NONINCREASE EVERYWHERE OF THE BROWNIAN MOTION PROCESS

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1. Introduction

The (linear, separable) Brownian motion process has been studied more than any other stochastic process. It has many applications and, at least since Bachelier, probabilists have been attracted by its delicate and curious properties. It furnished, in the hands of N. Wiener, the first instance of a satisfactorily defined nondiscrete stochastic process with continuous time parameter, and it is this work on Brownian motion (also known as Wiener space) that suggested the, now universally adopted, method of A. N. Kolmogorov for defining stochastic processes. Most advanced books on probability devote some space to this process but the more delicate results are beyond their scope. A notable exception is P. Lévy [2] which contains a very profound study of the process. However, though the proof of our principal result could be expedited by appealing to some advanced work on Brownian motion we preferred a presentation using only the simpler and better known properties of the process.

The Brownian motion process can be described as a probability space whose elements are all continuous functions defined on the whole real line and vanishing at the origin. The principal aim of this paper is to prove the, to us rather unexpected, result that *the probability of the set of functions which increase at least at one point is zero*. [A function is said to increase at a point if its values slightly to the right (left) of this point are not smaller (larger) than its value at the point.]

A formal statement of this result will be given in the next section and its significance will be discussed in the following one. Section 4 will give an interesting, though wrong and leading to a wrong result, heuristic argument. The

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next three sections will be devoted to the proof of the main result, while the last section will present some remarks.

2. Statement of main result

Notwithstanding its being so well known we will define the Brownian motion process, using the occasion to introduce notations employed in the sequel.

The separable Brownian process may be defined as a probability space $(\Omega, \mathcal{B}, P)$ whose elements $\omega \in \Omega$ consist of all real functions $x_\omega(t)$ defined and continuous on $-\infty < t < \infty$ and satisfying

$$x_\omega(0) = 0; \tag{2.1}$$

whose measurable sets are the elements of the sigma field generated by the sets

$$B(s, t; a) = \{\omega : x_\omega(t) - x_\omega(s) < a\}, \qquad -\infty < s < t < \infty, \; -\infty < a < \infty \tag{2.2}$$

completed with respect to the measure P defined hereafter; and whose probability measure P is determined by

$$P\{B(s, t; a)\} = \frac{1}{[2\pi(t-s)]^{1/2}} \int_{-\infty}^{a} \exp\left[\frac{-z^2}{2(t-s)}\right] dz; \tag{2.3}$$

and

$$P\left\{\bigcap_{i=1}^{n} B(s_i, t_i; a_i)\right\} = \prod_{i=1}^{n} P\{B(s_i, t_i; a_i)\} \tag{2.4}$$

for any finite number of triplets $(s_i, t_i; a_i)$, provided the open intervals (s_i, t_i), where $i = 1, \cdots, n$, are disjoint. Throughout this paper $\{\omega : \cdots\}$ denotes the set of ω satisfying $\cdots$. Sometimes ω: is omitted for brevity; this is done systematically in the braces following the probability sign P.

These conditions imply that the process is (strongly) Markovian. We also note in particular the symmetric distribution of the increments given by (2.3); the translation invariance, that is, the fact that the mapping

$$x_\omega(t) \to x_\omega(t + t_0) - x_\omega(t_0), \qquad -\infty < t < \infty, \tag{2.5}$$

is, for every $-\infty < t_0 < \infty$, a measure preserving mapping of Ω onto itself; and the homogeneity relation, that is, the property that the mapping

$$x_\omega(t) \to \frac{1}{\sigma} x_\omega(\sigma^2 t), \qquad -\infty < t < \infty, \tag{2.6}$$

is, for every $\sigma \neq 0$, a measure preserving mapping of Ω onto itself. All these properties of the Brownian process will be repeatedly used without specific mention.

Next we define carefully the notion of a point of increase (which might actually be better named a point of nondecrease).

DEFINITION. *A point t_0 is said to be a point of increase of size Δ, where $\Delta > 0$, of the real function $f(t)$, if $f(t)$ is defined in the interval $t_0 - \Delta \leqq t \leqq t_0 + \Delta$ and satisfies*

$$\max_{t_0-\Delta \leqq t \leqq t_0} f(t) = f(t_0) = \min_{t_0 \leqq t \leqq t_0+\Delta} f(t). \tag{2.7}$$

Also t_0 is said to be a point of increase of $f(t)$ or, equivalently, $f(t)$ is said to increase at t_0, if t_0 is a point of increase of size Δ of $f(t)$ for some $\Delta > 0$.

We can now state our main result.

THEOREM 1. *Almost all Brownian paths are nowhere increasing, that is,*

$$P\{x_\omega(t) \text{ has at least one point of increase}\} = 0. \tag{2.8}$$

3. Discussion

It is important to distinguish carefully between those properties of the process which hold almost surely (that is, with probability 1) everywhere and those that hold almost surely almost everywhere (that is, except for a set of t—possibly depending on ω—of Lebesgue measure zero). The latter kind of result is usually much easier to establish; indeed, it is in general sufficient to prove that a property holds almost surely for $t = 0$, apply (2.5) to deduce that it holds almost surely for any prescribed t_0 and then use Fubini's theorem to conclude that it holds almost surely almost everywhere. And, of course, if a property holds almost surely almost everywhere it may yet fail to hold almost surely everywhere.

