REMARKS ON PROCESSES OF CALLS

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1. Introduction

The theory of processes of calls is highly developed. In this paper I am going to consider some questions which, to my mind, have not yet been analyzed sufficiently from the measure theoretic point of view.

Palm [1] dealt with a special kind of conditional probability for stationary processes. Khinchin [2] presented and completed the ideas of Palm. Their methods were simple and elegant but they were of analytical character. In this paper I am going to give a different and, so far as I know, new approach to these problems. I shall confine myself to considering some basic notions and their properties.

As a by-product I have obtained a result from ergodic theory which seems to be of some interest in itself.

2. Discrete time

From the measure theoretic point of view, the theory of stationary processes of calls with discrete time is quite simple and consequently it is not dangerous to omit some technical details. Let us consider a doubly infinite stationary sequence of random variables $\cdots, \xi_{-2}, \xi_{-1}, \xi_0, \xi_1, \xi_2, \cdots$ taking only the value zero or one. It is easy to prove that either there are no calls or they occur infinitely many times in both directions. In symbols

$$(1) \qquad P\{\xi_i \equiv 0 \quad \text{or} \quad \overline{\lim_{i\to+\infty}}\ \xi_i = \overline{\lim_{i\to-\infty}}\ \xi_i = 1\} = 1,$$

where P denotes the probability measure. The first possibility is uninteresting from any point of view. Hence we may suppose that

$$(2) \qquad P\{\xi_i \equiv 0\} = 0.$$

Further, the general case can be reduced to this case by the introduction of a "new" probability measure, invariant under the shift transformation,

$$(3) \qquad P^*(\cdot) \overset{df}{=} P(\cdot|N), \qquad N = \{\xi_i \not\equiv 0\}.$$

We denote by A the event $\{\xi_0 = 1\}$ and put $\alpha = P(A)$. Under the condition that A has occurred, that is, that there is a call at time $t = 0$, we can define the sequence of random variables $\cdots, \eta_{-2}, \eta_{-1}, \eta_0, \eta_1, \cdots$, which are equal to the distances of the successive calls. The enumeration begins from the call ξ_0. This is illustrated in figure 1.

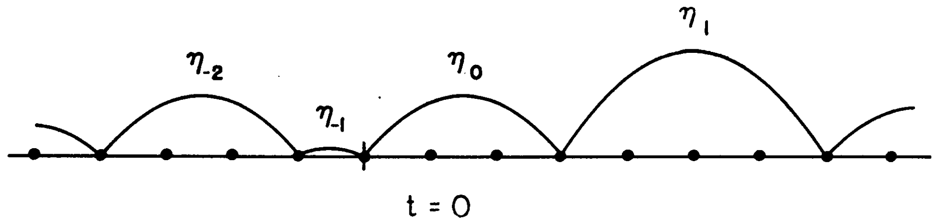

FIGURE 1

Sequence $\{\eta_i\}$ of distances of successive calls.

We denote by $p(\cdot)$ the conditional probability $p(\cdot) = P(\cdot|\xi_0 = 1) = P(\cdot|A)$. We can now assert

THEOREM 1. (i) *The sequence* $\{\eta_i\}$ *is*

(a) *stationary with respect to measure* p,

(b) *the random variables* η_i *admit only positive, integer values,*

(c) *the expectation of* η_i *is finite and is equal to* $(P\{\xi_0 = 1\})^{-1}$.

(ii) *The correspondence between* $\{\xi_i\}$ *and* $\{\eta_i\}$, *or more precisely between the probability measures* P *and* p, *is one to one.*

(iii) *Each sequence* $\{\eta_i\}$ *of random variables satisfying* (a) *to* (c) *can be obtained in this way.*

PROOF. Let $S_{-m}, S_{-m+1}, \cdots, S_0, S_1, \cdots, S_n$ be arbitrary positive numbers. In order to prove the equality

$$P\{\eta_{-m} = S_{-m}, \cdots, \eta_n = S_m\} = p\{\eta_{-m+1} = S_{-m}, \cdots, \eta_{n+1} = S_n\} \tag{4}$$

it is sufficient to apply the shift transformation S_{-1} times in the formula

$$P\{\xi_{s_{-m}} = 1, \xi_{s_{-m}+1} = 0, \cdots, \xi_{s_{-m+1}} = 1, \cdots, \xi_{s_{-1}} = 1, \xi_{s_{-1}+1} = 0, \cdots, \xi_{-1} = 0, \xi_0 = 1, \xi_1 = 0, \xi_{s_1} = 1, \cdots, \xi_{s_n} = 1\}. \tag{5}$$

