RECURRENT RANDOM WALK AND LOGARITHMIC POTENTIAL

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1. Introduction

This is an attempt to show that a potential theory is associated with certain recurrent Markov processes in a natural way. For transient Markov processes this fact has been studied intensely. Thus Hunt [9] bases a general potential theory on transient continuous parameter processes, Doob [7] and Hunt [10] use the theory to construct boundaries for discrete parameter processes, Itô and McKean [11] solve the problem of characterizing the recurrent sets for simple random walk in three and higher dimension within the framework of the associated potential theory.

The class of recurrent Markov processes considered here are defined as follows. The state space will be E, the set of all ordered pairs (lattice points) $x = (m, n)$ where m and n are integers. X is a random variable with values in E. Its characteristic function $\phi(\theta)$ is

$$\phi(\theta) = E[e^{i\theta\cdot X}] = \sum_{x\in E} e^{i\theta\cdot x} P\{X = x\}, \tag{1.1}$$

where $\theta = (\theta_1, \theta_2)$ and $\theta\cdot x = \theta_1 m + \theta_2 n$ if $x = (m, n)$. Here $X_1, X_2, \cdots$ is an infinite sequence of independent random variables with the same distribution as X. Each characteristic function $\phi(\theta)$ defines a Markov process

$$S_n = S_0 + X_1 + \cdots + X_n, \qquad n \geqq 1, \tag{1.2}$$

where the starting point S_0 is an arbitrary point in E.

For simplicity we assume throughout that $\phi(\theta) = \phi(-\theta)$, or equivalently that $P\{X = x\} = P\{X = -x\}$. Two further assumptions are essential to the theory. Let S denote the square $|\theta_1| \leqq \pi$, $|\theta_2| \leqq \pi$ in the θ-plane and let $\int [\ ]\, d\theta$ denote integration over S where $d\theta$ is two-dimensional Lebesgue measure. For $\theta \in S$, it is assumed that $\phi(\theta)$ satisfies

$$\phi(\theta) = 1 \Rightarrow \theta = 0 \tag{1.3}$$

$$\int \frac{d\theta}{1 - \phi(\theta)} = \infty. \tag{1.4}$$

A number theoretical argument may be used to show that (1.3) is equivalent to the condition that every x in E is a possible value of the process S_n, that is, that, given S_0 and x, there is a positive integer n such that $P\{S_n = x\} > 0$. Condition (1.4) is equivalent to

$$\sum_{n=0}^{\infty} P\{S_n = S_0\} = \infty \tag{1.5}$$

and if (1.3) holds, this is equivalent to

$$P\left\{\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} (S_k = x)\right\} = 1 \tag{1.6}$$

for every S_0, that is, every point in E is visited infinitely often with probability one. These things are discussed in [4].

Conditions (1.3) and (1.4) are obviously satisfied by the *simple random walk* with

$$\phi(\theta) = \frac{1}{2}\cos\theta_1 + \frac{1}{2}\cos\theta_2, \tag{1.7}$$

and, according to [4], a sufficient condition for (1.4) to hold is, using the assumption that $\phi(\theta) = \phi(-\theta)$,

$$\sum_{x\in E} |x|^2 P\{X = x\} = E(|X|^2) < \infty. \tag{1.8}$$

The potential theory will be based on three operators P, A, and K. P has the representation (kernel)

$$P(x, y) = P\{X = x - y\}, \tag{1.9}$$

and will act on the space $\mathfrak{L}$ of real-valued functions on E for which $P|f| < \infty$, that is, $P: \mathfrak{L} \to \mathfrak{L}$, where

$$Pf(x) = \sum_{y\in E} P(x, y)f(y), \qquad f \in \mathfrak{L} = \{f|P|f| < \infty\}. \tag{1.10}$$

If I is the identity then $P - I$ (being the second difference operator for simple random walk) is our version of the *Laplace operator*. We let Ω denote a finite subset of E and $\mathfrak{L}(\Omega)$ the vector space of real-valued functions on Ω whose dimension $|\Omega|$ is the number of points in Ω. In section 2, lemma 2.1, it is shown that, for each pair x, y in E, the integral

$$A(x, y) = (2\pi)^{-2} \int \frac{1 - \cos(x - y)\cdot\theta}{1 - \phi(\theta)}\, d\theta \tag{1.11}$$

exists and we define A as an operator (*potential kernel*) $A: \mathfrak{L}(\Omega) \to \mathfrak{L}$ by

$$Af(x) = \sum_{a\in\Omega} A(x, a)f(a), \qquad x \in E, \quad f \in \mathfrak{L}(\Omega). \tag{1.12}$$

The one-dimensional analogue of the kernel $A(x, y)$ was first studied by Kac [12] in a somewhat different context. In section 3, lemma 3.1 we use one of his results concerning $A(x, y)$. (This kernel is studied by W. Hoeffding in his paper "On sequences of sums of independent random vectors," given at this Symposium. In particular, many of the results in section 2 are also obtained by Hoeffding, and used in a context closely related to potential theory.)

Lemma 2.2 will show that $A\psi \in \mathfrak{L}$ when $\psi \in \mathfrak{L}(\Omega)$. It will show more, namely that $(P - I)A\psi(x) = 0$ when $x \in E - \Omega$ whereas $(P - I)A\psi(x) = \psi(x)$ when $x \in \Omega$. This is familiar from potential theory. If ψ is a nonnegative function (charge distribution on Ω) in $\mathfrak{L}(\Omega)$, $f = A\psi$ satisfies Laplace's equation $(P - I)f = 0$ off Ω whereas f satisfies Poisson's equation $(P - I)f = \psi$ on Ω. In the classical case the sign is reversed in the last equation. This could be remedied by using $-A$ as the potential kernel instead of A. Our notation has the advantage of a more natural probability interpretation which is given in section 4.

If the range of the operator A is restricted to $\mathfrak{L}(\Omega)$, A becomes a (matrix) operator mapping $\mathfrak{L}(\Omega)$ into $\mathfrak{L}(\Omega)$. In section 3, lemma 3.1, this mapping will be shown to be onto, that is, the restricted version of A has an inverse $K = K(x, y)$, $x, y \in \Omega$ defined by $K\colon \mathfrak{L}(\Omega) \to \mathfrak{L}(\Omega)$,

$$\begin{aligned} Kf(x) &= \sum_{a\in\Omega} K(x, a)f(a), && \text{for} \quad f \in \mathfrak{L}(\Omega), \quad x \in \Omega, \\ AKf(x) &= f(x), && \text{when} \quad f \in \mathfrak{L}(\Omega), \quad x \in \Omega. \end{aligned} \tag{1.13}$$

The first of several equivalent definitions, in section 8, of the *capacity* of the set Ω will be in terms of the quadratic form (e, Ke) where $e = e(x) \equiv 1$ for $x \in \Omega$.

This paper relies to some extent on the *imbedded Markov chain* induced by the finite set Ω. This chain is simply the process S_n observed only at those (random) times n when S_n is in Ω. Thus its transition matrix is the (stochastic) matrix Π of size $|\Omega,|$ defined by

$$\Pi(x, y) = \sum_{n=1}^{\infty} P\{S_\nu \notin \Omega \quad \text{for} \quad \nu = 1, \cdots, n - 1;\ S_n = y|S_0 = x\}, \qquad x, y \in \Omega. \tag{1.14}$$

In section 3, theorem 3.1, a formula for $\Pi(x, y)$ is obtained in terms of $A(x, y)$ and $K(x, y)$ by the method of the imbedded Markov chain, the foundation for which was laid in [14]. The method adds probabilistic interest to the potential theory and makes it more explicit, without being essential to its logical development. Thus, in section 5, theorem 5.2 gives an explicit formula for the *harmonic measure*. It was first discovered by the method of the imbedded Markov chain; nevertheless the proof which is given in section 5 consists of verifying that the formula obtained is the appropriate solution to the exterior Dirichlet problem, which has a unique solution in view of theorem 5.1. The last three sections, sections 6, 7, 8, imitate classical potential theory, giving explicit representations as well as the standard theorems for the *potential*, *Green function*, and *capacity* of finite sets of lattice points, corresponding to each process S_n, satisfying $\phi(\theta) = \phi(-\theta)$ as well as conditions (1.3) and (1.4).