A pretty obvious but pertinent example is the following. It is immediate that $P\{x_\omega(t) \leqq 0 \text{ for } 0 \leqq t \leqq \Delta\} = 0$ for every $\Delta > 0$; hence the set of (local) maximum points of $x_\omega(t)$ is, almost surely, a null set. On the other hand it is perhaps even more immediate that $P\{x_\omega(t) \text{ is monotone throughout an interval of length } 1\} = 0$ and hence, $x_\omega(t)$ being continuous, $P\{x_\omega(t) \text{ has at least one maximum}\} = 1$. Indeed, it can be shown with very little effort that the set of points of maximum is, almost surely, of the power of the continuum in every nonempty open interval.

To prove that a property holds almost surely everywhere we must usually find a way to reduce the problem to one involving a countable number of points. A simple illustration is furnished by the following:

$$P\left\{\limsup_{0<h\to 0} \frac{|x_\omega(t+h) - x_\omega(t)|}{h} = \infty \text{ for all } t\right\} = 1. \tag{3.1}$$

To establish (3.1) it is sufficient to prove that for every $K > 0$, $\Delta > 0$, almost surely, there are no points t for which $|x_\omega(t+h) - x_\omega(t)| \leqq Kh$ holds for all $0 \leqq h \leqq \Delta$. Moreover, it is enough to show this for $0 \leqq t \leqq 1$. Assume to the contrary there exists such a point t. Then, if $(i-1)/n \leqq t \leqq i/n$, for $i = 1, 2, \cdots, n$, and n is sufficiently large we would have

$$\begin{aligned} \left|x_\omega\left(\frac{i+1}{n}\right) - x_\omega\left(\frac{i}{n}\right)\right| &\leqq \frac{3}{n} K, \\ \left|x_\omega\left(\frac{i+2}{n}\right) - x_\omega\left(\frac{i+1}{n}\right)\right| &\leqq \frac{5}{n} K, \\ \left|x_\omega\left(\frac{i+3}{n}\right) - x_\omega\left(\frac{i+2}{n}\right)\right| &\leqq \frac{7}{n} K. \end{aligned} \tag{3.2}$$

These three conditions are independent and their joint probability is smaller than $[14K/(2\pi n)^{1/2}]^3$. This is a bound for the probability that any specific interval $[(i-1)/n, i/n]$ contains a point t with the above property. Since the unit interval contains n such small intervals the probability that there is a point t with the indicated property in $(0, 1)$ is $<$ constant $n^{-1/2}$. This being true for all large n, the proof is achieved.

An elaboration of the same argument can yield not only (3.1), which implies almost sure nonderivability everywhere, but also stronger results, for example, the fact that the Brownian paths almost surely satisfy nowhere a Lipschitz condition of any order $\alpha > 1/2$. All these results, the first ones showing the almost sure strong oscillatory character *everywhere* of the Brownian process, are due to Paley, Wiener and Zygmund [4]. (See Lévy [3] for further results of this nature.)

It will be noticed that the proof of (3.1) was deduced from the fact that having bounded right-hand derivatives at a point t implies certain properties of $x_\omega(t)$ in small intervals close to t and *not necessarily containing it* [otherwise the independence could not have been used to estimate the probability of (3.2)]. The property of being a point of increase is much more delicate and its trapping is considerably more troublesome.

4. A wrong heuristic argument

A most useful tool in the study of Brownian motion is the so-called reflection principle (which is, indeed, of wider applicability since the translation invariance is not needed and only the symmetric distribution of the increments, but not their specific form, is invoked). This principle asserts that the mapping (reflection at τ, where $\tau \geqq 0$)

$$x_\omega(t) \to \begin{cases} x_\omega(t) & \text{for} \quad t \leqq \tau \\ 2x_\omega(\tau) - x_\omega(t) & \text{for} \quad t \geqq \tau \end{cases} \tag{4.1}$$

is a measure preserving transformation of Ω onto itself.

A well known important consequence of the measure preserving character of (4.1) is the formula

$$P\{\max_{0 \leqq t \leqq T} x_\omega(t) > u\} = \frac{2}{\sqrt{2\pi T}} \int_u^\infty \exp\left(\frac{-z^2}{2T}\right) dz, \qquad u \geqq 0,\ T > 0. \tag{4.2}$$

Indeed, the set whose probability is evaluated may be written as a union of three disjoint sets

$$\begin{aligned} &\left\{\omega: \max_{0 \leqq t \leqq T} x_\omega(t) > u,\ x_\omega(T) > u\right\} \\ \cup &\left\{\omega: \max_{0 \leqq t \leqq T} x_\omega(t) > u,\ x_\omega(T) = u\right\} \\ \cup &\left\{\omega: \max_{0 \leqq t \leqq T} x_\omega(t) > u,\ x_\omega(T) < u\right\}. \end{aligned} \tag{4.3}$$

But the second of these sets has probability zero while the other two are mapped on one another by reflection at τ, where $\tau = \tau_\omega$ is the smallest positive value of

t for which $x_\omega(t) = u$. Hence the probability in question is twice the probability of the first set in (4.3), which is identical with $\{\omega : x_\omega(T) > u\}$ and the result follows from (2.3).

Now it is observed at once that if τ is a point where $x_\omega(t)$ has a (local) maximum then reflection at τ will transform it into a function for which τ is a point of increase. Since there are, almost surely, many maxima, it would seem to follow that there are also many points of increase.

It is, however, easy to discern a gap in this argument. We deliberately stated the reflection principle somewhat vaguely. To make it operative $\tau = \tau_\omega$ must be what is called a Markov time, that is, besides being a random variable (that is, a measurable function of ω) it must be defined in terms of $x_\omega(t)$ for $t \leqq \tau_\omega$. This is certainly the case if $\tau_\omega = \min \{t : t \geqq 0, x_\omega(t) = u\}$ (first crossing time) and hence the derivation of (4.2) is correct. But the fact that τ is a point of maximum depends necessarily on the behavior of $x_\omega(t)$ slightly to the right of τ.