Hence property (a) has been proved. Property (b) is obvious. Further, by the stationarity of $\{\xi_n\}$ and formulas (1) and (2) we have

$$\begin{aligned} E_p(\eta_0) &= \sum_{i=1}^{\infty} p\{\eta_0 \geqq i\} = \frac{1}{\alpha} \sum_{i=1}^{\infty} P\{\xi_0 = 1, \xi_1 = 0, \cdots, \xi_{i-1} = 0\} \\ &= \frac{1}{\alpha} \sum_{i=0}^{\infty} P\{\xi_{-i} = 1, \xi_{-i+1} = 0, \cdots, \xi_{-1} = 0, \xi_0 = 0\} = \frac{1}{\alpha}. \end{aligned} \tag{6}$$

Thus (c) is proved, which completes the proof of the first part of our assertion.

Now we shall consider the correspondence between the probability measures P and p. We have

$$P(\mathcal{E}) = \sum_{i=0}^{\infty} P\{\xi_{-i} = 1, \xi_{-i+1} = 0, \cdots, \xi_0 = 0, \mathcal{E}\} \tag{7}$$
$$= \sum_{i=0}^{\infty} P\{\xi_0 = 1, \xi_1 = 0, \cdots, \xi_i = 0, \mathcal{E}^{(i)}\},$$

where $\mathcal{E}^{(i)}$ denotes the event $\mathcal{E}$, shifted to the right i times. Finally, we obtain from (7) the following formula expressing the probability measure P by the probability measure p

$$P(\mathcal{E}) = \alpha \sum_{i=0}^{\infty} p\{\eta_0 > i, \mathcal{E}^{(i)}\}. \tag{8}$$

Now we have to prove only the last part of theorem 1. Namely, we suppose that conditions (a) to (c) hold, and formula (8) can be regarded as a definition of the measure P. Evidently we must put $\alpha \overset{df}{=} (\int \eta_0 \, dp)^{-1}$. We shall compute the value of $P(\mathcal{E}^{(1)})$. We have

$$P(\mathcal{E}^{(1)}) = \alpha \sum_{i=0}^{\infty} p\{\eta_0 > i, \mathcal{E}^{(i+1)}\} \tag{9}$$
$$= \alpha \sum_{i=0}^{\infty} [p\{\eta_0 > i, \mathcal{E}^{(i)}\} + p\{\eta_0 = i, \mathcal{E}^{(i)}\}] - \alpha p(\mathcal{E})$$
$$= P(\mathcal{E}) + \alpha\left[\sum_{i=0}^{\infty} p\{\eta_0 = i, \mathcal{E}^{(i)}\} - p(\mathcal{E})\right].$$

The last expression in brackets, in virtue of the stationarity of the sequence $\{\eta_n\}$ with respect to the measure p, is equal to zero. Hence we get $P(\mathcal{E}^{(1)}) = P(\mathcal{E})$ for all events $\mathcal{E}$, that is, the measure P is indeed invariant.

At the end of this section we give the law for forming statistical mixtures of the measures P and p.

THEOREM 2. *If a measure p_i corresponds, in the preceding sense, to a measure P_i, then the measure*

$$\left(\sum_i \alpha_i\lambda_i\right)^{-1} \sum_i \alpha_i\lambda_i p_i \tag{10}$$

corresponds to the measure $\sum_i \lambda_i P_i$, where $\lambda_i > 0$ and $\sum_i \lambda_i = 1$.

The proof of this theorem is not difficult. This rule has, however, an important consequence which is not quite evident.

THEOREM 3. *The sequence $\{\xi_i\}$ is metrically transitive with respect to the measure P if and only if the sequence $\{\eta_i\}$ has the same property with respect to p.*

By metrically transitive we mean that each event concerning the variables ξ_i which is invariant, under the transformation $\{\xi_i\} \to \{\xi_{i+1}\}$, has P-probability equal to zero or one.

