COMPETITION PROCESSES

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1. Introduction

Some of the simpler theoretical models which have been proposed for phenomena (for example, the competition between species or the occurrence of epidemics) which involve stochastic interactions between several populations have the common feature that they are Markov processes, homogeneous in time, with a countable set of states (m, n) where m and n represent the sizes of two populations. These processes are specified by prescribing the rates at which transitions occur, only transitions to "neighboring" states being allowed: a formal definition of such "competition processes" will be given in section 2.

It is usually difficult to find explicit formulas for the transition probabilities $p_{ij}(t)$ of a Markov process, or even for their limiting values π_{ij} as $t \to \infty$, when the process is defined in terms of the transition rates. However, there are simpler questions worth an answer, relating to recurrence and mean recurrence times and, if there are absorbing states, to absorption probabilities and mean absorption times. We shall discuss such problems for competition processes.

2. Definitions and statement of results

2.1. We consider a time-homogeneous Markov process with a countable set of states $i, j, k, \cdots$, continuous time parameter t, and transition matrix $\{p_{ij}(t)\}$. The process will be specified by prescribing the transition rates

$$q_{ij} = p'_{ij}(0) \tag{1}$$

subject to the conditions

$$\begin{aligned} q_{ij} &\geqq 0, \qquad i \neq j, \\ -q_{ii} &= q_i \geqq 0, \\ \sum_{j \neq i} q_{ij} &= q_i < \infty. \end{aligned} \tag{2}$$

At least one such transition matrix exists; if there is exactly one, we call the set $Q = \{q_{ij}\}$ and the unique associated transition matrix regular. (Thus regularity means that the prescribed transition rates do in fact specify the process: see [7] for further discussion.)

Suppose now that the states are labeled (m, n) where $m, n = 0, 1, 2, \cdots$, and that Q has the structure

$$
(3) \qquad \begin{array}{cc} j & q_{ij} \\ \hline (m+1, n) & a(m, n) \\ (m, n+1) & b(m, n) \\ (m-1, n) & c(m, n) \\ (m, n-1) & d(m, n) \\ (m-1, n+1) & e(m, n) \\ (m+1, n-1) & f(m, n) \\ \text{other } j \neq i & 0 \end{array} \qquad \text{for} \quad i = (m, n),
$$

where $a(m, n), \cdots, f(m, n) \geqq 0$, and q_{ii} is determined from (2), that is,

$$
(4) \qquad -q_{ii} = q_i = a(m, n) + b(m, n) + \cdots + f(m, n).
$$

Also, because there are no states with $m < 0$ or $n < 0$, we must have

$$
(5) \qquad \begin{aligned} c(0, n) &= e(0, n) = 0 \qquad \text{for} \quad n = 0, 1, 2, \cdots; \\ d(m, 0) &= f(m, 0) = 0 \qquad \text{for} \quad m = 0, 1, 2, \cdots. \end{aligned}
$$

More briefly, jumps from (m, n) always lead to one of the adjacent states $(m \pm 1, n)$, $(m, n + 1)$, $(m - 1, n + 1)$, $(m + 1, n - 1)$, but the boundaries $m = 0$ and $n = 0$ of the positive (m, n)-quadrant cannot be crossed.

A process for which Q has this form will be called a competition process; it is the natural analogue, in two dimensions, of the familiar one-dimensional birth and death process.

2.2. The limits

$$
(6) \qquad \pi_{ij} = \lim_{t \to \infty} p_{ij}(t)
$$

always exist; the structure of the set $\{\pi_{ij}\}$ and the resulting classification of states are described in [6], where also some methods for calculating the π_{ij} from the q_{ij} (for regular Q) are given. We have no explicit formulas for the π_{ij} arising from a competition process, but we shall give conditions on the q_{ij} which ensure that the process is "nondissipative" in the sense that

$$
(7) \qquad \sum_j \pi_{ij} = 1, \qquad \text{for all} \quad i.
$$

It will be convenient to supplement (3) to (5) by further assumptions about Q, relating to the absorbing states (with $q_i = 0$) and to communication between states. We say that j is accessible from i if there is a chain of states $k_1 = i$, $k_2, \cdots, k_{n-1}, k_n = j$ such that $q_{k_r k_{r+1}} > 0$ for $r = 1, \cdots, n - 1$ (this implies that $p_{ij}(t) > 0$ for $t > 0$), and that Q is irreducible if each j is accessible from every i. Now define two types of competition processes