One yet feels that the above objection could be circumvented. Though simple attempts such as reflecting at the first crossing time of a preassigned height fail (since the set of values at the points of maxima though, almost surely, of the power of the continuum is yet of measure zero), it seems that it should be possible to salvage the argument by more elaborate devices.

After all why should a path $x_\omega(t)$, at every point t_0 for which $x_\omega(t_0 - h) \leqq x_\omega(t_0)$ for sufficiently small $h > 0$, "prefer" going down, thus creating a maximum, over going up? (Of course, to the right of "most" such points the path does neither; it oscillates.)

In view of these considerations the theorem stated in section 2 is somewhat surprising. It turns out that the above attempted disproof is, not only delicately but quite definitely, wrong. We thought it instructive to bring the heuristic argument since it constitutes a fine example of the pitfalls courted by disregarding the Markovian injunction. It is also the only natural example known to us where the reflection principle, indiscriminately applied, yields wrong results.

In the last section we shall say a few words of explanation, based on our proof, about this perplexing preference of maxima over points of increase.

5. Outline and reduction of the proof

To establish the theorem it is enough to prove that, for any given $\Delta > 0$,

(5.1) $$P\{x_\omega(t) \text{ has at least one point of increase of size } \Delta\} = 0,$$

since the event in (2.8) is the intersection of the events in (5.1) with $\Delta = 1/2, 1/3, \cdots$. Moreover, by the homogeneity property, (5.1) for one $\Delta > 0$ implies it for all $\Delta > 0$. Thus it is enough to prove (5.1) for a specified Δ; we choose $\Delta = 2$.

Furthermore, by the translation invariance,

(5.2) $$P\{x_\omega(t) \text{ has at least one point of increase of size 2 in } (0, 1)\} = 0$$

implies the same statement without the restriction "in (0, 1)." Hence the proof of the theorem is reduced to that of (5.2). To simplify the writing we shall introduce the following notation, for $-\infty < a \leqq b < \infty$,

$$M_\omega(a, b) = \max_{a \leqq t \leqq b} x_\omega(t), \tag{5.3}$$

$$\Lambda_\omega(a, b) = \min_{a \leqq t \leqq b} x_\omega(t). \tag{5.4}$$

Then (5.2) may be rewritten as

$$P\{M_\omega(t-2, t) \leqq \Lambda_\omega(t, t+2) \text{ for at least one } t \text{ in } (0, 1)\} = 0. \tag{5.5}$$

We now introduce the events

$$A_k^{(n)} = \left\{\omega : M_\omega\left(\frac{k}{n} - 2, \frac{k-1}{n}\right) \leqq \Lambda_\omega\left(\frac{k}{n}, \frac{k-1}{n} + 2\right)\right\}, \qquad n = 1, 2, \cdots; k = 1, 2, \cdots, 2n. \tag{5.6}$$

Clearly, the event described in (5.5) implies at least one of the events $A_1^{(n)}, \cdots, A_n^{(n)}$. Hence (5.5), and our theorem, would follow from

$$\lim_{n=\infty} P\left\{\bigcup_{k=1}^{n} A_k^{(n)}\right\} = 0. \tag{5.7}$$

Let $c_1, c_2, \cdots, c_{10}$ denote universal finite positive constants. It is quite easy to establish (see lemma 3) that

$$P\{A_k^{(n)}\} < \frac{c_1}{n}, \qquad n = 1, 2, \cdots; \tag{5.8}$$

however, this does not suffice to prove (5.7). Since, on the other hand,

$$P\{A_k^{(n)}\} > \frac{c_2}{n} \tag{5.9}$$

by lemma 4, something more will be necessary. The idea is to show that the occurrence of an event $A_k^{(n)}$ increases the a priori probability of $A_{k+j}^{(n)}$ occurring. It will be more expeditious to work with expectations. Put

$$Y_k^{(n)} = Y_k^{(n)}(\omega) = \begin{cases} 1, & \omega \in A_k^{(n)} \\ 0, & \omega \notin A_k^{(n)} \end{cases} \qquad k = 1, 2, \cdots, 2n, \tag{5.10}$$

and

$$S_k^{(n)} = S_k^{(n)}(\omega) = \sum_{i=1}^{k} Y_i^{(n)}, \qquad k = 1, 2, \cdots, 2n. \tag{5.11}$$

Then (5.7) is equivalent to

$$\lim_{n=\infty} P\{S_n^{(n)} \geqq 1\} = 0. \tag{5.12}$$

Let E denote expectation. Since

$$E(S_{2n}^{(n)}) \geqq E(S_{2n}^{(n)} | S_n^{(n)} \geqq 1) P\{S_n^{(n)} \geqq 1\} \tag{5.13}$$

and since, by (5.8),

$$E(S_{2n}^{(n)}) < 2c_1, \tag{5.14}$$

equation (5.12) would follow from

$$\lim_{n=\infty} E(S_{2n}^{(n)} | S_n^{(n)} \geqq 1) = \infty. \tag{5.15}$$

Thus our problem is reduced to proving (5.15) and this will constitute our main effort.

From now on n is kept fixed; therefore the superscript in $A_k^{(n)}$, $Y_k^{(n)}$, $S_k^{(n)}$ will be omitted and the dependence on n will not be explicitly displayed in the random variables and events defined hereafter.