PROOF. The set of P-measures and the set of p-measures are convex. From theorem 2 it follows that the extremal points of one of the convex sets are mapped onto the extremal points of the other. On the other hand, extremality and transitivity are equivalent, which concludes the proof.

This elegant method, based on notions and theorems of the theory of convex sets, was successfully used in ergodic theory by Savage and Hewitt.

3. The recurrence transformation

We shall consider a probability space (Ω, B, μ) with a measure preserving transformation T, that is, a point transformation from the space Ω into itself, satisfying the following conditions: $T^{-1}(B) \subset B$ and $\mu[T^{-1}(\mathcal{E})] = \mu(\mathcal{E})$ for all $\mathcal{E} \in B$. We shall say that T is one-to-one measure preserving if it is one-to-one from the space Ω onto itself, if $T(B) = B$ and if T is measure preserving. Let $\mathcal{E}$ be a fixed measurable set of positive measure. By the famous Poincaré-Zermelo theorem for almost every point $\omega \in \mathcal{E}$ some of its iteration $T^k(\omega)$ for $k \geqq 1$ also belongs to the set $\mathcal{E}$. More precisely, we have the relation

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \chi_{\mathcal{E}}(T^k \omega) > 0, \qquad \text{a.e. } \omega \in \epsilon, \tag{11}$$

where $\chi_{\mathcal{E}}$ denotes the characteristic function of the set $\mathcal{E}$. Hence we can define on the set $\mathcal{E}$ the recurrence transformation $T_{\mathcal{E}}$ by condition

$$T_{\mathcal{E}}(\omega) \overset{df}{=} T^k(\omega) \in \mathcal{E} \quad \text{and} \quad T^i(\omega) \notin \mathcal{E}, \qquad 1 \leqq i < k. \tag{12}$$

More exactly, this transformation $T_{\mathcal{E}}$ must be considered on the set

$$\mathcal{E}_0 \overset{df}{=} \{\omega : \omega \in \mathcal{E} \quad \text{and} \quad \overline{\lim_k}\, \chi_{\mathcal{E}}(T^k \omega) = 1\}. \tag{13}$$

Now it is easy to verify that $\mathcal{E}$ and $\mathcal{E}_0$ are almost equal. In our probabilistic considerations, sets of measure zero may be neglected. We emphasize that the recurrence transformation depends on the choice of the set $\mathcal{E}$ and is defined only on it.

The basic property of the recurrence transformation is

THEOREM 4. *$T_{\mathcal{E}}$ preserves the measure μ in the new measure space $(\mathcal{E}, \mathcal{E} \cap B, \mu)$.*

In probabilistic applications we can also consider the normed conditional measure $\mu_{\mathcal{E}}(\cdot) \overset{df}{=} \mu(\cdot|\mathcal{E})$. For the proof it suffices to verify that

$$\mu(\mathfrak{X}) = \sum_{k=1}^{\infty} \mu[\mathcal{E} \cap T^{-1}(\mathcal{E}^c) \cap \cdots \cap T^{-k+1}(\mathcal{E}^c) \cap T^{-k}(\mathfrak{X})]. \tag{14}$$

It is easy to see that the right side of formula (14) is equal to $\mu[T_{\mathcal{E}}^{-1}(\mathfrak{X})]$.

On the other hand, we obtain by a simple computation

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \mu[\mathcal{E} \cap T^{-1}(\mathcal{E}^c) \cap \cdots \cap T^{-k+1}(\mathcal{E}^c) \cap T^{-k}(\mathfrak{X})] \\ &= \mu[F_n \cap T^{-n}(\mathfrak{X})], \end{aligned} \tag{15}$$

where $F_n = \mathcal{E} \cup \cdots \cup T^{-n}(\mathcal{E})$. We observe now that the limit set $\lim_n F_n = F = \bigcup_{n=0}^{\infty} T^{-n}(\mathcal{E})$ is almost T-invariant and contains the set $\mathfrak{X}$. Hence we can write

$$(16) \qquad \begin{aligned} S_n &= \mu[F \cap T^{-n}(\mathfrak{X})] - \mu[(F - F_n) \cap T^{-n}(\mathfrak{X})] \\ &= \mu(\mathfrak{X}) - \mu[(F - F_n) \cap T^{-n}(\mathfrak{X})]. \end{aligned}$$

To conclude the proof it suffices to remark that the second term on the right side converges to zero.

