*_n \leqq x\} = F(x) = 1 - \exp(-\lambda x)$ for $x \geqq 0$. Further we have

$$(169) \qquad P\{\nu_n = j | \chi^*_n = x\} = e^{-\lambda p x}\frac{(\lambda p x)^j}{j!}, \qquad j = 0, 1, 2, \cdots$$

for every n.

Since

$$(170) \qquad E\{\exp(-s\chi^*_n)z^{\nu_n}\} = \psi[s + \lambda p(1 - z)]$$

we get by (167) and (168) that

$$(171) \qquad U_{n+1}(s, z) = \psi[s + \lambda p(1 - z)]\left[\frac{U_n(s, z) - U_n(s, 0)}{z} + U_n(s, 0)\frac{\lambda}{\lambda + s}\right].$$

Hence

$$(172) \qquad \sum_{n=0}^{\infty} U_n(s, z)w^n = \frac{zU_0(s, z) - w\psi[s + \lambda p(1 - z)]\dfrac{s + \lambda(1 - z)}{\lambda + s}\sum_{n=0}^{\infty} U_n(s, 0)w^n}{z - w\psi[s + \lambda p(1 - z)]}.$$

The left side of (172) is a regular function of z if $|z| \leqq 1$, $\mathrm{Re}(s) \geqq 0$, and $|w| < 1$. In this domain the denominator of the right side of (172) has exactly one root $z = \gamma(s, w)$. This must also be a root of the numerator. Thus we have

$$(173) \qquad \sum_{n=0}^{\infty} U_n(s, 0)w^n = \frac{U_0[s, \gamma(s, w)]}{1 - \dfrac{\lambda}{\lambda + s}\gamma(s, w)}.$$

Putting (173) into (158) we get (165), as was to be proved.

In what follows let us suppose that $t = -0$ is a departure point and write

$$(174) \qquad P_{ik}(t) = P\{\xi(t) = k | \xi(0) = i\}.$$

THEOREM 17. *Let*

$$\prod_{ik}(s) = \int_0^\infty e^{-st}P_{ik}(t)dt \tag{175}$$

for $\mathrm{Re}(s) > 0$. *For* $\mathrm{Re}(s) > 0$ *and* $|z| \leqq 1$ *we have*

$$\begin{aligned}(176)\quad \sum_{k=0}^\infty \prod_{ik}(s)z^k \\ &= \frac{z^{i+1}\{1 - \psi[s + \lambda p(1 - z)]\}}{[s + \lambda p(1 - z)]\{z - \psi[s + \lambda p(1 - z)]\}} \\ &\quad + \frac{[\gamma(s)]^i}{s + \lambda[1 - \gamma(s)]}\left(1 - z\frac{[s + \lambda(1 - z)]\{1 - \psi[s + \lambda p(1 - z)]\}}{[s + \lambda p(1 - z)]\{z - \psi[s + \lambda p(1 - z)]\}}\right),\end{aligned}$$

where $\gamma(s)$ *is the root with smallest absolute value in* z *of the equation*

$$z = \psi[s + \lambda p(1 - z)]. \tag{177}$$

PROOF. Under the assumption $\xi(0) = i$ let us denote by $M_j(t)$ for $j = 0, 1, 2, \cdots$ the expectation of the number of transitions $E_j \to E_{j+1}$ occurring in the time interval $(0, t]$ and denote by $N_j(t)$ for $j = 0, 1, 2, \cdots$ the expectation of the number of transitions $E_{j+1} \to E_j$ occurring in the time interval $(0, t]$. We have evidently

$$M_0(t) = \lambda \int_0^t P_{i0}(u)du \tag{178}$$

and by (128)

$$\int_0^\infty e^{-st}dM_0(t) = \frac{\lambda[\gamma(s)]^i}{s + \lambda[1 - \gamma(s)]}. \tag{179}$$

Since clearly

$$N_j(t) = \sum_{n=1}^\infty P\{\tau_n' \leqq t, \xi_n = j\} \tag{180}$$

we obtain by (165) that

$$\begin{aligned}(181)\quad \sum_{j=0}^\infty z^j \int_0^\infty e^{-st}dN_j(t) &= \sum_{n=1}^\infty U_n(s, z) \\ &= \psi[s + \lambda p(1 - z)]\,\frac{z^i - \dfrac{[s + \lambda(1 - z)][\gamma(s)]^i}{s + \lambda[1 - \gamma(s)]}}{z - \psi[s + \lambda p(1 - z)]}.\end{aligned}$$