The methods and results of this paper should be compared to those of Itô and McKean [11]. Our method will yield part of their results, namely those concerned with finite sets, in a different and more explicit form. Thus the matrix Π of the imbedded Markov chain can be expressed in terms of the inverse of Itô and McKean's potential kernel, restricted to the set in question. (Π is of course substochastic in their theory.) Just as we do in section 8, the capacity of a finite set can be expressed as a quadratic form, and the equilibrium charge on a finite set Ω turns out to be the limit of $p(x, a)$ as $x \to \infty$, $p(x, a)$ being the conditional probability that the set Ω is first hit at a, starting at x, given that it is hit in a finite time.

The fact that we get more formulas than Itô and McKean is of dubious merit. The hard calculations in [11] are based on maximum principles, such as we obtain in section 7, enabling one to use integral estimates from classical potential theory rather than "explicit" formulas, involving matrix inversion.

Typical new problems arising from this work are, in increasing order of difficulty: (1) Its extension to nonlattice random walk in the plane of the same sum of independent variable type as here. The theory of integral equations with completely continuous kernels should imitate the present matrix theory without serious difficulty. (2) To understand the connection of the limit theorems (as time tends to infinity) of Kac [12] which lead to the same potential kernel as the present (time-independent) theory. (3) To discover the domain of validity of present key theorems among general irreducible recurrent Markov chains. For example let x_k, $k = 1, 2, \cdots$ be the states of such a chain and Ω a finite set of states. What are necessary and sufficient conditions, that is what does it mean, for $\lim_{n\to\infty} p(x_n, a)$ to exist for all a in Ω, for every Ω, where $p(x, a)$ is the harmonic measure, defined as in (5.2), of Ω?

2. Some Fourier analysis

To justify the definition in (1.11) we first show

LEMMA 2.1. *For $x, y \in E$, $[1 - \cos (x - y)\cdot\theta][1 - \phi(\theta)]^{-1}$ is Lebesgue integrable on S and*

$$A(x, y) = (2\pi)^{-2} \int \frac{1 - \cos (x - y)\cdot\theta}{1 - \phi(\theta)}\, d\theta \tag{2.1}$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N} [P\{S_n = 0|S_0 = 0\} - P\{S_n = x - y|S_0 = 0\}].$$

PROOF. Let $S_0 = 0$. In view of (1.3), $[1 - \phi(\theta)]^{-1}$ is continuous on S except at the origin. $\phi(\theta)$ is real. Suppose that

$$Q(\theta) = E[(\theta\cdot X)^2] = \theta_1^2\sigma_{11} + 2\theta_1\theta_2\sigma_{12} + \theta_2^2\sigma_{22} < \infty. \tag{2.2}$$

Then

$$\lim_{\theta\to 0} \frac{1 - \phi(\theta)}{Q(\theta)} = \frac{1}{2}. \tag{2.3}$$

Equation (1.3) implies that $Q(\theta)$ is a positive definite quadratic form. Consequently there is a constant $d > 0$ such that $[1 - \phi(\theta)]^{-1} \leqq d(\theta\cdot\theta)^{-1}$ for all θ in S. The integral defining $A(x, y)$ therefore exists in the sense of Lebesgue, since $[1 - \cos (x - y)\cdot\theta](\theta\cdot\theta)^{-1}$ is Lebesgue integrable on S. If on the other hand $Q(\theta) = +\infty$, for some $\theta \neq 0$ we have

$$E[|X|^2] = +\infty, \quad \lim_{\theta \to 0} \frac{1 - \phi(\theta)}{\theta \cdot \theta} = +\infty, \qquad |X|^2 = X \cdot X, \tag{2.4}$$

and the previous argument may be resumed.

$$\begin{aligned} &\sum_{n=0}^{N} [P\{S_n = 0\} - P\{S_n = x - y\}] \\ &\qquad = (2\pi)^{-2} \int [1 - \cos(x - y) \cdot \theta][1 + \phi(\theta) + \cdots + \phi(\theta)^N] \, d\theta \\ &\qquad = A(x, y) - (2\pi)^{-2} \int \frac{1 - \cos(x - y) \cdot \theta}{1 - \phi(\theta)} [\phi(\theta)]^{N+1} \, d\theta. \end{aligned} \tag{2.5}$$

Since $|\phi(\theta)| < 1$ except at most at a finite number of points in S, the last term tends to zero as N tends to infinity.

$A(x, y)$ will play the role of a potential kernel because of

LEMMA 2.2. *For $f \in \mathfrak{L}(\Omega)$, $Af \in \mathfrak{L}$, and*

$$(P - I)Af(x) = \begin{cases} f(x), & x \in \Omega \\ 0, & x \in E - \Omega. \end{cases} \tag{2.6}$$

(For the case of simple random walk this fact is due to Stöhr [15], as is the result of lemma 2.4.)

PROOF. Since $A(x, y)$ is a difference kernel we assume without loss of generality that Ω contains 0 (the origin). Let $f(0) = 1$, and $f(x) \equiv 0$ when $x \in \Omega$, $x \neq 0$. It remains to show that, for $x \in E$,

$$\sum_{t \in E} P(x, t) A(t, 0) - A(x, 0) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases} \tag{2.7}$$

The left side is

$$\begin{aligned} &(2\pi)^{-2} \int [1 - \phi(\theta)]^{-1} [1 - \sum_{t \in E} P(x, t) \cos t \cdot \theta - 1 + \cos x \cdot \theta] \, d\theta \\ &\qquad = (2\pi)^{-2} \int [1 - \phi(\theta)]^{-1} [-(\cos x \cdot \theta) \phi(\theta) + \cos x \cdot \theta] \, d\theta \\ &\qquad = (2\pi)^{-2} \int \cos x \cdot \theta \, d\theta = 1 \end{aligned} \tag{2.8}$$

when $x = 0$ and 0 otherwise.

By linearity (2.6) holds for all $f \in \mathfrak{L}(\Omega)$ and by the triangle inequality $P|Af| < \infty$ so that $Af \in \mathfrak{L}$.

The next lemma, concerning the asymptotic behavior of $A(x, y)$ is responsible for the asymptotic results in our potential theory. All we assume is that $\phi(\theta) = \phi(-\theta)$ and that $\phi(\theta)$ satisfies (1.3) and (1.4). Let

$$a(x) = A(x, 0) = (2\pi)^{-2} \int \frac{1 - \cos x \cdot \theta}{1 - \phi(\theta)} \, d\theta. \tag{2.9}$$

LEMMA 2.3.

$$\lim_{x \to \infty} a(x) = +\infty. \tag{2.10}$$

For x_1, x_2 *in* E

$$\lim_{x\to\infty} [A(x, x_1) - A(x, x_2)] = 0. \tag{2.11}$$

PROOF. Let S_ϵ be the subset of S defined by $\theta\cdot\theta < \epsilon^2$, with $0 < \epsilon < \pi$, and decompose

$$\begin{aligned}(2\pi)^2 a(x) &= \int_{S_\epsilon} \frac{1-\cos x\cdot\theta}{1-\phi(\theta)}\,d\theta + \int_{S-S_\epsilon} \frac{1-\cos x\cdot\theta}{1-\phi(\theta)}\,d\theta \\ &\geqq \int_{S-S_\epsilon} \frac{d\theta}{1-\phi(\theta)} - \int_{S-S_\epsilon} \frac{\cos x\cdot\theta}{1-\phi(\theta)}\,d\theta.\end{aligned} \tag{2.12}$$

With ϵ fixed let $x\cdot x = |x|^2$ tend to infinity, and the two-dimensional Riemann Lebesgue lemma [1] gives $\mathit{lim}_{x\to\infty}\, (2\pi)^2 a(x) \geqq \int_{S-S_\epsilon} d\theta/[1 - \phi(\theta)]$ for every $0 < \epsilon < \pi$. Because of (1.4), the integral on the right tends to infinity as ϵ tends to zero, and this proves (2.10).