It is easy to observe close connection of the recurrence transformations with the theory of stationary sequences of calls. For this purpose it is enough to consider the exponent k, in the definition (5), as a function of a point $\omega \in \mathcal{E}_0$. The measurability of this function $k = k(\omega)$ is clear. Now we form a sequence

$$(17) \qquad k(\omega),\ k(T_{\mathcal{E}}\omega),\ k(T_{\mathcal{E}}^2\omega),\ \cdots.$$

In view of theorem 4 this sequence is stationary with respect to the measure $\mu_{\mathcal{E}}$, and has exactly the same meaning as the random variables $\eta_0, \eta_1, \eta_2, \cdots$ considered above. We must put $\mathcal{E} \overset{df}{=} \{\xi_0 = 1\}$, $T \overset{df}{=}$ the shift transformation and we have $\mu \overset{df}{=} P$ and $\mu_{\mathcal{E}} = p$.

In addition we remark that for each one-to-one measure preserving transformation T the transformation $T_{\mathcal{E}}$ is also one-to-one, and therefore we can also form the negative iterations

$$(18) \qquad \cdots,\ k(T^{-3}\omega),\ k(T^{-2}\omega),\ k(T^{-1}\omega).$$

We are not going to give a systematic study of the recurrence transformations. We shall present some formulas and properties only.

(i) $T_{\mathcal{E}_1} = (T_{\mathcal{E}_2})_{\mathcal{E}_1}$ for measurable sets $\mathcal{E}_1 \subset \mathcal{E}_2$.

(ii) If the transformation T is metrically transitive then $T_{\mathcal{E}}$ is also transitive.

(iii) If we suppose in addition that the iterations of $\mathcal{E}$ cover the whole space Ω, then the inverse implication is also true.

The proof of (i) consists of a simple calculation based on the definition (12). The proof of (ii) and (iii) is the same as that of theorem 3.

We can raise different problems about the recurrence transformations. For instance, are various mixing properties hereditary from T on $T_{\mathcal{E}}$?

4. The conditional probability for arbitrary processes of calls

We start with a precise description of the measurable space for the process of calls. Let Ω denote the class of all countable subsets of the real axis R, which is the time axis, that are finite in each finite interval. The elements of Ω, which are the realizations of our process, are denoted by ω. By $N(\omega, Q)$, or simply by $N(Q)$ we denote the number of calls in the time set Q, that is,

$$(19) \qquad N(\omega, Q) \overset{df}{=} \text{card } (\omega \cap Q).$$

Now we define the class B of measurable subsets of Ω as the σ-field generated by all the functions $N(\cdot, Q)$, where Q is a Borel set, that is, B is the smallest σ-field with respect to which all functions $N(\cdot, Q)$ are measurable. It is obvious that in the preceding definition we can replace the family $\{Q\}$ of all Borel sets by the family of all intervals, or by the family of the intervals with rational endpoints. Evidently each ω can also be treated as a purely atomic measure, finite for bounded sets. We emphasize that (Ω, B) has good set-theoretic structure. Namely, it is not difficult to prove that Ω can be mapped by a one-to-one function onto the unit interval I and the class B onto the class of all Borel subsets of I. Hence the typical difficulties of the theory of conditional probability do not occur in our space (Ω, B).

Let a fixed probability measure P be defined on (Ω, B). Our next aim is to give a precise meaning to the notion: the probability of an event $\mathcal{E}$ under the condition that a call occurred at moment t. For this notion, not yet defined, we shall use the symbols $P(\mathcal{E}|t)$ or $P(\mathcal{E}|t \in \omega)$. Now, we introduce a new assumption, which is quite natural and at the same time seems to be necessary for our consideration. We suppose that

$$\int N(\omega, Q)P(d\omega) = E_P[N(Q)] < \infty \tag{20}$$

for all bounded sets Q. Consequently we put $\mu(Q) \overset{df}{=} E_P[N(Q)]$. Obviously μ is a Borel measure on the time axis.