Knowing $M_0(t)$ and $N_j(t)$ with $j = 0, 1, 2, \cdots$ for all $t \geqq 0$, the probabilities $P_{ik}(t)$ can be obtained as follows. If $k = 1, 2, \cdots$, then

$$\begin{aligned}(182)\quad P_{ik}(t) &= \delta_{ik}^* [1 - H(t)]e^{-\lambda pt}\frac{(\lambda pt)^{k-i}}{(k - i)!} \\ &\quad + \sum_{j=1}^k \int_0^t [1 - H(t - u)]e^{-\lambda p(t-u)}\frac{[\lambda p(t - u)]^{k-j}}{(k - j)!}dN_j(u) \\ &\quad + \int_0^t [1 - H(t - u)]e^{-\lambda p(t-u)}\frac{[\lambda p(t - u)]^{k-1}}{(k - 1)!}dM_0(u),\end{aligned}$$

where $\delta_{ik}^* = 1$ if $i = 1, 2, \cdots, k$ and $\delta_{ik}^* = 0$ otherwise. Further

$$P_{i0}(t) = \delta_{i0}e^{-\lambda t} + \int_0^t e^{-\lambda(t-u)}dN_0(u), \tag{183}$$

where $\delta_{i0} = 1$ if $i = 0$ and $\delta_{i0} = 0$ if $i > 0$.

In proving (182) we take into consideration that the event $\xi(t) = k$ with $k = 1, 2, \cdots$ can occur in the following mutually exclusive ways. At the instant u, where $0 \leqq u < t$, there occurs a transition $E_{j+1} \rightarrow E_j$ with $j = 1, 2, \cdots, k$ or a transition $E_0 \rightarrow E_1$. The servicing starting at this instant u does not terminate in the time interval $(u, t]$ and during $(u, t]$, respectively $k - j$ or $k - 1$ customers join the queue. Finally we obtain (182) if we also take into consideration that the transition $E_{j+1} \rightarrow E_j$ may be the 1st, 2nd, $\cdots$, nth, $\cdots$ departure and similarly the transition $E_0 \rightarrow E_1$ may be the 1st, 2nd, $\cdots$, nth, $\cdots$ arrival. In proving (183) we take into consideration that the event $\xi(t) = 0$ can occur in such a way that at the instant u (where $0 \leqq u < t$) there occurs a transition $E_1 \rightarrow E_0$ and during $(u, t]$ no customer joins the queue.

Let us form the Laplace transforms of (182) and (183) and write the generating function of $\Pi_{ik}(s)$; then we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} \Pi_{ik}(s)z^k & \\ = \frac{1 - \psi[s + \lambda p(1 - z)]}{s + \lambda p(1 - z)} & \left\{ \sum_{j=1}^{\infty} \int_0^{\infty} e^{-st}dN_j(t) + z \int_0^{\infty} e^{-st}dM_0(t) \right\} \\ + \frac{1}{(\lambda + s)} \int_0^{\infty} e^{-st}dN_0(t) & + \begin{cases} \dfrac{1}{(\lambda + s)}, & i = 0, \\ z^i \dfrac{1 - \psi[s + \lambda p(1 - z)]}{s + \lambda p(1 - z)}, & i > 0. \end{cases} \end{aligned} \tag{184}$$

By (178) and (183) we have

$$\int_0^{\infty} e^{-st}dM_0(t) = \delta_{i0}\frac{\lambda}{\lambda + s} + \frac{\lambda}{\lambda + s} \int_0^{\infty} e^{-st}dN_0(t) \tag{185}$$

and hence by (180), (181), (184), and (185) we obtain (176).

THEOREM 18. *If* $\lambda p\alpha < 1$, *then the limiting distribution* $\lim_{t\rightarrow\infty} P\{\xi(t) = k\} = P_k^*$ *for* $k = 0, 1, 2, \cdots$ *exists and is independent of the initial distribution. We have*

$$P_k^* = \begin{cases} \dfrac{pP_0}{1 - \lambda(1 - p)\alpha}, & k = 0, \\ \dfrac{P_k}{1 - \lambda(1 - p)\alpha}, & k = 1, 2, \cdots, \end{cases} \tag{186}$$

where $\{P_k\}$ *is defined by* (162). *If* $\lambda p\alpha \geqq 1$ *then* $\lim_{t\rightarrow\infty} P\{\xi(t) = k\} = 0$ *for every* k.