It is not difficult to show that $P\{A_k \cap A_{k+j}\} = E(Y_k Y_{k+j}) > c_3/nj$ for $j \geqq 1$, which implies $E(S_{2n}|Y_k = 1) > (c_3/c_2) \log n$, by (5.9). Unfortunately, very little can be deduced from this fact since the conditioning events are not disjoint. It will be necessary to prove an analogous result with the disjoint conditioning events

$$B_k = A_k - \bigcup_{i=1}^{k-1} A_i = \{Y_k = 1\} \cap \{S_k = 1\}, \qquad k = 1, 2, \cdots, n. \tag{5.16}$$

To abbreviate we put

$$Z_k = \begin{cases} 1 & \text{if} \quad Y_k = S_k = 1 \\ 0 & \text{otherwise} \end{cases} \qquad k = 1, 2, \cdots, n \tag{5.17}$$

and shall establish a result concerning $E(S_{2n}|Z_k = 1)$ from which we shall be able to deduce (5.15) and thus complete the proof.

6. Lemmas

These lemmas are formulated in a form suited to our needs and their generality and sharpness can easily be augmented. On the other hand we went to some trouble to prove them more or less elementarily. We mention in particular that lemma 8 should be directly deducible from Doob's results [1] (though this may not be too easy since the convergence of the series in [1] gets very poor when small parameters are involved).

LEMMA 1. *For every $\Delta \geqq \epsilon > 0$ we have*

$$P\{\Lambda_\omega(\epsilon, \epsilon + \Delta) \geqq 0\} > \frac{1}{2\pi}\left(\frac{\epsilon}{\Delta}\right)^{1/2}. \tag{6.1}$$

PROOF. From (4.2) we obtain

(6.2)

$$\begin{aligned} P\{\Lambda_\omega(\epsilon, \epsilon + \Delta) \geqq 0\} &= \frac{1}{\sqrt{2\pi\epsilon}} \int_0^\infty P\{\Lambda_\omega(0, \Delta) \geqq -u | x_\omega(\epsilon) = u\} \exp\left(\frac{-u^2}{2\epsilon}\right) du \\ &= \frac{1}{\sqrt{2\pi\epsilon}} \int_0^\infty \frac{2}{\sqrt{2\pi\Delta}} \int_0^u \exp\left(\frac{-z^2}{2\Delta}\right) dz \exp\left(\frac{-u^2}{2\epsilon}\right) du \\ &> \frac{1}{\sqrt{2\pi\epsilon}} \frac{2}{\sqrt{2\pi\Delta}} \int_0^\infty u \exp\left(-\frac{u^2}{2\Delta} - \frac{u^2}{2\epsilon}\right) du \\ &= \frac{1}{\sqrt{2\pi\epsilon}} \frac{2}{\sqrt{2\pi\Delta}} \frac{\epsilon\Delta}{\epsilon + \Delta} \\ &\geqq \frac{1}{2\pi}\left(\frac{\epsilon}{\Delta}\right)^{1/2}. \end{aligned}$$

LEMMA 2. *For every* $\Delta \geqq \epsilon > 0$ *we have*

$$P\{\Lambda_\omega(\epsilon, \epsilon + \Delta) \geqq \sqrt{\epsilon}\} > c_4 \left(\frac{\epsilon}{\Delta}\right)^{1/2}. \tag{6.3}$$

PROOF. We have

$$P\{\Lambda_\omega(\epsilon, \epsilon + \Delta) \geqq \sqrt{\epsilon}\} \geqq P\left\{x_\omega\left(\frac{\epsilon}{2}\right) \geqq \sqrt{\epsilon}\right\} P\left\{\Lambda_\omega\left(\frac{\epsilon}{2}, \frac{\epsilon}{2} + \Delta\right) \geqq 0\right\} \tag{6.4}$$

and (6.3) follows from the preceding lemma.

LEMMA 3. *We have, for every* $\epsilon > 0$, $\Delta > 0$,

$$P\{M_\omega(-\Delta, 0) \leqq \Lambda_\omega(\epsilon, \epsilon + \Delta)\} < \frac{\epsilon}{\pi\Delta}. \tag{6.5}$$

PROOF. Again by (4.2) we have

$$\begin{aligned} P\{M_\omega(-\Delta, 0) &\leqq \Lambda_\omega(\epsilon, \epsilon + \Delta)\} \\ &\leqq \frac{1}{\sqrt{2\pi\epsilon}} \int_0^\infty P\{M_\omega(-\Delta, 0) \leqq u\} P\{\Lambda_\omega(\epsilon, \epsilon + \Delta) \\ &\qquad\qquad \geqq 0 | x_\omega(\epsilon) = u\} \exp\left(\frac{-u^2}{2\epsilon}\right) du \\ &= \frac{1}{\sqrt{2\pi\epsilon}} \int_0^\infty \left[\frac{2}{\sqrt{2\pi\Delta}} \int_0^u \exp\left(\frac{-z^2}{2\Delta}\right) dz\right]^2 \exp\left(\frac{-u^2}{2\epsilon}\right) du \\ &< \frac{2}{\pi\Delta} \frac{1}{\sqrt{2\pi\epsilon}} \int_0^\infty u^2 \exp\left(\frac{-u^2}{2\epsilon}\right) du \\ &= \frac{\epsilon}{\pi\Delta}. \end{aligned} \tag{6.6}$$

REMARK. Inequality (6.5) with $\epsilon = 1/n$ and $\Delta = 2 - 1/n$ yields (5.8).