Type I. $a(m, n) = \cdots = f(m, n) = 0$ when $m = 0$ or $n = 0$, so that all states $(m, 0)$ and $(0, n)$ are absorbing. Further,

$$
(8) \qquad a(m, n) + b(m, n) > 0, \qquad c(m, n) + d(m, n) > 0
$$

when $m > 0$ and $n > 0$; for each nonabsorbing state i, some absorbing state j and every nonabsorbing state j is accessible from i.

Type II. Q is irreducible (so that there are no absorbing states) and (8) holds for all $(m, n) \neq (0, 0)$. Note that, necessarily, $c(0, 0) = d(0, 0) = 0$ because of (5), but that $a(0, 0) + b(0, 0) > 0$ because $(0, 0)$ is not absorbing.

These definitions are designed to include two interesting examples.

Example 1 (see section 3.4 in Kendall [5]). This example, relating to competition between two species, has

$$
\begin{aligned}
&a(m, n) = \alpha m, \qquad b(m, n) = \beta n, \qquad c(m, n) = \gamma mn, \\
&d(m, n) = \delta mn, \qquad e(m, n) = f(m, n) = 0, \qquad \alpha, \cdots, \delta > 0.
\end{aligned} \tag{9}
$$

This, as it stands, does not belong to type I, but it will be noted that once m or n becomes zero it remains zero; then one population becomes extinct and the other grows according to a simple birth process. Thus the most immediately interesting question is that of extinction probabilities, and if in treating this question one adopts the standard device of "freezing" the states $(m, 0)$ and $(0, n)$ by making them absorbing then the process becomes of type I. The nondissipation property (7) then states that extinction of one or other of the populations is almost certain. If this is so, one will naturally ask also whether the mean time for extinction is finite.

Example 2 (Bartlett [1], section 2). This example relates to the growth of an epidemic; here m and n are the respective numbers of susceptible and infectious persons, and if one allows for immigration then

$$
\begin{aligned}
&a(m, n) = \alpha, \quad b(m, n) = \beta, \quad c(m, n) = \gamma m, \\
&d(m, n) = \delta n, \quad e(m, n) = \epsilon mn, \quad f(m, n) = 0, \qquad \alpha, \beta \geqq 0; \gamma, \delta, \epsilon > 0.
\end{aligned} \tag{10}
$$

Here an immediate question (when $\alpha + \beta > 0$, that is, immigration occurs) is whether the process is "positive recurrent" (in the terminology of [6]), or equivalently, whether it is nondissipative.

2.3. To state our results, put

$$
\begin{aligned}
r_k &= \max\,[a(m, n) + b(m, n)], \\
s_k &= \min\,[c(m, n) + d(m, n)],
\end{aligned} \tag{11}
$$

max and min being taken over (m, n) with $m + n = k$ and $m > 0$, $n > 0$ (type I), $m \geqq 0$, $n \geqq 0$ (type II). Thus r_k, s_k are defined and positive when $k \geqq 2$ for type I, defined when $k \geqq 0$ and positive except for s_0 for type II.

THEOREM 1. *A sufficient condition for regularity is*

$$
\sum_{k=2}^{\infty} \left(\frac{1}{r_k} + \frac{s_k}{r_k r_{k-1}} + \cdots + \frac{s_k \cdots s_3}{r_k \cdots r_2} \right) = \infty. \tag{12}
$$