Since $A(x, y)$ is a difference kernel it clearly suffices to prove (2.11) for $x_2 = 0$, with x_1 arbitrary in E. Letting $u_1 = (1, 0)$, $u_2 = (0, 1)$, $x = mu_1 + nu_2$, it is clear that

$$\begin{aligned}&\lim_{x\to\infty} [A(x_1, x) - A(0, x)] \\ &\qquad = m \lim_{x\to\infty} [A(u_1, x) - A(0, x)] + n \lim_{x\to\infty} [A(u_2, x) - A(0, x)],\end{aligned} \tag{2.13}$$

provided one can show that the last two limits exist. Since they are of the same type it suffices to consider only the first one and the proof of (2.11) is completed by showing that it is zero.

$$\begin{aligned}(2\pi)^2[A(u_1, x) - A(0, x)] &= \int \frac{\cos x\cdot\theta - \cos (x - u_1)\cdot\theta}{1-\phi(\theta)}\,d\theta \\ &= -2\int \frac{\sin\frac{\theta_1}{2}\sin\left(x - \frac{u_1}{2}\right)\cdot\theta}{1-\phi(\theta)}\,d\theta.\end{aligned} \tag{2.14}$$

But $f(\theta) = -2\sin(\theta_1/2)[1 - \phi(\theta)]^{-1}$ is integrable on S since $|\theta_1|(\theta\cdot\theta)^{-1}$ is integrable. Letting $x - u_1/2 = y$ and noting that y tends to infinity with x, we have

$$\lim_{x\to\infty} (2\pi)^2[A(u_1, x) - A(0, x)] = \lim_{y\to\infty} \int f(\theta)\sin y\cdot\theta\,d\theta = 0 \tag{2.15}$$

by the two-dimensional Riemann-Lebesgue lemma.

Sharper results concerning the asymptotic behavior of $A(x, y)$ require further assumptions concerning the random walk S_n, or its characteristic function $\phi(\theta)$. Under the assumption that $E(|X|^2) < \infty$ it can be shown that $a(x) \sim c \log |x|$ as $|x| \to \infty$, where c is a positive constant. (Theorem 5.1 of Hoeffding's Symposium paper.) However, the behavior of the difference $a(x) - c\log|x|$ depends on the direction in which x tends to infinity, unless we assume the random walk to be isotropic, that is, that the quadratic form $Q(\theta)$ has no preferred principal axes. To get a result as strong as in classical potential theory it also seems necessary to assume that $|X|$ has a finite moment of order greater than two.

LEMMA 2.4. *If* $Q(\theta) = E[(\theta\cdot X)^2] = \sigma^2\theta\cdot\theta < \infty$, *and if* $E(|X|^{2+\delta}) < \infty$ *for some* $\delta > 0$, *then*

$$\lim_{x\to\infty}\left[a(x) - \frac{1}{\pi\sigma^2}\log|x|\right] = c + \frac{1}{\pi\sigma^2}d = k, \tag{2.16}$$

$$0 < c = (2\pi)^{-2}\int\left[\frac{1}{1-\phi(\theta)} - \frac{2}{Q(\theta)}\right]d\theta < \infty, \tag{2.17}$$

where $\gamma = .5772\cdots$ *is Euler's constant,* $\lambda = \sum_{n=0}^{\infty}(-1)^n(2n+1)^{-2}$ *is Catalan's constant, and* $d = \gamma + \log\pi - 2\lambda/\pi$. *For the simple random walk* $\sigma^2 = 1/2$, $k = (1/\pi)\log 8 + 2\gamma/\pi$.

PROOF. First we check that the integrand

$$\psi(\theta) = [1-\phi(\theta)]^{-1}Q^{-1}(\theta)\{Q(\theta) - 2[1-\phi(\theta)]\} \tag{2.18}$$

in (2.17) is Lebesgue integrable. Since $[1-\phi(\theta)]^{-1}Q^{-1}(\theta)$ has a positive constant times $(\theta\cdot\theta)^{-2}$ as a lower bound, it suffices to prove integrability for

$$\begin{aligned}\chi(\theta) &= (\theta\cdot\theta)^{-2}\left[\frac{\sigma^2}{2}(\theta\cdot\theta) - 1 + \phi(\theta)\right] \\ &= (\theta\cdot\theta)^{-2}E\left[\frac{1}{2}(\theta\cdot X)^2 - 1 + \cos\theta\cdot X\right].\end{aligned} \tag{2.19}$$

For some constant $h > 0$, by the Schwarz inequality,

$$\begin{aligned}|\chi(\theta)| &\leqq h(\theta\cdot\theta)^{-2}E(|\theta\cdot X|^{2+\delta}) \\ &\leqq h(\theta\cdot\theta)^{-2}E\{[(\theta\cdot\theta)(X\cdot X)]^{1+\delta/2}\} \\ &\leqq h(\theta\cdot\theta)^{-1+\delta/2}E(|X|^{2+\delta}),\end{aligned} \tag{2.20}$$

which is integrable.

Now we decompose

$$\begin{aligned}a(x) &= (2\pi)^{-2}\int\frac{1-\cos x\cdot\theta}{1-\phi(\theta)}d\theta \\ &= \frac{2}{\sigma^2}(2\pi)^{-2}\int\frac{1-\cos\theta\cdot x}{\theta\cdot\theta}d\theta + (2\pi)^{-2}\int(1-\cos\cdot x\ \theta)\psi(\theta)\,d\theta.\end{aligned} \tag{2.21}$$

The last integral tends to c by the two-dimensional Riemann Lebesgue lemma and it only remains to verify that

$$\begin{aligned}&\lim_{x\to\infty}\left(\frac{1}{2\pi}\int\frac{1-\cos x\cdot\theta}{\theta\cdot\theta}d\theta - \log|x|\right) \\ &\qquad = \gamma + \log\pi - \frac{2}{\pi}\lambda\,\frac{1}{2\pi}\int\frac{1-\cos x\cdot\theta}{\theta\cdot\theta}d\theta = I_1(x) + I_2(x),\end{aligned} \tag{2.22}$$

where

$$I_1(x) = \frac{1}{2\pi} \int\limits_{0 \leqq \sqrt{\theta \cdot \theta} < \pi} \frac{1 - \cos x \cdot \theta}{\theta \cdot \theta} \, d\theta, \tag{2.23}$$

and $I_2(x)$ is the integral over the remaining part of the square S
By introducing polar coordinates it is shown that $I_1(x) - \log |x| \rightarrow \gamma + \log (\pi/2)$, using the formula

$$\gamma = \int_0^1 \frac{1 - \cos t}{t} \, dt - \int_1^\infty \frac{\cos t}{t} \, dt. \tag{2.24}$$

The second integral $I_2(x)$, by the Riemann-Lebesgue lemma, tends to a limit as $x \rightarrow \infty$ which may be expressed in terms of Catalan's constant λ. The details are omitted as is the calculation of k for simple random walk.

3. The imbedded Markov chain

For the purpose of this section only let $A = A(x, y)$ be restricted to x, y in $\Omega \subset E$, and let $2 \leqq |\Omega| < \infty$. The case when $|\Omega| = 1$ is trivial. Imitating an argument of Kac [12], we obtain more than is needed to assert that A has an inverse K according to (1.13). A real number λ is an eigenvalue of A if there is a function f in $\mathfrak{L}(\Omega)$ such that $Af = \lambda f$ on Ω. Thus the inverse K exists if $\lambda = 0$ is not an eigenvalue. Also $e = e(x) \equiv 1$ for x in Ω and $(f, g) = \sum_{x \in \Omega} f(x) g(x)$.

LEMMA 3.1. *A has one simple positive eigenvalue, its other eigenvalues are negative and the quadratic form* $(e, Ke) = \sum_{x \in \Omega} \sum_{y \in \Omega} K(x, y) \neq 0$.