LEMMA 4. *For every* $\Delta \geqq \epsilon > 0$ *we have*

$$P\{M_\omega(-\Delta, 0) \leqq \Lambda_\omega(\epsilon, \epsilon + \Delta)\} > c_2 \frac{\epsilon}{\Delta}. \tag{6.7}$$

PROOF. As before

$$\begin{aligned} P\{M_\omega(-\Delta, 0) &\leqq \Lambda_\omega(\epsilon, \epsilon + \Delta)\} \\ &= P\left\{M_\omega\left(-\Delta - \frac{\epsilon}{2}, -\frac{\epsilon}{2}\right) \leqq \Lambda_\omega\left(\frac{\epsilon}{2}, \frac{\epsilon}{2} + \Delta\right)\right\} \\ &\geqq P\left\{M_\omega\left(-\Delta - \frac{\epsilon}{2}, -\frac{\epsilon}{2}\right) \leqq 0\right\} P\left\{\Lambda_\omega\left(\frac{\epsilon}{2}, \frac{\epsilon}{2} + \Delta\right) \geqq 0\right\} \\ &= \left[P\left\{\Lambda_\omega\left(\frac{\epsilon}{2}, \frac{\epsilon}{2} + \Delta\right) \geqq 0\right\}\right]^2 \end{aligned} \tag{6.8}$$

and (6.7) follows from (6.1).

REMARK. Inequality (5.9) follows from (6.7) with $\epsilon = 1/n$ and $\Delta = 2 - 1/n$.

LEMMA 5. *We have, for all* $0 \leqq u \leqq 1/2$,

$$P\left\{-u < \Lambda_\omega\left(0, \frac{1}{2}\right) \leqq M_\omega\left(0, \frac{1}{2}\right) < 1 - u\right\} \geqq c_5 u. \tag{6.9}$$

PROOF. The complementary probability to that in (6.9) is

$$P\left\{\Lambda_\omega\left(0, \frac{1}{2}\right) \leqq -u\right\} + P\left\{\Lambda_\omega\left(0, \frac{1}{2}\right) > -u, M_\omega\left(0, \frac{1}{2}\right) \geqq 1 - u\right\}. \tag{6.10}$$

The second probability in (6.10) is smaller than the probability that $x_\omega(t)$, considered for all $t \geqq 0$, attains the value $1 - u$ before it does the value $-u$. This is well known to be exactly u. Recalling the classical problem of the gambler's ruin, calling it $p(u)$, the assertion follows immediately from $p(u) + p(1 - u) = 1$ and $p[(u_1 + u_2)/2] = [p(u_1) + p(u_2)]/2$. From this and (4.2) it follows that (6.10) is smaller than $1 - (2/\sqrt{\pi}) \int_0^u \exp(-z^2)\, dz + u$. Hence the probability in (6.9) is greater than

$$\frac{2}{\sqrt{\pi}} \int_0^u e^{-z^2} dz - u \geqq \left(\frac{2}{\sqrt{\pi}} e^{-u^2} - 1\right) u. \tag{6.11}$$

This proves lemma 5 for sufficiently small u. Since the left side of (6.9) is clearly symmetric about $u = 1/2$ and monotone from 0 to 1/2 [because of the continuity of $x_\omega(t)$] the result is fully established.

LEMMA 6. *We have, for all* $0 \leqq u \leqq 1/2$,

$$P\{-u - t \leqq x_\omega(t) \leqq 1 \quad \textit{for} \quad 0 \leqq t \leqq 1 | x_\omega(1) = 0\} \geqq c_6 u. \tag{6.12}$$

PROOF. To prove (6.12) we have to use the result that the probabilities in the "tied" Brownian motion process [that is, conditioned by $x_\omega(1) = 0$] may be evaluated by remarking that the mapping $(1 + t)x_\omega[t/(1 + t)] \to x_\omega(t)$ is a measure preserving mapping of the tied process on (0, 1) onto the ordinary process on $(0, \infty)$. (See, for example, Lévy [2], [3] or Doob [1]; the result is easily verified on observing that the two processes are Gaussian and comparing the covariance functions.) Therefore the probability in (6.12) is equal to

$$\begin{aligned} &P\left\{-u - \frac{t}{1 + t} \leqq \frac{x_\omega(t)}{1 + t} \leqq 1 \text{ for } t \geqq 0\right\} \\ &\qquad \geqq P\{-u - t \leqq x_\omega(t) \leqq 1 + t \text{ for } t \geqq 0\} \\ &\qquad \geqq P\left\{-u \leqq x_\omega(t) \leqq 1 \text{ for } t \leqq \frac{1}{2}\right\} \\ &\min_{-u \leqq v \leqq 1} P\left\{-u - t \leqq x_\omega(t) \leqq 1 + t \text{ for } t \geqq \frac{1}{2} \Big| x_\omega\left(\frac{1}{2}\right) = v\right\} \end{aligned} \tag{6.13}$$

which, by (6.9), is greater than

$$c_5 u P\left\{\left|x_\omega(t) - x_\omega\left(\frac{1}{2}\right)\right| \leqq \frac{1}{2} + \left(t - \frac{1}{2}\right) \text{ for } t \geqq \frac{1}{2}\right\} \tag{6.14}$$

and the second factor is $P\{|x_\omega(t)| \leqq 1/2 + t \text{ for } t \geqq 0\} > 0$ (we omit the proof of this well known fact which can be derived directly or, also, by use of the transformation mentioned in the beginning of the proof of this lemma).