THEOREM 2. *For a regular process of type I, let A denote the set of absorbing states, D the remaining set of states (m, n) with $m > 0$ and $n > 0$. Then $\pi_{ij} = 1$ if $i = j \in A$, while $\pi_{ij} = 0$ if $j \in D$, and π_{ij} for $i \in D$ and $j \in A$ is the probability that the process, starting at i, will ultimately enter (and then remain in) the state j. Hence $\alpha_i = \sum_j \pi_{ij}$, with $i \in D$, is the probability that some absorbing state is* *reached from i. Either $\alpha_i < 1$ for all $i \in D$ or $\alpha_i = 1$ for all $i \in D$; the latter occurs if*

$$\sum_{k=2}^{\infty} \frac{s_2 \cdots s_k}{r_2 \cdots r_k} = \infty. \tag{13}$$

THEOREM 3. *If in theorem 2 $\alpha_i = 1$ for all $i \in D$, let τ_1 be the mean time to reach A, starting at $i \in D$. Then $\tau_i < \infty$ for all $i \in D$ if*

$$\sum_{k=2}^{\infty} \frac{r_2 \cdots r_{k-1}}{s_2 \cdots s_k} < \infty. \tag{14}$$

THEOREM 4. *For a regular process of type II, either $\pi_{ij} = 0$ for all i, j, or $\pi_{ij} = \pi_j > 0$ is independent of i and $\sum_j \pi_j = 1$; the latter occurs if*

$$\sum_{k=1}^{\infty} \frac{r_1 \cdots r_{k-1}}{s_1 \cdots s_k} < \infty. \tag{15}$$

For the two examples mentioned in section 2.2, theorems 1 and 4 show that the process in example 2 is regular and positive recurrent (when $\alpha, \beta > 0$); for the process in example 1, theorem 1 proves regularity but theorems 2 and 3 do not always apply. We therefore prove separately

THEOREM 5. *For the process in example 1, absorption is certain and the mean absorption times τ_i are finite.*

3. Proofs of theorems

3.1. For theorem 1 we use the regularity criterion (theorem 6 in [7]): (A) *$Q = \{q_{ij}\}$ is regular if, for each $\lambda > 0$, the equations*

$$\lambda z_i = \sum_j q_{ij} z_j, \tag{16}$$

where $0 \leqq z_i \leqq 1$, have only the trivial solution $z_i \equiv 0$.

To deduce theorem 1 from (A), suppose first that we are dealing with type II, let (12) hold, and let $\{z_i\}$ satisfy (16) and $0 \leqq z_i \leqq 1$. Writing $z(m, n)$ for z_i when $i = (m, n)$,

$$\begin{aligned}[\lambda + a(m, n) + \cdots + f(m, n)]\, z(m, n) &\leqq a(m, n)\, z(m + 1, n) + b(m, n)\, z(m, n + 1) \\ &\quad + c(m, n)\, z(m - 1, n) + d(m, n)\, z(m, n - 1) \\ &\quad + e(m, n)\, z(m - 1, n + 1) + f(m, n)\, z(m + 1, n - 1)\end{aligned} \tag{17}$$

for $m \geqq 0$ and $n \geqq 0$; the values assigned to $z(-1, n)$ and $z(m, -1)$ are immaterial because of (5). Now put

$$Z_k = \max z(m, n), \tag{18}$$

max being taken over (m, n) with $m \geqq 0$, $n \geqq 0$, $m + n = k$. If this maximum is attained at (m_k, n_k), then (17) gives

$$[\lambda + a_k + \cdots + f_k]Z_k \leqq (a_k + b_k)Z_{k+1} + (c_k + d_k)Z_{k-1} + (e_k + f_k)Z_k, \tag{19}$$

where $a_k = a(m_k, n_k), \cdots, f_k = f(m_k, n_k)$. Thus

$$(a_k + b_k)(Z_{k+1} - Z_k) \geqq (c_k + d_k)(Z_k - Z_{k-1}) + \lambda Z_k \tag{20}$$

for $k \geqq 0$ if we define Z_{-1} arbitrarily. It follows, by induction, since $Z_k \geqq 0$ for $k \geqq 0$, that $Z_{k+1} - Z_k \geqq 0$, and therefore, because $a_k + b_k \leqq r_k$ and $c_k + d_k \geqq s_k$ [compare (11)], that