For each event $\mathcal{E} \in B$ the integral

$$\int \chi_{\mathcal{E}}(\omega)N(\omega, Q)P(d\omega), \tag{21}$$

treated as a function of the set Q, is an absolutely continuous measure with respect to μ. Hence by the Radon-Nikodym theorem we can write

$$\int \chi_{\mathcal{E}}(\omega)N(\omega, Q)P(d\omega) = \int_Q P(\mathcal{E}|t)\mu(dt). \tag{22}$$

For each fixed $\mathcal{E}$ the Radon-Nikodym derivative $P(\mathcal{E}|t)$ is unique only a.e. with respect to μ, and we can always suppose that it is a "true" measure with respect to sets $\mathcal{E} \in B$. This follows from the previously mentioned property that a measurable space (Ω, B) is a Borel space.

Formula (22) can be generalized to

$$\int f(\omega)N(\omega, Q)P(d\omega) = \int_Q \mu(dt) \int f(\omega)P(d\omega), \tag{23}$$

where f is a P-integrable function.

It seems that this way of introducing the probability $P(\mathcal{E}|t)$ is appropriate. We shall only remark that

(i) If there is exactly one call then $P(\mathcal{E}|t)$ is identical with the ordinary conditional probability.

(ii) If some process of calls is realized by the sequence of random variables $x_1, x_2, \cdots$, for which $P\{x_i \neq x_j\} = 1$ for $i \neq j$, in the following sense $N(Q) \overset{df}{=} \sum_i \chi_Q(x_i)$ for all Borel sets Q, then our assumption (20) takes on the form $\sum_i \mu_i(Q) = \mu(Q) < \infty$ for all bounded Q, where $\mu_i(Q) \overset{df}{=} P\{x_i \in Q\}$. Moreover, the probability $P(\mathcal{E}|t)$ can be written

$$P(\mathcal{E}|t) = \sum_i P(\mathcal{E}|x_i = t)P(x_i = t|t), \tag{24}$$

where $P(x_i = t|t)$ is equal to the Radon-Nikodym derivative $d\mu_i/d\mu$ and can be interpreted as the conditional probability that the ith call occurs at the moment t, given that a call appears at this moment.

5. The stationary process

Now we shall consider stationary processes of calls. We shall use the following notations for shifts

$$\begin{aligned} \omega^t &\overset{df}{=} \omega + t, \qquad \omega \in \Omega;\ -\infty < t < \infty, \\ \omega \in \mathcal{E}^t &\overset{df}{=} \omega^{-t} \in \mathcal{E}. \end{aligned} \tag{25}$$

We add the new assumption $P(\mathcal{E}) = P(\mathcal{E}^t)$ for all $\mathcal{E} \in B$ and $-\infty < t < \infty$. As in the case of the discrete time, we have

$$P\{N(-\infty, +\infty) = 0 \quad \text{or} \quad N(-\infty, 0) = N(0, \infty) = \infty\} = 1, \tag{26}$$

and in what follows we always assume that

$$P\{N(-\infty, +\infty) = 0\} = 0. \tag{27}$$

Hence we can restrict our attention to the realizations ω with infinitely many calls in both time directions. First we shall establish the form of the conditional probability $P(\mathcal{E}|t)$, from the preceding section.

THEOREM 5. *There exists one and only one probability measure P_0 defined on the space (Ω, B) for which the measure function*

$$\Pi\ (\mathcal{E}|t) \overset{df}{=} P_0(\mathcal{E}^{-t}), \tag{28}$$

depending on the parameter t, satisfies the condition (22) for all $\mathcal{E} \in B$ and Q.