LEMMA 7. *We have, for all* $y \geqq 1$ *and* $0 \leqq u \leqq 1/2$,

$$(6.15)\qquad P\{-u \leqq \Lambda_\omega(0, 1) \leqq M_\omega(0, 1) \leqq 1 + y|x_\omega(1) = y\} \geqq c_6 u.$$

PROOF. Remarking (see, for example, Lévy [2], [3]) that the process conditioned by $x_\omega(1) = y$ is transformed into that conditioned by $x_\omega(1) = 0$ on replacing $x_\omega(t)$ by $x_\omega(t) - ty$ we see that the probability to be evaluated is

$$(6.16)\qquad P\{-u - ty \leqq x_\omega(t) \leqq 1 + (1 - t)y \text{ for } 0 \leqq t \leqq 1|x_\omega(t) = 0\},$$

and (6.15) follows at once from (6.12).

LEMMA 8. *We have, for all* $0 \leqq u \leqq 1/2$ *and* $0 \leqq v \leqq 1/2$,

$$(6.17)\qquad P\{-u \leqq \Lambda_\omega(0, 2) \leqq M_\omega(0, 2) \leqq x_\omega(2) + v,\ x_\omega(2) \geqq 2\} \geqq c_7 uv.$$

PROOF. The probability to be evaluated is greater than the product

$$(6.18)\qquad P\{-u \leqq \Lambda_\omega(0, 1) \leqq M_\omega(0, 1) \leqq x_\omega(1) + 1,\ x_\omega(1) \geqq 1\}$$
$$P\{-1 \leqq \Lambda_\omega(1, 2) - x_\omega(1) \leqq M_\omega(1, 2) - x_\omega(2) \leqq v,\ x_\omega(2) - x_\omega(1) \geqq 1\}$$

which, by the previous lemma, is greater than

$$(6.19)\qquad c_5 u P\{x_\omega(1) \geqq 1\} c_5 v P\{x_\omega(2) - x_\omega(1) \geqq 1\}.$$

LEMMA 9. *We have, for all* $\delta > 0$ *and* $0 \leqq u \leqq (\delta/8)^{1/2}$ *and* $0 \leqq v \leqq (\delta/8)^{1/2}$,

$$(6.20)\qquad P\{-u \leqq \Lambda_\omega(0, \delta) \leqq M_\omega(0, \delta) \leqq x_\omega(\delta) + v,\ x_\omega(\delta) \geqq \sqrt{\delta}\} \geqq c_7 \frac{uv}{\delta}.$$

PROOF. This follows at once from (6.17) and the homogeneity property.

7. Completion of the proof

Returning to the notations of section 5 we shall estimate $E(Y_{k+j}|Z_k = 1) = P\{A_{k+j}|B_k\}$.

Since $A_k = \{Y_k = 1\}$ implies $\Lambda_\omega[i/n, (i - 1)/n + 2] = \Lambda_\omega(i/n, k/n)$ for $i = 1, 2, \cdots, k - 1$ and $k = 1, 2, \cdots, n$ we may write $B_k = A_k \cap C_k$ where

$$(7.1)\qquad C_k = \left\{\omega : M_\omega\left(\frac{i}{n} - 2, \frac{i-1}{n}\right) > \Lambda_\omega\left(\frac{i}{n}, \frac{k}{n}\right) \text{ for } i = 1, 2, \cdots, k - 1\right\};$$

Therefore,

$$(7.2)\qquad P\{B_k\} = P\{C_k\} \int_0^\infty P\left\{\Lambda_\omega\left(\frac{k}{n}, \frac{k-1}{n} + 2\right) \geqq x_\omega\left(\frac{k}{n}\right) - u\right\} dF_k(u)$$

where

$$(7.3)\qquad F_k(u) = P\left\{x_\omega\left(\frac{k}{n}\right) - M_\omega\left(\frac{k}{n} - 2, \frac{k-1}{n}\right) \leqq u|C_k\right\}.$$

[All this is possible since C_k is determined in terms of $x_\omega(t)$ for $t \leqq k/n$.]

Similarly, we may write, for $j = 1, 2, \cdots, 2n - k$,

$$(7.4)\qquad B_k \cap A_{k+j} = C_k \cap C'_{k,j} \cap C_{k,j} \cap C''_{k,j}$$

where

$$(7.5)\qquad C'_{k,j} = \left\{\omega : M_\omega\left(\frac{k-1}{n}, \frac{k}{n}\right) \leqq \Lambda_\omega\left(\frac{k+j}{n}, \frac{k+j-1}{n} + 2\right)\right\},$$

$$(7.6)\qquad C_{k,j} = \left\{\omega : M_\omega\left(\frac{k}{n} - 2, \frac{k-1}{n}\right) \leqq \Lambda_\omega\left(\frac{k}{n}, \frac{k+j-1}{n}\right) \leqq M_\omega\left(\frac{k}{n}, \frac{k+j-1}{n}\right) \leqq \Lambda_\omega\left(\frac{k+j}{n}, \frac{k+j-1}{n} + 2\right)\right\}$$

and

$$(7.7)\qquad C''_{k,j} = \left\{\omega : M_\omega\left(\frac{k}{n} - 2, \frac{k-1}{n}\right) \leqq \Lambda_\omega\left(\frac{k+j-1}{n}, \frac{k+j}{n}\right)\right\}.$$

Put

$$(7.8)\qquad D_{k,j} = \left\{\omega : x_\omega\left(\frac{k+j-1}{n}\right) - x_\omega\left(\frac{k}{n}\right) \geqq \left(\frac{j-1}{n}\right)^{1/2}\right\}.$$