PROOF. Suppose that λ_1 and λ_2 are two distinct positive eigenvalues of A with eigenvectors u_1 and u_2 in $\mathfrak{L}(\Omega)$. Since A is symmetric the functions u_1 and u_2 may be taken real and such that $(u_1, u_1) = (u_2, u_2) = 1$, $(u_1, u_2) = 0$. One may choose real constants α_1, α_2, not both zero, such that $(v, e) = 0$ where $v = \alpha_1 u_1 + \alpha_2 u_2$. Direct computation yields

$$(v, Av) = \alpha_1^2 \lambda_1 + \alpha_2^2 \lambda_2 > 0. \tag{3.1}$$

On the other hand, for every v such that $(v, e) = 0$ the definition of $A(x, y)$ in (1.11) gives

$$(v, Av) = -(2\pi)^{-2} \int \Big| \sum_{x \in \Omega} v(x) \, e^{ix \cdot \theta} \Big|^2 \, |1 - \phi(\theta)|^{-1} \, d\theta \leqq 0 \tag{3.2}$$

with strict inequality if and only if $v \equiv 0$ on Ω. Thus (3.2) contradicts (3.1) so that A has at most one positive eigenvalue, and there must be one, since the trace of A is zero. This eigenvalue is simple by the theory of positive matrices of Frobenius (for every $\epsilon > 0$, $A + \epsilon I$ is a strictly positive matrix whose largest eigenvalue is simple).

A is nonsingular, for suppose $Au = 0$, $(u, u) = 1$. Then either $(u, e) = 0$ in which case (3.2) applied to u gives $(u, Au) < 0$ which is impossible, or $(u, e) \neq 0$. In the latter case let $\mu > 0$, $Av = \mu v$, $(v, v) = 1$. By (3.2) $(v, e) \neq 0$. Hence we can choose real $\alpha \neq 0$, $\beta \neq 0$, such that $(\alpha u + \beta v, e) = 0$ and calculate, for $w = \alpha u + \beta v$, that $(w, Aw) = \beta^2 \mu < 0$ which is impossible.

Finally let K be the inverse of A which has just been shown to exist, and take $v = Ke$. Since K has an inverse, v is not the zero vector. If $(v, e) = 0$, then (3.2) implies $(v, Av) < 0$ which is impossible since $(v, Av) = (e, Ke) = (v, e)$. Hence $(v, e) = (e, Ke) \neq 0$.

Unfortunately the eigenvalues and eigenfunctions of A do not seem to have a natural place in the potential theory. An additional fact which does, to be proved in section 5, is that

$$Ke(x) \geqq 0 \quad \text{for} \quad x \in \Omega, \quad \text{and} \quad (e, Ke) > 0. \tag{3.3}$$

Lemma 3.1 is used to derive a representation for the transition matrix $\Pi = \Pi(x, y)$, defined in (1.14), of the imbedded Markov chain. For $f, g \in \mathcal{L}(\Omega)$ we write $f \otimes g$ for the matrix $f(x)\, g(y)$, with $x, y \in \Omega$.

Theorem 3.1. *If* $2 \leqq |\Omega| < \infty$,

$$\Pi = I + K - \frac{Ke \otimes Ke}{(e, Ke)}. \tag{3.4}$$

Proof. Define $\Pi_t = \Pi_t(x, y)$ for $0 \leqq t < 1$, by

(3.5)

$$\Pi_t(x, y) = \sum_{n=1}^{\infty} t^n P\{S_\nu \notin \Omega \quad \text{for} \quad \nu = 1, \cdots, n-1; S_n = y | S_0 = x\},$$

$$x, y \in \Omega.$$

Then $\Pi(x, y) = \lim_{t \to 1^-} \Pi_t(x, y)$ and a simple renewal argument, given in detail in [14], shows that

$$(I - \Pi_t)^{-1}(x, y) = \sum_{n=0}^{\infty} t^n P\{S_n = y | S_0 = x\}, \quad 0 \leqq t < 1; x, y \in \Omega. \tag{3.6}$$

This renewal argument gives $I + \Pi_t + \Pi_t^2 + \cdots$ on the left, but by (3.5) Π_t has all row sums less than one when $0 \leqq t < 1$ so that one obtains (3.6). Let

$$s(t) = (2\pi)^{-2} \int \frac{d\theta}{1 - t\phi(\theta)}, \qquad 0 \leqq t < 1. \tag{3.7}$$

Using lemma 2.1, equation (3.6) may be written for $0 \leqq t < 1$,

$$(I - \Pi_t)^{-1} = s(t) e \otimes e - A + B_t, \tag{3.8}$$

where

$$\begin{aligned} B_t &= B_t(x, y) \\ &= (2\pi)^{-2} \int [1 - \cos (x - y) \cdot \theta] \left[\frac{1}{1 - \phi(\theta)} - \frac{1}{1 - t\phi(\theta)} \right] d\theta. \end{aligned} \tag{3.9}$$

A matrix M of the form $M = f \otimes f + C$, where C is nonsingular and $(f, C^{-1} f) + 1 \neq 0$, has the elegant property [8] that ($\tilde{f}$ being the transpose of the vector f)

$$M^{-1} = C^{-1} - \frac{C^{-1}f\tilde{f}C^{-1}}{1 + \tilde{f}C^{-1}f}. \tag{3.10}$$

Thanks are due to I. Olkin, who pointed out this fact to me. (3.10) is easily verified, and it may be used to invert the matrix $(I - \Pi_t)^{-1}$ in (3.8) if one can verify that, for t close enough to one, the conditions for (3.10) are satisfied. We shall take $\sqrt{s(t)}\, e = f$ and $B_t - A = C$. Since A is nonsingular and since B_t tends to the zero matrix as t tends to one, by (3.9) and lemma 2.1, it is clear that $B_t - A$ is nonsingular for t sufficiently close to one. Also

$$1 + s(t)[e, (B_t - A)^{-1}e] \neq 0 \tag{3.11}$$

for t close enough to one by the above argument and lemma 3.1.

Observing that $B_t - A$ is symmetric, one obtains from (3.8) and (3.10) for some $\delta > 0$

$$I - \Pi_t = (B_t - A)^{-1} - \frac{(B_t - A)^{-1}e \otimes (B_t - A)^{-1}e}{[s(t)]^{-1} + (e, (B_t - A)^{-1}e)} \tag{3.12}$$

for $1 - \delta \leqq t < 1$, and passing to the limit

$$\begin{aligned} \lim_{t\to 1^-} [I - \Pi_t] = I - \Pi &= -A^{-1} + \frac{A^{-1}e \otimes A^{-1}e}{(e, A^{-1}e)} \\ &= -K + \frac{Ke \otimes Ke}{(e, Ke)}, \end{aligned} \tag{3.13}$$

proving theorem 3.1.

4. Probability interpretation of the potential kernel

Taking x, y, z in E, not necessarily distinct, we let ${}_zE_{xy}$ denote the expected number of visits of the process S_n, with $S_0 = x$, to the point y, before the first visit to z. Formally

$${}_zE_{xy} = \sum_{n=1}^{\infty} P\{S_\nu \neq z, \quad \text{for} \quad \nu = 1, \cdots, n-1;\ S_n = y | S_0 = x\}. \tag{4.1}$$

It turns out that

$${}_zE_{xy} = A(z, y) + A(z, x) - A(x, y) + \delta(z, y) + \delta(z, x) - \delta(x, y), \tag{4.2}$$

where $\delta(a, b) = 1$ if $a = b$ and 0 otherwise. In section 7, theorem 7.1, the Green function $g_\Omega(x, y)$ is obtained explicitly and that result will generalize (4.2) as ${}_zE_{xy} = g_\Omega(x, y)$ with Ω consisting of the single point z, when x, y, z are distinct. Equation (4.2) will not be proved there as its proof requires only trivial modifications of that of theorem 2 in [14] which concerns one-dimensional random walk. A simple direct proof may be based on theorem 3.1 for arbitrary sets Ω with $|\Omega| = 2$ and $|\Omega| = 3$ and requires only certain calculations based on the geometric distribution of the number of returns to a point before visiting a different, prescribed, point.

