7. PROOFS OF THEOREMS 3.1 AND 3.2.

The first step is to show that for a parameter point p_0 which satisfies Condition A or B of Ch. 3 there exist large rectangles for which the crossing probabilities in both the horizontal and vertical direction are bounded away from zero. The RSW theorem will then show that with P_{p_0}-probability one there exist arbitrarily large occupied circuits on $\mathcal{G}$ surrounding the origin. From this it follows that there are no infinite vacant clusters on $\mathcal{G}^*$ under P_{p_0}[1]. An interchange of the roles of $\mathcal{G}$ and $\mathcal{G}^*$ and of occupied and vacant then shows that there is also no percolation on $\mathcal{G}$ under P_{p_0}. This is just the content of (3.43), which is the most important statement in Theorem 3.1(i). Clearly the above implies that for p', p'' such that $p'(i) \le p_0(i) \le p''(i)$, $i = 1,\dots,\lambda$ also

$$P_{p'}\{\#W(v) = \infty\} = 0 \quad \text{and} \quad P_{p''}\{\#W^*(v) = \infty\} = 0.$$

The above conclusions are basically already in Harris' beautiful paper (Harris (1960)). The first proof that percolation actually occurs for $p'' \gg p_0$ is in Kesten (1980a). The proof given below is somewhat simpler because we now use Russo's formula (Prop. 4.2) which only appeared in Russo (1981). Actually we prove the dual statement that for $p' \ll p_0$ infinite vacant clusters occur on $\mathcal{G}^*$. An easy argument shows that it suffices to show $E_{p'}\{\#W(v)\} < \infty$, and by Theorem 5.1 this will follow once we prove that the crossing-probabilities $\tau(\overline{N};1,p')$ and $\tau(\overline{N};2,p')$ of some large rectangles are small for $p' \ll p$. This is done by showing that $\frac{d}{dt}\tau(\overline{N};i,p(t))$ is "large" for $0 \le t \le 1$, $p(t) = (1-t)p' + tp_0$. By Russo's formula, this amounts to showing that there are many pivotal sites (see Def. 4.2) for the events

$$(7.1) \qquad A(\overline{N};i) = \{\exists \text{ occupied crossing in the i-direction of } T(\overline{N};i)\}.$$

[1] In this part we shall use some simplifications suggested by S. Kotani.

This last step is the same as in Kesten (1980a). The pivotal sites for $A(\overline{N};1)$ for instance are found more or less as the sites on the lowest occupied horizontal crossing of $T(\overline{N};1)$ which have a vacant connection on $\mathcal{G}^*$ to the upper edge of $T(\overline{N};1)$ (see Ex. 4.2(iii)). To enable us to talk about a "lowest horizontal crossing" the actual proofs are carried out on $\mathcal{G}_{p\ell}$, $\mathcal{G}^*_{p\ell}$, rather than on $\mathcal{G}$, $\mathcal{G}^*$. For the remainder of this chapter $(\mathcal{G},\mathcal{G}^*)$ is a matching pair of periodic graphs imbedded in $\mathbb{R}^2$, based on $(\mathcal{M},\mathcal{F})$ and $\mathcal{G}_{p\ell}$, $\mathcal{G}^*_{p\ell}$ their planar modifications (see Ch.2.2, 2.3). $\mathcal{V}_1,\dots,\mathcal{V}_\lambda$ is a periodic partition of the vertices of $\mathcal{G}$ (and hence of the vertices of $\mathcal{G}^*$). For $p \in \mathcal{P}_\lambda$, P_p is the corresponding λ-parameter periodic probability measure (see Ch.3.2). P_p is always extended to a measure on the occupancy configurations of $\mathcal{G}_{p\ell}$ and $\mathcal{G}^*_{p\ell}$ by taking the central vertices of $\mathcal{G}_{p\ell}$ occupied and those of $\mathcal{G}^*_{p\ell}$ vacant, i.e.,

(7.2) $\quad P_p\{\omega(v) = +1\} = 1$ if v is a central vertex of some face $F \in \mathcal{F}$,

(7.3) $\quad P_p\{\omega(v) = -1\} = 1$ if v is a central vertex of some face $F \in \mathcal{F}^*$.