Then,

$$(7.9)\qquad B_k \cap A_{k+j} \supset B_k \cap A_{k+j} \cap D_{k,j} \supset C_k \cap D'_{k,j} \cap C_{k,j} \cap D_{k,j} \cap D''_{k,j},$$

where

$$(7.10)\qquad D'_{k,j} = \left\{\omega : M_\omega\left(\frac{k-1}{n}, \frac{k}{n}\right) \leqq x_\omega\left(\frac{k}{n}\right) + \left(\frac{j-1}{n}\right)^{1/2}\right\}$$

and

$$(7.11)\qquad D''_{k,j} = \left\{\omega : \Lambda_\omega\left(\frac{k+j-1}{n}, \frac{k+j}{n}\right) \geqq x_\omega\left(\frac{k+j-1}{n}\right) - \left(\frac{j-1}{n}\right)^{1/2}\right\}.$$

Hence

$$(7.12)\qquad P\{B_k \cap A_{k+j}\} \geqq P\{C_k \cap C_{k,j} \cap D_{k,j}\} - P\{\Omega - D'_{k,j}\} - P\{\Omega - D''_{k,j}\}.$$

We have

$$(7.13)\qquad P\{C_k \cap C_{k,j} \cap D_{k,j}\} = P\{C_k\} \int_0^\infty P\left\{C_{k,j} \cap D_{k,j} | x_\omega\left(\frac{k}{n}\right) - M_\omega\left(\frac{k}{n} - 2, \frac{k-1}{n}\right) = u\right\} dF_k(u)$$

where $F_k(u)$ is given by (7.3). The integrand in (7.13) is greater than the product

$$(7.14)\qquad P\left\{x_\omega\left(\frac{k}{n}\right) - u \leqq \Lambda_\omega\left(\frac{k}{n}, \frac{k+j-1}{n}\right) \leqq M_\omega\left(\frac{k}{n}, \frac{k+j-1}{n}\right) \leqq x_\omega\left(\frac{k+j-1}{n}\right) + \frac{1}{\sqrt{n}}, x_\omega\left(\frac{k+j-1}{n}\right) - x_\omega\left(\frac{k}{n}\right) > \left(\frac{j-1}{n}\right)^{1/2}\right\}$$
$$P\left\{\Lambda_\omega\left(\frac{k+j}{n}, \frac{k+j-1}{n} + 2\right) \geqq x_\omega\left(\frac{k+j-1}{n}\right) + \frac{1}{\sqrt{n}}\right\}.$$

By lemmas 9 and 2 with $\delta = (j-1)/n$ and $v = 1/\sqrt{n}$ and $\epsilon = 1/n$, this is, for $j > 1$, not smaller than $c_4c_7u/(j-1)\sqrt{2}$ for $u \leqq [(j-1)/8n]^{1/2}$. Hence

$$(7.15)\qquad P\{C_k \cap C_{k,j} \cap D_{k,j}\} \geqq \frac{c_8}{j-1} P\{C_k\} \int_0^{[(j-1)/8n]^{1/2}} u\, dF_k(u).$$

By (4.2),

(7.16)
$$\begin{aligned}P\{\Omega - D'_{k,j}\} = P\{\Omega - D''_{k,j}\} &= \frac{2}{(2\pi/n)^{1/2}} \int_{[(j-1)/n]^{1/2}}^{\infty} \exp\left(\frac{-u^2}{2/n}\right) du \\ &< \left[\frac{2}{\pi(j-1)}\right]^{1/2} \int_{[(j-1)/n]^{1/2}}^{\infty} nu \exp\left(\frac{-u^2}{2/n}\right) du \\ &= \left[\frac{2}{\pi(j-1)}\right]^{1/2} \exp\left(-\frac{j-1}{2}\right)\end{aligned}$$

and, therefore,

$$P\{\Omega - D'_{k,j}\} = P\{\Omega - D''_{k,j}\} < \frac{1}{n^3} \tag{7.17}$$

for values of j for which $j > 1 + 6 \log n$. Combining (7.12), (7.15) and (7.17) we obtain

$$P\{B_k \cap A_{k+j}\} > \frac{c_8}{j-1} P\{C_k\} \int_0^{[(j-1)/8n]^{1/2}} u\, dF_k(u) - \frac{2}{n^3} \tag{7.18}$$

for all values of j satisfying

$$1 + 6 \log n < j \leqq 2n - k. \tag{7.19}$$

Returning to (7.2) we have

(7.20)
$$\begin{aligned}P\{B_k\} \leqq P\{C_k\} \int_0^{[(j-1)/8n]^{1/2}} P\left\{\Lambda_\omega\left(\frac{k}{n}, \frac{k-1}{n} + 2\right) \geqq x_\omega\left(\frac{k}{n}\right) - u\right\} dF_k(u) \\ + P\left\{x_\omega\left(\frac{k}{n}\right) - x_\omega\left(\frac{k-1}{n}\right) > \left(\frac{j-1}{8n}\right)^{1/2}\right\}.\end{aligned}$$

By (4.2) the integrand in this expression is smaller than

$$P\left\{\Lambda_\omega\left(\frac{k}{n}, \frac{k}{n} + 1\right) \geqq x_\omega\left(\frac{k}{n}\right) - u\right\} = \frac{2}{\sqrt{2\pi}} \int_0^u e^{-u^2/2}\, du \leqq u, \tag{7.21}$$

while, as before, the last summand is for sufficiently large n certainly less than $1/n^3 < 1/n^2$ for j satisfying (7.19). Thus we have from (7.20) that

$$P\{B_k\} < P\{C_k\} \int_0^{[(j-1)/8n]^{1/2}} u\, dF_k(u) + \frac{1}{n^2} \tag{7.22}$$

for j satisfying (7.19). Hence we have, for these values of j,

$$\text{either } P\{B_k\} < \frac{2}{n^2} \quad \text{or} \quad P\{B_k\} < 2P\{C_k\} \int_0^{[(j-1)/8n]^{1/2}} u\, dF_k(u). \tag{7.23}$$

From this and (7.18) we obtain the inequality

$$P\{B_k \cap A_{k+j} | B_k\} > \frac{c_8}{2(j-1)} - \frac{1}{n} \tag{7.24}$$

whenever

$$P\{B_k\} > \frac{2}{n^2} \tag{7.25}$$

and j satisfies (7.19).