$$r_k(Z_{k+1} - Z_k) \geqq s_k(Z_k - Z_{k-1}) + \lambda Z_k. \tag{21}$$

If $z(m, n)$ is not identically zero, let k_0 be the first k for which $Z_k > 0$. Then (21), combined with the fact that Z_k increases with k, leads to

$$Z_{k+1} - Z_k \geqq \frac{s_k}{r_k}(Z_k - Z_{k-1}) + \frac{\lambda}{r_k} Z_{k_0}, \qquad k \geqq k_0, \tag{22}$$

whence

$$\begin{aligned} Z_{k+1} - Z_k &\geqq Z_{k_0}\left[\frac{1}{r_k} + \frac{s_k}{r_k r_{k-1}} + \cdots + \frac{s_k \cdots s_{k_0+1}}{r_k \cdots r_{k_0}}\right] \\ &\geqq B\left[\frac{1}{r_k} + \frac{s_k}{r_k r_{k-1}} + \cdots + \frac{s_k \cdots s_3}{r_k \cdots r_2}\right], \end{aligned} \tag{23}$$

where $B > 0$ is independent of k. But from (12) it then follows that $\sum(Z_{k+1} - Z_k)$ diverges so that $Z_k \to \infty$, contrary to the assumption that $z(m, n) \leqq 1$. Thus $z(m, n) \equiv 0$, as required.

The proof of theorem 1 for type I is almost identical: it is only necessary to observe, in using (A), that any $z_i \geqq 0$ satisfying (16) must be equal to zero whenever $q_i = -q_{ii} = 0$ (that is, i is an absorbing state), so that absorbing states can in effect be ignored. Thus the argument for type II, referring to states with $m \geqq 0$ and $n \geqq 0$, merely needs rewording for type I so as to refer only to states with $m > 0$ and $n > 0$.

3.2. For theorem 2, all but the last assertion, that (13) is sufficient to ensure $\alpha_i = 1$ for all $i \in D$, follows from the description of the structure of $\{\pi_{ij}\}$ given in [6], section 4. To prove the last assertion, we use a test for nondissipation which is the analogue for continuous time processes of a test for discrete time processes due to Foster [2] and Kendall [4].

(B) *Suppose Q regular. If there exist $u_i \geqq 0$ such that $u_i \to \infty$ as $i \to \infty$ and*

$$\sum_j q_{ij} u_j \leqq 0 \qquad \textit{for all} \quad i, \tag{24}$$

then

$$\sum_j \pi_{ij} = 1 \qquad \textit{for all} \quad i. \tag{25}$$

We postpone the proof of (B) to section 3.5; to deduce theorem 2 from (B) we try to find $u_i = u(m, n) = U_{m+n}$ which satisfy (24). Since (24) holds automatically for absorbing i, we require that

$$\begin{aligned} [a(m, n) + \cdots + f(m, n)]U_{m+n} \quad &\geqq [a(m, n) + b(m, n)]U_{m+n+1} \\ &+ c(m, n) + d(m, n)]U_{m+n-1} + [e(m, n) + f(m, n)]U_{m+n} \end{aligned} \tag{26}$$

for $m > 0$ and $n > 0$. This will be so if we define U_k recursively by

$$s_k(U_k - U_{k-1}) = r_k(U_{k+1} - U_k) \qquad \text{for} \quad k \geqq 2, \tag{27}$$

with $U_1 = 0$ and $U_2 = 1$, say. This gives

$$U_k = 1 + \frac{s_2}{r_2} + \cdots + \frac{s_2 \cdots s_{k-1}}{r_2 \cdots r_{k-1}} \tag{28}$$

and therefore $U_k \to \infty$ as $k \to \infty$ when (13) holds, and (B) can be applied with

$$u_i = U_{m+n} \qquad \text{for} \quad i = (m, n). \tag{29}$$

3.3. For theorems 3 and 5, we use the criterion (C): *Suppose Q regular; denote the sets of nonabsorbing and absorbing states by D and A; let α_i, with $i \in D$, be the probability of reaching some state in A from i, and τ_i, with $i \in D$, the expected time to reach A from i. If there exist finite $u_j \geqq 0$ such that*

$$\sum_j q_{ij} u_j + 1 \leqq 0, \qquad i \in D, \tag{30}$$

then $\alpha_i = 1$ and $\tau_i \leqq u_i < \infty$.