PROOF. The transitions $E_{j+1} \rightarrow E_j$ for $j = 0, 1, 2, \cdots$ and similarly the transitions $E_0 \rightarrow E_1$ form a recurrent process. The distances between successive transitions are identically distributed independent random variables having nonlattice distributions. Therefore by a theorem of Blackwell [1] it follows that the following limits exist for every $h > 0$ and agree with the corresponding right sides

$$\lim_{t\to\infty} \frac{N_j(t+h) - N_j(t)}{h} = \lim_{t\to\infty} \frac{N_j(t)}{t}, \qquad j = 0, 1, 2, \cdots, \tag{187}$$

and

$$\lim_{t\to\infty} \frac{M_0(t+h) - M_0(t)}{h} = \lim_{t\to\infty} \frac{M_0(t)}{t}. \tag{188}$$

Furthermore it is easy to see that these limits are independent of the initial state. Forming the Riemann-Stieltjes sums approximating the integrals (182) and (183), respectively, and using (187) and (188) we obtain that the limit

$$\lim_{t\to\infty} P_k(t) = P_k^*, \qquad k = 0, 1, 2, \cdots, \tag{189}$$

always exists and is independent of the initial state. Specifically we have

$$P_0^* = \frac{1}{\lambda}(N_0 + M_0) \tag{190}$$

and for $k = 1, 2, \cdots$

$$\begin{aligned} P_k^* = \sum_{j=1}^{k} N_j \int_0^\infty [1 - H(x)]e^{-\lambda px}\frac{(px)^{k-j}}{(k-j)!}dx \\ + M_0 \int_0^\infty [1 - H(x)]e^{-\lambda px}\frac{(\lambda px)^{k-1}}{(k-1)!}dx, \end{aligned} \tag{191}$$

where

$$\lim_{t\to\infty} \frac{N_j(t)}{t} = N_j \quad \text{and} \quad \lim_{t\to\infty} \frac{M_0(t)}{t} = M_0. \tag{192}$$

Now it is valid that the difference of the numbers of transitions $E_k \to E_{k+1}$ and $E_{k+1} \to E_k$ occurring in the time interval $(0, t]$ is at most one. Hence we have

$$|M_k(t) - N_k(t)| \leqq 1 \tag{193}$$

for every k and $t \geqq 0$. On the other hand obviously

$$M_k(t) = \begin{cases} \lambda p \int_0^t P_k(u)du, & k = 1, 2, \cdots, \\ \lambda \int_0 P_0(u)du, & k = 0. \end{cases} \tag{194}$$

Having proved that $\lim_{t\to\infty} P_k(t) = P_k^*$ for $k = 0, 1, 2, \cdots$ always exists we can conclude that

$$\lim_{t\to\infty} \frac{M_k(t)}{t} = \begin{cases} \lambda p P_k^*, & k = 1, 2, \cdots, \\ \lambda P_0^*, & k = 0. \end{cases} \tag{195}$$

By the theory of Markov chains it follows that

$$\lim_{t\to\infty} \frac{N_k(t)}{N(t)} = P_k, \qquad k = 0, 1, \cdots, \tag{196}$$

where $N(t)$ denotes the expectation of the number of the departing customers in the time interval $(0, t]$, that is,

$$N(t) = \sum_{k=0}^{\infty} N_k(t). \tag{197}$$

We have proved that if $\lambda p\alpha < 1$ then $P_k > 0$ for every k and $\{P_k\}$ is a probability distribution, while if $\lambda p\alpha \geqq 1$ then $P_k = 0$ for every k. Now by (193)

$$\lim_{t\to\infty} \frac{M_k(t)}{t} = \lim_{t\to\infty} \frac{N_k(t)}{t}, \tag{198}$$

and thus in virtue of (195) and (196) we get

$$P_k^* = \begin{cases} \dfrac{P_k}{\lambda p} \lim\limits_{t\to\infty} \dfrac{N(t)}{t}, & k = 1, 2, \cdots, \\ \dfrac{P_0}{\lambda} \lim\limits_{t\to\infty} \dfrac{N(t)}{t}, & k = 0. \end{cases} \tag{199}$$

If we suppose that $\lambda p\alpha < 1$, then by (162) and (131) we obtain

$$P_0^* = \frac{pP_0}{1 + \lambda(1 - p)\alpha} \tag{200}$$

and hence if we write $k = 0$ in (199) we obtain

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{\lambda p}{1 + \lambda(1 - p)\alpha} \tag{201}$$

and by (199)

$$P_k^* = \frac{pP_k}{1 + \lambda(1 - p)\alpha} \qquad k = 1, 2, \cdots. \tag{202}$$

Formulas (200) and (202) prove (186).

If $\lambda p\alpha \geqq 1$ then every $P_k = 0$ and since $\lim_{t\to\infty} N_k(t)/t$ is evidently finite it follows by (199) that $P_k^* = 0$ for every k. This completes the proof of the theorem.

11. The process of departures

Let us suppose that there is a departure at $t = -0$ and denote by $\tau_1', \tau_2', \cdots, \tau_n', \cdots$ the sequence of the successive departures. Further denote by $N(t)$ the expected number of the departures occurring in the time interval $(0, t]$.