PROOF. For the stationary process the measure μ is proportional to the Lebesgue measure $\mu(dt) = \alpha\, dt$, where α is the intensity of calls. Roughly speaking, the matter is quite simple: the new measure $P_0(\mathcal{E})$ is equal to $P(\mathcal{E}|0)$ and formula (28) is a special case of

$$P(\mathcal{E}|t) = P(\mathcal{E}^s|t + s), \tag{29}$$

which seems to be obvious in view of the assumed stationarity. For a precise proof, let $P(\mathcal{E}|t)$ be any conditional probability measure satisfying (22). It follows from the stationarity that for each $\mathcal{E} \in B$, $s \in R$, and for almost every t, in the sense of the Lebesgue measure, the relation (29) holds. In view of the Fubini theorem we know that there is a number t_0 such that, for almost every s and for all $\mathcal{E} \in B$, we have $P(\mathcal{E}|t_0) = P(\mathcal{E}^s|t_0 + s)$. The quantifier "for all $\mathcal{E}$" can be put at the end because $P(\cdot|t)$ is a measure and it suffices to consider only some countable class of sets $\mathcal{E}$ generating the whole field B. It is easy to see that the measure $P_0(\mathcal{E}) \overset{df}{=} P(\mathcal{E}^{t_0}|t_0)$ satisfies the assertion of theorem 5. The proof of the uniqueness of the measure P_0 is omitted because it is quite simple.

REMARK 1. Now our conditional probability measure $P(\cdot|t)$ can be determined in a unique manner by equation (29). This "regular" $P(\cdot|t)$ will be used later.

REMARK 2. Now the equation (23) takes the form

$$\int f(\omega)N(\omega, Q)P(d\omega) = \alpha \int P_0(d\omega) \int_Q f(\omega^t)\, dt. \tag{30}$$

Next we give another description of P_0. We say that a B-measurable function $f(\omega)$ defined on Ω is continuous if and only if it is bounded and if for each fixed $\omega \in \Omega$ the function $f(\omega^t)$ of the real variable t is continuous.

THEOREM 6. *If f is continuous, then*

$$\lim_{|I|\to 0} \frac{1}{\alpha|I|} \int_{\{N(I)\geqq 1\}} f(\omega)P(d\omega) = \int f(\omega)P_0(d\omega), \tag{31}$$

where $0 \in I$, an interval. Or, in another form,

$$\lim_{|I|\to 0} E_P[f|N(I) \geqq 1] = EP_0(f). \tag{32}$$

PROOF. From formula (30) we have

$$\begin{aligned}\lim_{|I|\to 0} \frac{1}{\alpha|I|} \int f(\omega)N(\omega, I)P(d\omega) \\ = \int P_0(d\omega) \lim_{|I|\to 0} \frac{1}{|I|} \int f(\omega^t)\, dt = \int P_0(d\omega)f(\omega),\end{aligned} \tag{33}$$

and on the other hand

$$\begin{aligned}\left|\frac{1}{\alpha|I|} \int f(\omega)N(\omega, I)P(d\omega) - \frac{1}{\alpha|I|} \int_{\{N(I)\geqq 1\}} f(\omega)P(d\omega)\right| \\ \leqq \sup_{\omega\in\Omega} |f(\omega)| \frac{1}{\alpha|I|} \int_{\{N(I)\geqq 1\}} [N(\omega, I) - 1]P(d\omega).\end{aligned} \tag{34}$$

The right side tends to zero together with the length of I (compare the theorem of Koroliuk, p. 39 of [2]).

COROLLARY. $P_0(\Omega_0) = 1$, *where* $\Omega_0 \overset{df}{=} \{\omega: 0 \in \omega\}$.

For the proof of the preceding very intuitive equality, we consider a sequence $\{f_n(\omega)\}$ of continuous functions defined as

$$f_n(\omega) = \begin{cases} 0, & d(\omega) \geqq \frac{1}{n}, \\ 1 - n\, d(\omega), & d(\omega) < \frac{1}{n}, \end{cases} \tag{35}$$

where $d(\omega)$ is the distance of the set ω from the point $t = 0$. By theorem 6 we have $\int f_n(\omega)P_0(d\omega) = 1$ for $n = 1, 2, \cdots$, and hence in the limit we obtain $P_0(\Omega_0) = 1$.

We introduce now a sequence $\{\eta_n(\omega)\}$ of functions defined on the subspace Ω_0 sketched in figure 2.