$A(x, y)$ has the probability interpretation given by

THEOREM 4.1. *If* $x \neq y$,

$$ {}_yE_{xx} = {}_xE_{yy} = 2A(x, y) - 1 \tag{4.3} $$

$$ \lim_{z \to \infty} {}_xE_{yz} = \lim_{z \to \infty} {}_yE_{xz} = A(x, y). \tag{4.4} $$

PROOF. (4.3) is just a special case of (4.2) and (4.4) follows from lemma 2.3 applied to (4.2).

5. The harmonic measure

Given a finite subset Ω of E, the harmonic measure of Ω is defined as a function $p(x, a)$ on the product space $E \times \Omega$ by $x \in E$, $a \in \Omega$,

$$ p(x, a) = \delta(x, a) \qquad \text{when} \quad x \in \Omega \tag{5.1} $$

$$ p(x, a) = \sum_{n=1}^{\infty} P\{S_\nu \notin \Omega \quad \text{for} \quad \nu = 1, \cdots, n-1;\ S_n = a | S_0 = x\} \tag{5.2} $$

$$ \text{when} \quad x \in E - \Omega. $$

Thus the harmonic measure defines, for each x, a probability measure on Ω which assigns to each point a the probability that Ω is first visited at a, if $S_0 = x$.

This is well known, as is the fact that the *harmonic measure* yields the solution to the *exterior Dirichlet problem*, that is, the boundary value problem

$$ (P - I)f = 0 \quad \text{on } E - \Omega, \tag{5.3'} $$

$$ f = \phi \quad \text{on } \Omega, \tag{5.3''} $$

$$ f \in \mathfrak{L} \quad \text{is bounded on } E. \tag{5.3'''} $$

Here $\phi \in \mathfrak{L}(\Omega)$ is a known function and (5.3) turns out to have a unique solution $f \in \mathfrak{L}$. It is given by

THEOREM 5.1. *The exterior Dirichlet problem has the unique solution*

$$ f(x) = \sum_{a \in \Omega} p(x, a)\phi(a). \tag{5.4} $$

PROOF. The definition of $p(x, a)$ in (5.1) and (5.2) and a simple renewal argument (based on the strong Markov property of the process S_n) imply that (5.4) is indeed a solution of (5.3). To prove uniqueness, suppose f_1 and f_2 are two solutions of (5.3) and let $h = f_1 - f_2$. Then h satisfies (5.3) with $\phi \equiv 0$ on Ω and we have to show that $h \equiv 0$ on E. Define

$$ \begin{aligned} P_\Omega^{(1)}(x, y) &= P(x, y), \\ P_\Omega^{(n+1)}(x, y) &= \sum_{t \in E - \Omega} P(x, t) P_\Omega^{(n)}(t, y) \qquad \text{for} \quad n \geq 1. \end{aligned} \tag{5.5} $$

As $h \equiv 0$ on Ω, and $Ph = h$ on $E - \Omega$,

$$ h(x) = \sum_{y \in E - \Omega} P_\Omega^{(1)}(x, y)h(y) = \cdots = \sum_{y \in E - \Omega} P_\Omega^{(n)}(x, y)h(y), \tag{5.6} $$

for all $n \geqq 1$ and every x in $E - \Omega$. Choosing M such that $|h(x)| \leqq M$ on E, we have for every $x \in E - \Omega$,

$$(5.7)\qquad |h(x)| \leqq M \sum_{y \in E - \Omega} P_\Omega^{(n)}(x, y) = M\, P\{S_\nu \in E - \Omega \quad \text{for} \quad \nu = 1, \cdots, n | S_0 = x\}$$

for all $n \geqq 1$. As the process S_n is recurrent the right side tends to zero as n approaches infinity, so that $h(x) \equiv 0$.

An equally short proof comes from the following observation of Doob. If $S_0 = x$ and if f satisfies (5.3) and if S_n' is the process S_n, stopped when it first reaches Ω, then $f(S_n')$ is a bounded martingale [6]. (5.4) is the martingale theorem:

$$(5.8)\qquad f(x) = E[f(S_n')] = E[\lim_{n\to\infty} f(S_n')] = E[\phi(\lim_{n\to\infty} S_n')] = \sum_{a \in \Omega} p(x, a)\phi(a).$$

The next theorem gives an explicit formula for the harmonic measure $p(x, a)$ which will be quite indispensable for the further development of the potential theory. It need be given only for $2 \leqq |\Omega| < \infty$. When $|\Omega| = 1$, $p(x, a) \equiv 1$ for all $x \in E$, and when Ω is the empty set the only bounded (harmonic) functions satisfying $Pf = f$ on E are the constants by Derman's theorem, giving the uniqueness of invariant measure [5].

THEOREM 5.2. *When* $2 \leqq |\Omega| < \infty$,

$$(5.9)\qquad p(x, a) = \frac{Ke(a)}{(e, Ke)} + \sum_{b \in \Omega} A(x, b)[\Pi(b, a) - \delta(b, a)].$$

PROOF. As mentioned in section 1 this result was first derived probabilistically as a consequence of theorem 3.1. (It is not hard to see how $p(x, a)$ is related to the imbedded Markov chain corresponding to the set Ω' obtained by adjoining the point x to the set Ω.) We shall be content to verify that (5.9) is correct. In view of theorem 5.1 it suffices to verify that the right side in (5.9) satisfies (5.3) with $\phi(x) = 1$ when $x = a$ and $\phi(x) = 0$ when $x \in \Omega - \{a\}$. As a is fixed we write $h(\cdot a)$ for the function $h(x, a)$ on the right in (5.9), and $(P - I)h(\cdot a)(x)$ for the image of $h(\cdot a)$ after applying $P - I$ to it. The first term on the right in (5.9) is independent of x so that $P - I$ maps it into zero. Using lemma 2.2 we get

$$(5.10)\qquad (P - I)h(\cdot a)(x) = \sum_{b \in \Omega} [(P - I)A](x, b)[\Pi(b, a) - \delta(b, a)] = 0$$

when $x \notin \Omega$, so that (5.3′) holds. Using theorem 4.1

$$(5.11)\qquad h(x, a) = \frac{Ke(a)}{(e, Ke)} + \sum_{b \in \Omega} A(x, b)\left[K(b, a) - \frac{Ke(b)Ke(a)}{(e, Ke)}\right].$$

The definition of K in (1.13) applied to (5.11) shows that $h(x, a) = \delta(x, a)$ for x in Ω so that (5.3″) holds. To verify that $h(x, a)$ is bounded in x, let $c(b) = \Pi(b, a) - \delta(b, a)$. Note that for each fixed a, $\sum_{b \in \Omega} c(b) = 0$, as Π is a symmetric stochastic matrix. (Even if S_n were unsymmetric but recurrent, with $\phi(\theta)$ complex this conclusion would hold as Π would still be doubly stochastic without being symmetric.) A trivial extension of equation (2.11) in lemma 2.3 implies that

$$\lim_{x\to\infty} \sum_{b\in\Omega} A(x, b)[\Pi(b, a) - \delta(b, a)] = 0 \tag{5.12}$$

so that

$$\lim_{x\to\infty} h(x, a) = \frac{Ke(a)}{(e, Ke)} \tag{5.13}$$

which is independent of x, so that (5.3‴) is verified. Summarizing the observations which led to equations (5.10), (5.11) and (5.12) and introducing the notation

$$p_\infty = p_\infty(a) = \frac{Ke(a)}{(e, Ke)} \tag{5.14}$$

for the harmonic measure of Ω "seen from infinity" which will play an important role in the sequel, we can state

THEOREM 5.3. *The unique solution of* 5.4 *is*

$$f = (p_\infty, \phi) + A(\Pi - I)\phi \tag{5.15}$$

$$\lim_{x\to\infty} f(x) = (p_\infty, \phi) \tag{5.16}$$

$$(P - I)f = (\Pi - I)\phi \quad \text{on} \quad \Omega. \tag{5.17}$$

Here (5.15) follows from (5.4) and (5.9), (5.16) from (5.12), and (5.17) from (5.10) with x in Ω.