To prove theorem 3, suppose that (14) holds and take $u_i = U_{m+n} \geqq 0$ when $i = (m, n)$. Then (30) will hold if

$$(a + b)U_{m+n+1} + (c + d)U_{m+n-1} + 1 \leqq (a + b + c + d)U_{m+n} \tag{31}$$

when $m > 0$, $n > 0$, where $a = a(m, n), \cdots$. Thus it suffices to have U_k increasing and

$$r_k(U_{k+1} - U_k) + 1 \leqq s_k(U_k - U_{k-1}) \tag{32}$$

for $k \geqq 2$. This can be achieved by setting $U_{k+1} - U_k = V_k$, choosing $U_1 \geqq 0$, $V_1 \geqq 0$, and defining V_k recursively by $r_k V_k + 1 = s_k V_{k-1}$, with $k \geqq 2$, so that

$$V_k = \frac{s_2 \cdots s_k}{r_1 \cdots r_k}\left\{V_1 - \left(\frac{1}{s_2} + \frac{r_2}{s_2 s_3} + \cdots + \frac{r_2 \cdots r_{k-1}}{s_2 \cdots s_k}\right)\right\}. \tag{33}$$

Since (14) holds, we can ensure that $V_k \geqq 0$ by choosing V_1 sufficiently large.

The u_i used in the proof of theorem 3 are too crude to prove theorem 5. In example 1,

$$\begin{aligned} r_k &= (k - 2)\max(\alpha, \beta) + (\alpha + \beta), \\ s_k &= (\gamma + \delta)(k - 1), \end{aligned} \tag{34}$$

and therefore (14) is violated if $\max(\alpha, \beta) > \gamma + \delta$. The u_i we use are defined, when $i = (m, n)$, by

$$u(m, n) = m + n + \frac{A}{1 - \rho}[(1 - \rho^m) + (1 - \rho^n)], \tag{35}$$

where the choice of A (large) and ρ (small) is described later. For (30) we then require

$$\begin{aligned} &\gamma mn(1 + A\rho^n) + \delta mn(1 + A\rho^m) \\ &\qquad - \alpha m(1 + A\rho^{m-1}) - \beta n(1 + A\rho^{n-1}) \geqq 1. \end{aligned} \tag{36}$$

We choose $\rho \leqq 1$ so that

$$(37) \qquad \rho\beta \leqq \frac{1}{2}\delta, \qquad \rho\beta \leqq \frac{1}{2}\gamma,$$

then choose the integer N so that

$$(38) \qquad \rho N\delta \geqq \alpha + 1, \qquad \rho N\gamma \geqq \beta + 1,$$

and finally A so that

$$(39) \qquad \frac{1}{2}\delta A\rho^{N-1} \geqq \alpha + 1, \qquad \frac{1}{2}\gamma A\rho^{N-1} \geqq \beta + 1.$$

We now check (36), whose left member we call $v(m, n)$, in four possible cases.

(i) $m \leqq N, \quad n \leqq N.$

$$(40) \qquad \begin{aligned} v(m, n) &\geqq \gamma n(1 + A\rho^{n-1}) - \beta n(1 + A\rho^{n}) \\ &\qquad + \delta m(1 + A\rho^{n-1}) - \alpha m(1 + A\rho^{m}) \\ &= m[\delta - \alpha + A\rho^{m-1}(\delta - \rho\alpha)] + n[\gamma - \beta + A\rho^{n-1}(\gamma - \rho\beta)]. \end{aligned}$$

Here we have

$$(41) \qquad \begin{aligned} \delta - \alpha + A\rho^{m-1}(\delta - \rho\alpha) &\geqq \delta - \alpha + A\rho^{N-1}\left(\frac{1}{2}\delta\right) \\ &\geqq \delta - \alpha + \alpha + 1 > 1, \end{aligned}$$

similarly

$$(42) \qquad \gamma - \beta + A\rho^{n-1}(\gamma - \rho\beta) > 1,$$

hence $v(m, n) > m + n > 1$.