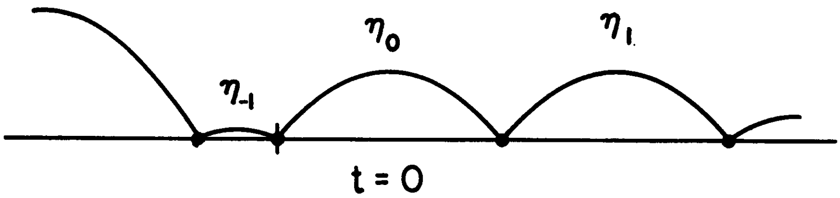

FIGURE 2

Sequence $\{\eta_n(\omega)\}$ of distances of successive calls.

THEOREM 7. *The random variables* $\{\eta_n\}$ $n = 0, \pm 1, \pm 2, \cdots$ *in the measure space* $(\Omega_0, \Omega_0 \cap B, P_0)$ *are*

(i) *nonnegative* $\eta_n > 0$,

(ii) *they have finite first moment* $E_{P_0}(\eta_n) = \alpha^{-1} < \infty$,

(iii) $\cdots, \eta_1, \eta_2, \cdots$ *form a stationary sequence with respect to the measure* P_0.

Conversely each sequence of random variables $\{\eta_n\}$ *satisfying* (i) *to* (iii) *can be obtained in this way. Moreover, the correspondence between* P *and* P_0 *is one to one and it is given by*

$$\int f(\omega)P(d\omega) = \alpha \int_{\Omega_0} P_0(d\omega) \int_0^{\eta_0} f(\omega^{-t})\, dt, \tag{36}$$

which is valid for all measurable and P*-integrable function* f.

PROOF. Let a be a positive number. We have, in virtue of the stationarity,

$$\begin{aligned}\int_{\{N(-a,0)>0\}} f(\omega)P(d\omega) &= \sum_{k=1}^{n} \int_{\{N(I_{-k})>0, N(I_{-k+1})=0, \cdots, N(I_{-1})=0\}} f(\omega)P(d\omega) \\ &= \frac{1}{\delta}\sum_{k=1}^{n} \int_{\{N(I_{-1})>0, N(I_0)=0, \cdots, N(I_{k-2})=0\}} f(\omega^{-(k-1)\delta})P(d\omega),\end{aligned} \tag{37}$$

where $\delta \overset{df}{=} a/n$ and $I_k \overset{df}{=} [k\delta, (k+1)\,\delta]$. Consequently, we obtain

$$\int_{\{N(-a,0)>0\}} f(\omega)P(d\omega) = \frac{\alpha}{\alpha\delta}\int P(d\omega)\,\delta \sum_{k=1}^{m_n(\omega)} f(\omega^{-(k-1)\delta}), \tag{38}$$

where $m_n \overset{df}{=} n[\varphi/n] - 1$ and $\varphi \overset{df}{=} \min(\eta_0, a)$. When $n \to \infty$, the Riemann sums

$$\delta \sum_{k=1}^{m_n} f[\omega^{-(k-1)\delta}] \to \int_0^{\varphi} f(\omega^{-t})\, dt \tag{39}$$

in a bounded manner with respect to the variable ω. In virtue of theorem 6 we obtain

$$\int_{\{N(-a,0)\,>\,0\}} f(\omega)P(d\omega) = \alpha \int P_0(d\omega) \int_0^{\min(\eta_0,a)} f(\omega^{-t})\, dt. \tag{40}$$

Finally, if $a \to +\infty$ in the last formula, we obtain the equality (36) for each continuous f and, consequently, for each P-integrable function.

Now from formula (36) we obtain, by putting $f \equiv 1$,

$$1 = \alpha \int_{\Omega_0} \eta_0 P_0(d\omega) \quad \text{and} \quad E_{P_0}(\eta_0) = \alpha^{-1}. \tag{41}$$

Next we must prove property (iii). Let $\chi_\delta(\omega)$ be the characteristic function of the event $\{\omega : N(\omega, I) \geqq 1\}$, where $I = \langle 0, \delta \rangle_0$.