A final observation is the

COROLLARY. *Equation* (3.3) *holds for* $2 \leqq |\Omega| < \infty$.

PROOF. The probability interpretation of $p(x, a)$ yields that $p_\infty(a) \geqq 0$. To get (3.3) we have to rule out the possibility that $Ke(a) \leqq 0$ and $(e, Ke) < 0$. That is impossible since $\sum_{a\in\Omega} A(b, a)Ke(a) = e(b) = 1$ for $b \in \Omega$ and $A(b, a) \geqq 0$. Therefore $Ke(a) \geqq 0$ on Ω and $(e, Ke) > 0$.

6. Logarithmic potential

When $1 \leqq |\Omega| < \infty$ and when $0 \leqq \psi \in \mathcal{L}(\Omega)$, we call the function $f(x) = \sum_{a\in\Omega} A(x, a)\psi(a)]$ the *potential* due to the *charge distribution* ψ. The total charge on Ω is (ψ, e). The potential $f = A\psi$ has the properties (1) $(P - I)f = 0$ on $E - \Omega$, (2) $(P - I)f = \psi$ on Ω, (3) $f(x) - (\psi, e)a(x) \to 0$ as $x \to \infty$. Properties (1) and (2) are immediate consequences of lemma 2.2, and, recalling that $a(x) = A(x, 0)$, property (3) follows from lemma 2.3. Thus $f = A\psi$ is similar (except for a change in sign) to classical *logarithmic potential* [13], if the additional conditions of lemma 2.4 are met, so that

$$a(x) - \frac{1}{\pi\sigma^2} \log |x| \to k \tag{6.1}$$

with k given in lemma 2.4, as x tends to infinity. Note that in any case the potential $f(x) = A\psi(x)$ of a $\psi \geqq 0$ tends to infinity with x, unless $\psi \equiv 0$ on Ω. That follows from (2.10) in lemma 2.3.

It is also true that properties (1), (2), (3) characterize our potentials:

THEOREM 6.1. *If* $0 \leqq \psi \in \mathcal{L}(\Omega)$ *then* $f = A\psi$ *satisfies* (1), (2), (3). *Conversely if* f *satisfies* (1), (2), (3) *with* $0 \leqq \psi \in \mathcal{L}(\Omega)$, *then* $f = A\psi$.

PROOF. Having already proved the first part, we assume that a function $g \in \mathcal{L}$, as well as $f = A\psi$ satisfies (1), (2), (3) with $0 \leqq \psi \in \mathcal{L}(\Omega)$. Letting $h = f - g$, (1) and (2) imply that $(P - I)h \equiv 0$ on E and (3) implies that h is bounded, in fact that $\lim_{x\to\infty} h(x) = 0$. From the uniqueness theorem for invariant measure [5] it follows that $h \equiv 0$. That completes the proof.

Much more than theorem 6.1 could be desired. The second (converse) part could probably be altered by assuming that $f \geqq 0$ and that f satisfies (1) and (2) with $0 \leqq \psi \in \mathcal{L}(\Omega)$. One would then expect the conclusion that $f =$ constant $+ A\psi$.

Our methods break down, as well as the simple argument in section 3 of [11] where it is shown for simple random walk in three and higher dimension that $f \geqq 0$ and $(P - I)f \leqq 0$ on E implies that f is a constant plus a (Newtonian) potential. *Note added in proof.* It will be shown elsewhere that the above conjecture is correct for the random walks considered here. For one-dimensional random walks the situation turns out to be more complicated.

In the next section we shall require the quite obvious extension of theorem 6.1

THEOREM 6.2. *If* $(P - I)u = 0$ *on* $E - \Omega$, $(P - I)u = \psi \geqq 0$ *on* Ω, $(\psi, e) = 1$, $|u(x) - a(x)| \leqq M$ *on* E *for some* $M > 0$, *then* $u =$ constant $+ A\psi$.

A number of inequalities which imitate maximum principles from classical potential theory will be proved in the next section (theorem 7.3).

7. The Green function

The Green function for the exterior domain $E - \Omega$ is first defined probabilistically as the function $g_\Omega(x, y)$ on $E \times E$

When $y \in \Omega$, $g_\Omega(x, y) = 0$;

When $y \in E - \Omega$ and $x \in \Omega$, $g_\Omega(x, y) = 0$;

When $y \in E - \Omega$, $x \in E - \Omega$, $x \neq y$;

$$g_\Omega(x, y) = \sum_{n=1}^{\infty} P\{S_\nu \in E - \Omega \text{ for } \nu = 1, \cdots, n-1; S_n = y | S_0 = x\}. \tag{7.1}$$

When $x = y \in E - \Omega$

$$g_\Omega(x, y) = 1 + \sum_{n=1}^{\infty} P\{S_\nu \in E - \Omega \text{ for } \nu = 1, \cdots, n-1; S_n = x | S_0 = x\}. \tag{7.1'}$$

When $x \neq y$ and both x and y are outside Ω, then $g_\Omega(x, y)$ is the expected number of visits to y of the process S_n with $S_0 = x$, before the first visit to Ω. In spite of the unsymmetric definition we shall show that $g_\Omega(x, y) = g_\Omega(y, x)$ in the course of finding an explicit formula for the Green function. This could be done probabilistically, using theorem 3.1 and the method of the imbedded Markov chain, but we shall appeal instead to the potential theory developed in section 6.

First we need a definition.

$$f(x) = \sum_{a \in \Omega} A(x, a) p_\infty(a) \tag{7.2}$$

will denote the potential due to the (nonnegative unit charge) distribution p_∞ on Ω. In terms of f we shall prove

THEOREM 7.1. *For* $2 \leqq |\Omega| < \infty$ *and* $x, y \in E$

$$g_\Omega(x, y) = -A(x, y) + f(x) + f(y) - \frac{1}{(e, Ke)} + \sum_{a \in \Omega} \sum_{b \in \Omega} A(x, a)[\Pi(a, b) - \delta(a, b)] A(b, y). \tag{7.3}$$

When $|\Omega| = 1$, theorem 7.1 is meaningless since K is undefined, but in this case (suppose Ω consists of the single point $\{a\}$) equation (4.2) and the definition (7.1) gives

$$g_\Omega(x, y) = -A(x, y) + A(x, a) + A(a, y). \tag{7.4}$$

This agrees with (7.3) if we define $(e, Ke)^{-1} = 0$ when $|\Omega| = 1$.

Before proceeding to the proof of theorem 7.1 we need an analogue of the classical theorem [13] that the harmonic measure of Ω is the normal derivative of the Green function at the boundary of Ω. Here the normal derivative of a function $u(t)$ evaluated at a in Ω becomes $\sum_{t \in E - \Omega} P(a, t) u(t) - u(a)$ and applied to the Green function, which vanishes on Ω, this is

THEOREM 7.2. *For* $a \in \Omega$, $x \in E - \Omega$,

$$p(x, a) = \sum_{t \in E} P(a, t) g_\Omega(t, x). \tag{7.5}$$

One can also obtain Π from g_Ω, since for $a, b \in \Omega$,

$$\Pi(a, b) = \sum_{s \in E} P(a, s) p(s, b) = \sum_{s \in E} \sum_{t \in E} P(a, t) g_\Omega(t, s) P(s, b). \tag{7.6}$$

PROOF of (7.5). In view of (7.1) the right side in (7.5) is

$$\sum_{n=1}^{\infty} P\{S_\nu = E - \Omega \quad \text{for} \quad \nu = 1, \cdots, n-1; S_n = x | S_0 = a\}. \tag{7.7}$$

The symmetry of the process S_n, $\phi(\theta) = \phi(-\theta)$, enables one to set up a probability preserving one to one correspondence between paths going from a to x in time n without hitting Ω and paths going from x to a in time n without hitting Ω. Therefore the above sum is

$$\sum_{n=1}^{\infty} P\{S_\nu \in E - \Omega \quad \text{for} \quad \nu = 1, \cdots, n-1; S_n = a | S_0 = x\} \tag{7.8}$$

and this is the harmonic measure $p(x, a)$ according to (5.2).