(ii) $m > N, \quad n > N.$

$$(43) \qquad \begin{aligned} &v(m, n) \\ &\geqq (\gamma N - \beta)n + A(\rho\gamma N - \beta)n\rho^{n-1} + (\delta N - \alpha)m + A(\rho\delta N - \alpha)m\rho^{m-1} \\ &\geqq m + n > 1, \end{aligned}$$

since $\gamma N \geqq \beta + 1$, $\delta N \geqq \alpha + 1$ and $\rho\gamma N \geqq \beta$, $\rho\delta N \geqq \alpha$.

(iii) $m \leqq N, \quad n > N.$

$$(44) \qquad \begin{aligned} &v(m, n) \\ &\geqq \gamma n(1 + A\rho^{n-1}) - \beta n(1 + A\rho^{n}) + (\delta N - \alpha)m + A(\rho\delta N - \alpha)m\rho^{m-1} \\ &\geqq m + n > 1, \end{aligned}$$

as in (i) and (ii).

(iv) $m > N, \quad n \leqq N.$

Similar to case (iii).

3.4. In theorem 4 the first assertion again follows from general theory (compare [6]). To prove that (15) is sufficient for nondissipation, we cannot apply (B), because (24) can be shown (as in [4]) to imply the existence of a finite closed set of states. Instead we use a criterion whose analogue for discrete time is due to Foster [3].

(D) *Suppose Q regular and irreducible. If there exist a state I and $U_i \geqq 0$ such that*

$$\sum_j q_{ij}u_j + 1 \leqq 0, \qquad i \neq I, \tag{45}$$
$$\sum_j q_{ij}u_j \quad \text{finite}, \qquad i = I,$$

then $\sum_j \pi_{ij} = 1$ for all i.

The construction of such u_i, taking $I = (0, 0)$, and assuming (15) to hold, is almost identical with that in section 3.3.

Proofs of (D), together with those of (B) and (C), follow in section 3.5.

3.5. We now prove the criteria (B) to (D) used above.

For (B) and (D) we need two facts connected with $\varphi_{ij}(\lambda)$, the Laplace transform of the $p_{ij}(t)$ associated with a regular Q (compare [7]). First

$$\pi_{ij} = \lim_{\lambda \downarrow 0} \lambda\, \varphi_{ij}(\lambda) \tag{46}$$

and secondly

(E) *If $u_i \geqq 0$, v_i is bounded below, $\lambda > 0$, and*

$$\lambda u_i \geqq v_i + \sum_j q_{ij}u_j, \tag{47}$$

then

$$u_i \geqq \sum_j \varphi_{ij}(\lambda)v_j. \tag{48}$$

(E) is an extension of theorem 5 in [7], and can be proved similarly.

To prove (B) note that (24) implies, for each $\lambda > 0$, that

$$\lambda u_i \geqq \lambda u_i + \sum_j q_{ij}u_j; \tag{49}$$

thus (E), with $v_i = \lambda u_i$, gives

$$u_i \geqq \sum_j \lambda\varphi_{ij}(\lambda)u_j. \tag{50}$$

Letting $\lambda \downarrow 0$ and using (46), we get

$$u_i \geqq \sum_j \pi_{ij}u_j \tag{51}$$

and the argument of Kendall in [4] then shows that $\sum_j \pi_{ij} = 1$ for all i.

To prove (D), write (45) as

$$0 \geqq c_i + \sum q_{ij}u_j, \tag{52}$$

where $c_i = 1$ for $i \neq I$ and $c_I = -\sum_j q_{Ij}u_j$. Then

$$\lambda u_i \geqq c_i + \sum_j q_{ij}u_j, \qquad \lambda > 0, \tag{53}$$

whence by (E)

$$\begin{aligned} u_i &\geqq \sum_j \varphi_{ij}(\lambda)c_j \\ &= \sum_{j \neq I} \varphi_{ij}(\lambda) + \varphi_{iI}(\lambda)c_I \\ &= \lambda^{-1} + (c_I - 1)\varphi_{iI}(\lambda) \end{aligned} \tag{54}$$