From (31) and (36) we have for each continuous f

$$\begin{aligned} \int f(\omega) P_0(d\omega) &= \lim_{\delta \to 0} \frac{1}{\alpha\delta} \int f(\omega)\chi_\delta(\omega) P(d\omega) \\ &= \lim_{\delta \to 0} \int_{\Omega_0} P_0(d\omega) \frac{1}{\delta} \int_0^{\eta_0} f(\omega^{-t})\chi_\delta(\omega^{-t})\, dt \\ &= \lim_{\delta \to 0} \int_{\Omega_0} P_0(d\omega) \frac{1}{\delta} \int_{\max(\eta_0 - \delta, 0)}^{\eta_0} f(\omega^{-t})\, dt \\ &= \int_{\Omega_0} f(\omega^{-\eta_0}) P_0(d\omega). \end{aligned} \tag{42}$$

Hence we have obtained the important equality

$$\int_{\Omega_0} f(\omega) P_0(d\omega) = \int_{\Omega_0} f(\omega^{-\eta_0}) P_0(d\omega), \tag{43}$$

valid for all continuous f, and consequently for all bounded measurable functions since, by iterated passage to the limit, we obtain all measurable functions from continuous functions. The last formula expresses the stationarity of $\{\eta_n\}$. Hence statements (i) to (iii) are proved.

We shall now give the proof of the inverse implication. We suppose that (i) to (iii) hold and a probability measure P_0 satisfies (43). We define the measure P by formula (36).

$$\int f(\omega) P(d\omega) \overset{df}{=} \beta \int_{\Omega_0} P_0(d\omega) \int_0^{\eta_0} f(\omega^{-t})\, dt, \tag{44}$$

where

$$\beta \overset{df}{=} \Big[\int_{\Omega_0} \eta_0(\omega) P_0(d\omega)\Big]^{-1}. \tag{45}$$

From (43) we obtain

$$\int f(\omega) P(d\omega) = \beta \int_{\Omega_0} P_0(d\omega) \int_{\eta_0 + \cdots + \eta_{k-1}}^{\eta_0 + \cdots + \eta_k} f(\omega^{-t})\, dt \tag{46}$$

for $k = 1, 2, \cdots$ and consequently,

$$(47) \qquad \int f(\omega)P(d\omega) = \beta \lim_{k\to\infty} \int_{\Omega_0} P_0(d\omega) \frac{1}{k+1} \int_0^{\eta_0+\cdots+\eta_k} f(\omega^{-t})\, dt,$$

and by analogy for arbitrary real s

$$(48) \qquad \int f(\omega^{-s})P(d\omega) = \beta \lim_{k\to\infty} \int_{\Omega_0} P_0(d\omega) \frac{1}{k+1} \int_s^{\eta_0+\cdots+\eta_k+s} f(\omega^{-t})\, dt.$$

Hence

$$(49) \qquad \int f(\omega^{-s})P(d\omega) = \int f(\omega)P(d\omega),$$

that is, the measure P is invariant.

In addition we must prove that the conditional probability induced by P and denoted for the moment by P^* is identical with P_0. We have by (36)

$$(50) \qquad \int f(\omega)P(d\omega) = \alpha \int P^*(d\omega) \int_0^{\eta_0} f(\omega^{-t})\, dt.$$

Let χ_δ have the same meaning as before. We obtain

$$(51) \qquad \int \chi_\delta(\omega)P(d\omega) = \beta \int_{\Omega_0} P_0(d\omega) \min(\delta, \eta_0),$$

and

$$(52) \qquad \lim_{\delta\to 0} \frac{1}{\delta} \int \chi_\delta(\omega)P(d\omega) = \alpha, \qquad \lim_{\delta\to 0} \frac{1}{\delta} \beta \int_{\Omega_0} P_0(d\omega) \min(\delta, \eta_0) = \beta.$$

Therefore, $\alpha = \beta$.

From (44) we have, by the previous reasoning, the equality

$$(53) \qquad \int P^*(d\omega)f(\omega) = \int P_0(d\omega)f(\omega^{-\eta_0})$$

which is valid for all continuous f. Then $P^* = P_0$.

Finally we give an analogy of theorem 3 which can be proved in a similar way.

THEOREM 8. *The measure P is metrically transitive if and only if the measure P_0 is metrically transitive.*

REFERENCES

[1] C. PALM, "Intensitätsschwankungen im Fernsprechverkehr," *Ericsson Technics*, Vol. 44 (1943), pp. 1–89.

[2] A. I. KHINCHIN, *Mathematical Methods of the Theory of Mass Service*, published as Vol. 49 of *Trudy Mat. Inst. Steklov.*, Moscow, 1955.