PROOF of (7.3). Take a fixed y in $E - \Omega$. A simple renewal argument (based on the strong Markov property of S_n) and the definition (7.1) gives

$$(P - I) g_\Omega(\cdot y)(x) = \begin{cases} 0, & x \neq y, x \in E - \Omega, \\ -1, & x = y \in E - \Omega. \end{cases} \tag{7.9}$$

Let $u(x) = u_y(x) = g_\Omega(x, y) + A(x, y)$. By lemma 2.2 and (7.9)

$$(P - I)u = 0 \qquad \text{on} \quad E - \Omega, \tag{7.10}$$

and by theorem 7.2 and lemma 2.2

$$(P - I)u(x) = \sum_{t \in E} P(x, t)g_\Omega(t, y) = p(y, x), \qquad x \in \Omega. \tag{7.11}$$

We wish to solve the system consisting of (7.10) and (7.11) to find an explicit formula for $u = u_y$. To do so one must first check that u satisfies the conditions in theorem 6.2. The first two conditions in theorem 6.2 are equivalent to (7.10) and (7.11) with $\psi(x) = p(y, x) \geqq 0$ for $x \in \Omega$. Clearly $(\psi, e) = 1$. For each fixed y, $A(x, y) - a(x)$ is bounded, as it tends to a finite limit when x tends to infinity by lemma 2.2. Thus the result of theorem 6.2 is applicable if we show that $g_\Omega(x, y)$ is bounded in x for each fixed y. To do so let Ω' be a subset of Ω consisting of a single point. Then the probability interpretation of the definition (7.1) of the Green function implies that $0 \leqq g_\Omega(x, y) \leqq g_{\Omega'}(x, y)$, and either equation (4.2) or theorem 4.1 proves that $g_{\Omega'}(x, y)$ is bounded in x for each fixed y.

Applying the result of theorem 6.2 we have $u = \text{constant} + A\psi$, where $\psi(x) = p(y, x)$ for $x \in \Omega$ and we let $h(y)$ be the constant, which may of course depend on y. Then theorem 5.2 together with (7.2) yields

$$\begin{aligned} u(x) = u_y(x) &= g_\Omega(x, y) + A(x, y) \\ &= h(y) + f(x) + \sum_{a \in \Omega} \sum_{b \in \Omega} A(x, a)[\Pi(a, b) - \delta(a, b)]A(b, y), \\ &\qquad\qquad x \in E,\ y \in E - \Omega. \end{aligned} \tag{7.12}$$

Setting $x = c \in \Omega$ in (7.12) gives for $y \in E - \Omega$,

$$\begin{aligned} A(c, y) &= h(y) + \frac{1}{(e, Ke)} + \sum_{a \in \Omega} \sum_{b \in \Omega} A(c, a)[K(a, b) - (e, Ke)p_\infty(a)p_\infty(b)]A(b,y) \\ &= h(y) + \frac{1}{(e, Ke)} + A(c, y) - f(y), \end{aligned} \tag{7.13}$$

or

$$h(y) = f(y) - \frac{1}{(e, Ke)}. \tag{7.14}$$

Equations (7.12) and (7.14) together complete the proof of theorem 7.1 for the case when $y \in E - \Omega$. For the case when $y \in \Omega$ the proof is completed by straightforward calculation which shows that the right side in (7.3) vanishes.

The function $h(x)$ defined by equation (7.14) is important as it follows from theorem 7.1 and lemma 2.3 that $g_\Omega(x, y)$ has a limit as y tends to infinity, and this limit is $h(x)$,

$$\begin{aligned} \lim_{y \to \infty} g_\Omega(x, y) = g_\Omega(x, \infty) = g_\Omega(\infty, x) = h(x) = f(x) - \frac{1}{(e, Ke)}, \\ f(x) = Ap_\infty(x) = \sum_{a \in \Omega} A(x, a)p_\infty(a). \end{aligned} \tag{7.15}$$

We use the resulting inequality

$$h(x) = f(x) - \frac{1}{(e, Ke)} \geqq 0, \qquad x \in E, \tag{7.16}$$

to establish a maximum (or rather minimum) principle for potentials.

THEOREM 7.3. *For ψ_1 and ψ_2 in $\mathfrak{L}(\Omega)$, $A\psi_1 \geqq A\psi_2$ on Ω implies that $A\psi_1 \geqq A\psi_2$ on E, and that $(\psi_1, e) \geqq (\psi_2, e)$. If $0 \leqq \psi \in \mathfrak{L}(\Omega)$, then*

$$A\psi(x) \geqq \min_{a \in \Omega} A\psi(a) \qquad \textit{for all } x \textit{ in } E. \tag{7.17}$$

PROOF. For $a, t \in \Omega$, theorem 3.1 gives

$$K(t, a) = \frac{Ke(t)Ke(a)}{(e, Ke)} + \Pi(t, a) - \delta(t, a). \tag{7.18}$$

For $x \in E - \Omega$ and $a \in \Omega$, therefore, using theorem 5.2,

$$\begin{aligned} \sum_{t \in \Omega} A(x, t)K(t, a) &= f(x)Ke(a) + p(x, a) - \frac{Ke(a)}{(e, Ke)} \\ &= \left[f(x) - \frac{1}{(e, Ke)}\right]Ke(a) + p(x, a). \end{aligned} \tag{7.19}$$

In view of (7.16), and (1.13)

$$\sum_{t \in \Omega} A(x, t)K(t, a) \geqq p(x, a) \geqq 0 \quad \text{when} \quad x \in E, a \in \Omega. \tag{7.20}$$

Hence, if $0 \leqq g \in \mathfrak{L}(\Omega)$,

$$\begin{aligned} &AK\, g(x) \\ &\quad = \sum_{t \in \Omega} \sum_{a \in \Omega} A(x, t)K(t, a)g(a) \geqq \sum_{a \in \Omega} p(x, a)g(a) \geqq 0, \qquad x \in E. \end{aligned} \tag{7.21}$$

If $A\psi_1 \geqq A\psi_2$ on Ω, let $g = A(\psi_1 - \psi_2)$ restricted to Ω, giving, for all $x \in E$,

$$A\psi_1(x) - A\psi_2(x) \geqq \sum_{a \in \Omega} \sum_{b \in \Omega} p(x, a)A(a, b)[\psi_1(b) - \psi_2(b)] \geqq 0. \tag{7.22}$$

Since $g \geqq 0$ and $Ke \geqq 0$ on Ω we also have $(g, Ke) \geqq 0$, but $(g, Ke) = (A(\psi_1 - \psi_2), Ke) = (\psi_1 - \psi_2, e) = (\psi_1, e) - (\psi_2, e) \geqq 0$. This proves the first part of theorem 7.3. For the second part let $\psi_2(a) = cp_\infty(a)$ for $a \in \Omega$ where $c > 0$ is chosen so that

$$\min_{a \in \Omega} A\psi(a) = A\psi_2(t) = \frac{c}{(e, Ke)} \tag{7.23}$$

for all t in Ω. (It follows from the definition of p_∞ that its potential $f(x) = Ap_\infty(x)$ is constant on Ω.) Since $A\psi_2$ is constant on Ω we have $A\psi \geqq A\psi_2$ on Ω and by the first part of the theorem $A\psi \geqq A\psi_2$ on E. But

$$A\psi_2(x) = cf(x) \geqq \frac{c}{(e, Ke)} = \min_{a \in \Omega} A\psi(a) \tag{7.24}$$

because of (7.16) and (7.23).