because $\sum_j p_{ij}(t) = 1$ so that $\sum_j \varphi_{ij}(\lambda) = \lambda^{-1}$. Hence

$$\lambda u_i \geqq 1 + (c_I - 1)\lambda\varphi_{iI}(\lambda), \tag{55}$$

and $\lambda \downarrow 0$ gives

$$0 \geqq 1 + (c_I - 1)\pi_{iI}. \tag{56}$$

This shows that $\pi_{iI} \neq 0$ for all i, and the irreducibility of Q then implies that $\sum_j \pi_{ij} = 1$ for all i.

Finally we prove (C). There is no real loss in assuming that there is just one absorbing state, $i = 0$ say (otherwise we modify Q so as to combine all absorbing states into one). The fact that $\alpha_i = 1$ follows from the argument in (D) above. Since $c_0 = 0$, relation (56) gives

$$1 - \pi_{i0} \leqq 0 \tag{57}$$

so that $\alpha_i = \pi_{i0} = 1$. Next, note that $p_{i0}(t)$ is increasing and $p'_{i0}(t)$ is the probability density of the absorption time. Hence

$$\tau_i = \int_0^\infty t p'_{i0}(t)\, dt. \tag{58}$$

We can express this in terms of $\varphi_{i0}(\lambda)$ by observing that $\lambda^{-1}(1 - e^{-\lambda t}) \uparrow t$ as $\lambda \downarrow 0$, so that

$$\begin{aligned} \tau_i &= \lim_{\lambda \downarrow 0} \int_0^\infty \lambda^{-1}(1 - e^{-\lambda t}) p'_{i0}(t)\, dt \\ &= \lim_{\lambda \downarrow 0} \frac{1 - \lambda\varphi_{i0}(\lambda)}{\lambda}, \end{aligned} \tag{59}$$

where the last evaluation follows by integrating the previous integral by parts. Now suppose that $u_i \geqq 0$ and

$$\sum_j q_{ij} u_j + 1 \leqq 0, \qquad i \neq 0. \tag{60}$$

Then, for $\lambda > 0$,

$$\begin{aligned} \lambda u_i &\geqq (1 - \delta_{i0}) + \sum_j q_{ij} u_j, \\ u_i &\geqq \sum_{j \neq 0} \varphi_{ij}(\lambda) = \lambda^{-1}[1 - \lambda\varphi_{i0}(\lambda)] \end{aligned} \tag{61}$$

so that, letting $\lambda \downarrow 0$, we get

$$\tau_i = \lim_{\lambda \downarrow 0} \lambda^{-1}[1 - \lambda\varphi_{i0}(\lambda)] \leqq u_i. \tag{62}$$

REFERENCES

[1] M. S. Bartlett, "Deterministic and stochastic models for recurrent epidemics," *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley and Los Angeles, University of California Press, 1956, Vol. 4, pp. 81–109.

[2] F. G. Foster, "Markoff chains with an enumerable number of states and a class of cascade processes," *Proc. Cambridge Philos. Soc.*, Vol. 47 (1951), pp. 77–85.

[3] ———, "On the stochastic matrices associated with certain queuing processes," *Ann. Math. Statist.*, Vol. 24 (1953), pp. 355–360.

[4] D. G. KENDALL, "On non-dissipative Markoff chains with an enumerable infinity of states," *Proc. Cambridge Philos. Soc.*, Vol. 47 (1951), pp. 633–634.

[5] ———, "Les processus stochastiques de croissance en biologie," *Ann. Inst. H. Poincaré*, Vol. 12 (1952), pp. 43–108.

[6] D. G. KENDALL and G. E. H. REUTER, "The calculation of the ergodic projection for Markov chains and processes with a countable infinity of states," *Acta Math.*, Vol. 97 (1957), pp. 103–144.

[7] G. E. H. REUTER, "Denumerable Markov processes and the associated contraction semi-groups on 1," *Acta Math.*, Vol. 97 (1957), pp. 1–46.