Since $B_k = \{Z_k = 1\}$ and $A_{k+j} = \{Y_{k+j} = 1\}$, we have from (7.24)

$$E(S_{2n}|B_k) > \sum_{1+6\log n<j\leq 2n-k} \left[\frac{c_8}{2(j-1)} - \frac{1}{n}\right] \tag{7.26}$$

whenever (7.25) holds. Therefore we have for $k = 1, 2, \cdots, n$ and all $n > c_9$ sufficiently large

$$E(S_{2n}|B_k) > c_{10} \log n \tag{7.27}$$

for the values of k satisfying (7.25).

Denoting by B the union of those B_k for which (7.25) is valid we deduce from (7.27) and the disjointness of the B_k that

$$E(S_{2n}|B) > c_{10} \log n. \tag{7.28}$$

Now,

$$P\{B\} \geqq P\{B_1\} = P\{A_1\} \geqq \frac{c_2}{n} \tag{7.29}$$

by (5.9) while

$$P\left\{\bigcup_{k=1}^{n} B_k - B\right\} \leqq n\frac{2}{n^2} = \frac{2}{n} \tag{7.30}$$

since the B_k not in B have each probability $\leqq 2/n^2$. As

$$E\left(S_{2n}|\bigcup_{k=1}^{n} B_k\right) \geqq E(S_{2n}|B)\, P\left\{B|\bigcup_{k=1}^{n} B_k\right\} \tag{7.31}$$

we have from (7.28), (7.29) and (7.30),

$$E\left(S_{2n}|\bigcup_{k=1}^{n} B_k\right) > \frac{c_2c_{10}}{c_2+2} \log n \tag{7.32}$$

for $n > c_9$.

Since the conditioning in (7.32) is exactly $S_n \geqq 1$, inequality (7.32) implies (5.15) thus completing the proof of theorem 1.

8. Remarks

8.1. Very slight modifications are required in the details of our proof that, for any given real number a, the functions $x_\omega(t) + at$ are also almost surely nowhere increasing (this may also be deduced directly from theorem 1). Therefore, almost surely, there are no points t for which $x_\omega(t+h) - x_\omega(t) + ah$ has the same sign as h, for all $-h(t, \omega) < h < h(t, \omega)$, with $h(t, \omega) > 0$. Hence

$$\liminf_{h\to 0} \frac{x_\omega(t+h) - x_\omega(t)}{h} \leqq -a \tag{8.1}$$

almost surely everywhere. This, being true for every a, implies that the lower derivative of $x_\omega(t)$ is $-\infty$.

Since throughout we may have treated points of decrease equally well as points of increase we have

THEOREM 2. *Almost all Brownian paths have everywhere lower derivative $-\infty$ and upper derivative $+\infty$, that is,*

(8.2)

$$P\left\{\liminf_{h\to 0}\frac{x_\omega(t+h)-x_\omega(t)}{h} = -\infty,\ \limsup_{h\to 0}\frac{x_\omega(t+h)-x_\omega(t)}{h} = \infty \text{ for all } t\right\} = 1.$$

8.2. From this theorem and (3.1) it follows that the four derived numbers D_-, D^-, D_+, D^+ of $x_\omega(t)$ satisfy, almost surely, everywhere the relations

$$\begin{gathered}\max(|D_-|, |D^-|) = \max(|D_+|, |D^+|) = \max(D^-, D^+) = \infty,\\ \min(D_-, D_+) = -\infty.\end{gathered} \tag{8.3}$$

The asymmetry between interchanging left and right ($-$ and $+$) and lower and upper (subscript and superscript) is apparent. This is again of the same nature as that discussed in section 4 and similar to that occurring in the Denjoy-Young-Saks theorem in an entirely nonprobabilistic context. If a function $f(t)$ has a (strict) maximum of size Δ at a point t_0 then it cannot have other maxima between $t_0 - \Delta$ and $t_0 + \Delta$; however, if t_0 is a point of increase it does not preclude other arbitrary near points from being also points of increase. As a matter of fact it makes it "easier" for them to be points of increase, since the fact that t_0 is a point of increase of size Δ makes part of the conditions in the definition of a point of increase redundant. Hence, maxima of prescribed size necessarily occur separated whereas points of increase tend to occur, if at all, in bunches. It is precisely these considerations that were utilized in our proof of theorem 2 and that lie behind the apparent paradox discussed in section 4.

8.3. Our results can be extended to certain other stochastic processes. They also can be sharpened somewhat; we mention the problem to what extent can h in the denominator in (8.2) be replaced by an odd function approaching zero less rapidly than h. Another interesting question is whether theorem 1 remains valid if instead of considering points of increase we consider points t where $[x_\omega(t+h) - x_\omega(t)]h \geqq 0$ for all sufficiently small h for $t + h$ is rational (t need not be rational).

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