8. Capacity

For a discussion of *capacity* in classical logarithmic potential theory see Nevanlinna [13] and for the modern theory of capacity (of the Newtonian type) Choquet [2] and Choquet and Deny [3]. To preserve the analogy of the present to the classical theory, the capacity of a finite set Ω should be defined as $\exp[(e, Ke)^{-1}]$ where K is the (matrix) operator given by (1.13). The quantity $-(e, Ke)^{-1}$ then is our analogue of Robin's constant. But for the sake of convenience we call $(e, Ke)^{-1}$ the capacity of Ω.

DEFINITION. *If* $|\Omega| = 1$ *the capacity of* Ω *is* $C(\Omega) = 0$. *If* $2 \leqq |\Omega| < \infty$ *the capacity of* Ω *is*

$$C(\Omega) = \frac{1}{(e, Ke)}. \tag{8.1}$$

The first theorem has the nature of a maximum principle. If $0 \leqq \psi \in \mathcal{L}(\Omega)$ and $(\psi, e) = 1$ then the potential $A\psi$ of ψ is greater than or equal to $C(\Omega)$ somewhere on Ω and less than or equal to $C(\Omega)$ somewhere on Ω. But there is a unique ψ (namely p_∞) such that $0 \leqq \psi \in \mathcal{L}(\Omega)$, $(\psi, e) = 1$, and such that $A\psi$ is constant on Ω. This constant is $C(\Omega)$ and p_∞ is called the *equilibrium unit charge*. A finite set Ω has the same capacity as its boundary,

$$\partial\Omega = [x | x \in \Omega, p_\infty(x) > 0]. \tag{8.2}$$

THEOREM 8.1. *If* $0 \leqq \psi \in \mathcal{L}(\Omega)$ and $(\psi, e) = 1$, *then*

$$\min_{a\in\Omega} A\psi(a) \leqq C(\Omega) \leqq \max_{a\in\Omega} A\psi(a) \tag{8.3}$$

$$C(\Omega) = Ap_\infty(x) = f(x) \qquad \textit{for} \quad x \in \Omega \tag{8.4}$$

$$C(\Omega) = C(\partial\Omega). \tag{8.5}$$

PROOF. When $|\Omega| = 1$ theorem 8.1 is obvious, therefore let $2 \leqq |\Omega| < \infty$. Suppose (8.3) fails, in such a way that the second inequality is violated. Then

$$A\psi < \frac{e}{(e, Ke)} \quad \text{on} \quad \Omega. \tag{8.6}$$

This inequality is preserved under the inner product with Ke, so that

$$(Ke, A\psi) = (e, KA\psi) = (e, \psi) > \frac{(Ke, e)}{(e, Ke)} = 1. \tag{8.7}$$

This contradicts the hypothesis $(e, \psi) = 1$. If the first inequality fails the same argument works.

Equation (8.4) follows immediately from (5.14), (7.2) and (8.1). Equation (8.5) is true because p_∞, restricted to $\partial\Omega$ is a unit charge distribution on $\partial\Omega$; the potential due to p_∞ is constant on $\partial\Omega$ and has the value

$$\sum_{a\in\partial\Omega} A(x, c) \frac{Ke(c)}{(e, Ke)} = \frac{1}{(e, Ke)} = C(\Omega), \qquad x \in \Omega, \tag{8.8}$$

where K is the inverse of A restricted to Ω, but by (8.3) $C(\Omega)$ is the capacity of $\partial\Omega$.

Equation (8.3) may be expressed in the language of *game theory*. Let u, v range over $\mathcal{L}(\Omega)$ with $(u, e) = (v, e) = 1$, $u \geqq 0$, $v \geqq 0$, and let $(u, Av) = \sum_{a\in\Omega}\sum_{b\in\Omega} u(a)A(a, b)v(b)$. Then (8.3) becomes

$$\max_v \min_u (u, Av) = C(\Omega) = \min_u \max_v (u, Av). \tag{8.9}$$

Thus $C(\Omega)$ is the *value* of the game whose matrix is A.

As in classical potential theory the capacity of Ω may also be defined through the asymptotic behavior of the Green function.

THEOREM 8.2. *Let* $g_\Omega(x, \infty) = \lim_{y\to\infty} g_\Omega(x, y)$, $a(x) = A(x, 0)$. *Then*

$$C(\Omega) = \lim_{x\to\infty} [a(x) - g_\Omega(x, \infty)]. \tag{8.10}$$

PROOF. When $|\Omega| = 1$, $\Omega = \{a\}$, equation (7.5) and lemma (2.2) give $g_\Omega(x, \infty) = A(x, a)$. Lemma (2.3) gives $C(\Omega) = 0$ as we defined it. When $2 \leqq |\Omega| < \infty$, (7.15) and (8.1) give $g_\Omega(x, \infty) = f(x) - C(\Omega)$, $a(x) - g_\Omega(x, \infty) = C(\Omega) + a(x) - f(x)$ and by lemma (2.3) and (7.2)

$$\lim_{x\to\infty} [a(x) - f(x)] = 0. \tag{8.11}$$

This characterization of capacity furnishes simple proofs of its behavior under set theoretical operations. Typical results are

THEOREM 8.3. *Let* F, G *be two finite subsets of* E, *and* $C(F \cap G) = -\infty$ *if* $F \cap G$ *is the empty set. Then*

$$C(F \cap G) + C(F \cup G) \leqq C(F) + C(G). \tag{8.12}$$

If $F \subset G \subset E$,

$$C(F) \leqq C(G). \tag{8.13}$$

If $\{x\}$ *is the set consisting of the single point* $x \in E - \Omega$, $|\Omega| < \infty$,

$$C(\Omega) \leqq C(\Omega \cup \{x\}) \leqq C(\Omega) + \frac{1}{2}\min_{y\in\Omega} A(x, y). \tag{8.14}$$

PROOF. If F and G have nonempty intersection, the definition (7.1) of $g_\Omega(x, y)$ implies that

$$g_{F\cap G}(x, y) - g_G(x, y) \geqq g_F(x, y) - g_{F\cup G}(x, y), \tag{8.15}$$

for all x, y in E. Letting first y tend to infinity, and then x, one obtains in view of theorem 8.2

$$\begin{aligned} &\lim_{x\to\infty} [g_{F\cap G}(x, \infty) - g_G(x, \infty)] \\ &\qquad = C(G) - C(F \cap G) \geqq \lim_{x\to\infty} [g_F(x, \infty) - g_{F\cup G}(x, \infty)] \\ &\qquad = C(F \cup G) - C(F). \end{aligned} \tag{8.16}$$

The proof of (8.13) is based on the simpler inequality $g_F(x, y) \geqq g_G(x, y)$ when $F \subset G$. The first inequality in (8.14) follows from (8.13). The second one comes from (8.12) with $F = \Omega$ and $G = \{x\} \cup \{y\}$ where $y \in \Omega$, $x \in E - \Omega$. Equation (8.12) yields

$$C(\{y\}) + C(\Omega \cup \{x\}) = C(\Omega \cup \{x\}) \leqq C(\Omega) + C(\{x\} \cup \{y\}). \tag{8.17}$$

For every pair of distinct points x, y in E, (1.13) and (8.1) give $C(\{x\} \cup \{y\}) = A(x, y)/2$. Since (8.17) holds for all $y \in \Omega$, this proves (8.14).

Still another definition of capacity follows from theorem 7.1 and equation (4.2). Let Ω be an arbitrary finite subset of E, and a an arbitrary fixed point in E. Then

$$\lim_{x \to \infty} [g_{\{a\}}(x, x) - g_{\Omega}(x, x)] = C(\Omega). \tag{8.18}$$

This formula has a comparatively pleasant probability interpretation, as $g_{\Omega}(x, x)$ (or $g_{\{a\}}(x, x)$) is the expected number of returns to x before the first visit to the set Ω (or the set $\{a\}$). Thus $C(\Omega)$ is a measure of the difference in "size" of the sets Ω and $\{a\}$, seen from infinity.

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