THEOREM 19. *If* $\mathrm{Re}(s) > 0$ *then*

$$\int_0^\infty e^{-st} dN(t) = \frac{\psi(s)}{1 - \psi(s)} \left\{1 - \frac{sU_0[\gamma(s)]}{s + \lambda[1 - \gamma(s)]}\right\}, \tag{203}$$

where $\gamma(s)$ *is the root with smallest absolute value in* z *of the equation*

$$z = \psi[s + \lambda p(1 - z)], \tag{204}$$

and

$$U_0(z) = E\{z^{\xi(0)}\} \tag{205}$$

is the generating function of the initial queue size.

PROOF. If we put $z = 1$ in (165) we get

$$\sum_{n=1}^{\infty} E\{e^{-s\tau_n'}\}w^n = \frac{1 - \dfrac{sw\psi(s)U_0[\gamma(s, w)]}{s + \lambda[1 - \gamma(s, w)]}}{1 - w\psi(s)} - 1. \tag{206}$$

Since

$$N(t) = \sum_{n=1}^{\infty} P\{\tau_n' \leqq t\} \tag{207}$$

we obtain (203) from (206) letting $w \to 1$.

Now let us suppose that $\lambda p\alpha < 1$ and the initial distribution of the queue size agrees with the stationary distribution defined by (162). In this case

$$U_0(z) = \frac{(1 - \lambda p\alpha)(1 - z)\psi[\lambda p(1 - z)]}{\psi[\lambda p(1 - z)] - z} \tag{208}$$

and by (203)

$$\int_0^{\infty} e^{-st}dN(t) = \frac{\psi(s)}{1 - \psi(s)}\left\{1 - \frac{(1 - \lambda p\alpha)s[1 - \gamma(s)]\psi\{\lambda p[1 - \gamma(s)]\}}{\{s + \lambda[1 - \gamma(s)]\}(\psi\{\lambda p[1 - \gamma(s)]\} - \gamma(s))}\right\}. \tag{209}$$

If we suppose in particular that $H(x) = 1 - \exp(-\mu x)$ for $x \geqq 0$, when $\psi(s) = \mu/(\mu + s)$ and $\alpha = 1/\mu$, then by (209) we obtain

$$\int_0^{\infty} e^{-st}dN(t) = \frac{\mu}{s}\left[1 - \frac{1 - \dfrac{\lambda p}{\mu}}{1 - \dfrac{\lambda(1 - p)}{\mu}\gamma(s)}\right]. \tag{210}$$

In the case of $p = 1$ (210) reduces to

$$\int_0^{\infty} e^{-st}dN(t) = \frac{\lambda}{s}, \tag{211}$$

whence

$$N(t) = \lambda t. \tag{212}$$

The latter equation is in agreement with the fact that in this case the departures follow a Poisson process with density λ. This theorem was proved by P. J. Burke [4] and E. Reich [19]. In the general case the output process $\{\tau_n'\}$ cannot be characterized in a similar simple way.

The interdeparture times can always be expressed as follows,

$$\tau_{n+1}' - \tau_n' = \begin{cases} \chi_{n+1}^*, & \xi_n \geqq 1, \\ \chi_{n+1}^* + \vartheta_{n+1}^*, & \xi_n = 0, \end{cases} \tag{213}$$

where $\{\chi_n^*\}$ and $\{\vartheta_n^*\}$ are independent sequences of identically distributed independent random variables with distribution functions $P\{\chi_n^* \leqq x\} = H(x)$ and

$P\{\vartheta_n^* \leqq x\} = F(x) = 1 - \exp(-\lambda x)$ if $x \geqq 0$, and further ξ_n is the queue size immediately after the nth departure. Then we have

$$P\{\tau'_{n+1} - \tau'_n \leqq x\} = [1 - P\{\xi_n = 0\}]H(x) + P\{\xi_n = 0\}H(x) * F(x). \tag{214}$$

If $\lambda p\alpha < 1$ then $\lim_{n\to\infty} P\{\xi_n = 0\} = 1 - \lambda p\alpha$, whence

$$\lim_{n\to\infty} P\{\tau'_{n+1} - \tau'_n \leqq x\} = \lambda p\alpha H(x) + (1 - \lambda p\alpha)H(x) * F(x), \tag{215}$$

and

$$\lim_{n\to\infty} E\{\tau'_{n+1} - \tau'_n\} = \frac{1 + \lambda\alpha(1-p)}{\lambda}. \tag{216}$$

The Laplace-Stieltjes transform of the limiting distribution (215) is

$$\psi(s) \frac{\lambda(1 + p\alpha s)}{\lambda + s}. \tag{217}$$

The distribution function (215) is an exponential distribution if and only if $p = 1$ and $H(x) = 1 - \exp(-x/\alpha)$ if $x \geqq 0$.

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