Lastly, we assume that the second coordinate axis is an axis of symmetry for $\mathcal{G}$, $\mathcal{G}^*$ and the partition $\mathcal{V}_1,\dots,\mathcal{V}_\lambda$. As we saw in Comment 2.4 (iii) we may then also take $\mathcal{G}_{p\ell}$ and $\mathcal{G}^*_{p\ell}$ symmetric with respect to the vertical line $x(1)=0$, and by periodicity also with respect to any line $x(1)=k$, $k \in \mathbb{Z}$. We always assume that $\mathcal{G}_{p\ell}$ and $\mathcal{G}^*_{p\ell}$ have been imbedded in this symmetric way.

<u>Lemma 7.1</u>. <u>Assume that</u> $p_0 \in \mathcal{P}_\lambda$ <u>satisfies Condition A in Ch. 3.3 and that</u>

(7.4) $$\overline{0} << p_0 << \overline{1}\ .$$

<u>Then there exists a vector</u> $\overline{\Lambda} = (\Lambda(1),\Lambda(2))$ <u>and a sequence</u> $\{\overline{m}_n = (m_{n1},m_{n2})\}_{n\geq 1}$ <u>of integral vectors such that</u> $m_{ni} \to \infty$ $(n \to \infty$, $i = 1,2)$ <u>and such that for all large</u> n <u>(with</u> δ <u>as in Condition A)</u>

(7.5) $$\sigma(\overline{m}_n;1,p_0) \geq \delta \quad \text{and} \quad \sigma(\overline{m}_n+\overline{\Lambda};2,p_0) \geq \delta\ .$$

<u>Also, for some sequence</u> $\{m^*_n\}_{n\geq 1}$ <u>of integral vectors with</u> $\overline{m}^*_n(i) \to \infty$ $(n \to \infty;\ i = 1,2)$ <u>and a vector</u> $\overline{\Lambda}^*$

(7.6) $\quad \sigma^*(\overline{m}_n^*;1,p_0) \geq \delta \quad$ and $\quad \sigma^*(\overline{m}_n^* + \overline{\Lambda}^*;2,p_0) \geq \delta$

eventually.

Proof: First we show that there exists a constant $C = C(\mathcal{G},p_0)$ such that

(7.7) $\quad \lim_{n\to\infty} \sigma((n,e^{Cn});1,p_0) = 1 \quad$ and $\quad \lim_{n\to\infty} \sigma((e^{Cn},n);2,p_0) = 1.$

We prove the first relation in (7.7). Choose $\Lambda_3 = \Lambda_3(\mathcal{G})$ such that each horizontal (vertical) strip of height Λ_3 (width Λ_3) possesses a horizontal (vertical) crossing on $\mathcal{M}$ (and hence also on $\mathcal{G}$, as well as on $\mathcal{G}^*$). Such a Λ_3 exists by Lemma A.3. We also pick a constant $\Lambda = \Lambda(\mathcal{G},\mathcal{G}^*)$ for which

(7.8) $\quad$ diameter (e) $\leq \Lambda$ for each edge e of $\mathcal{G}$ or $\mathcal{G}^*$ or $\mathcal{G}_{p\ell}$ or $\mathcal{G}_{p\ell}^*$.

We set

(7.9) $\quad \mu = \mu(\mathcal{G},\mathcal{G}^*)$ = number of vertices which belong to $\mathcal{G}$ or $\mathcal{G}^*$ in $[0,1)\times[0,1)$.

Now consider the strips

$$S_k := [0,n]\times[2k\Lambda_3,(2k+1)\Lambda_3] .$$

Each S_k contains a horizontal crossing of at most

$$(n+2\Lambda)\Lambda_3\mu \leq 2\Lambda_3\mu n$$

vertices, for n sufficiently large. Therefore

$$\begin{aligned} &P_{p_0}\{\ \exists \text{ occupied horizontal crossing on } \mathcal{G} \text{ of } S_k\} \\ &\quad \geq [\min_{v\in\mathcal{G}} P_{p_0}\{v \text{ is occupied}\}]^{2\Lambda_3\mu n} \\ &\quad \geq e^{-\gamma n} \end{aligned}$$

for some $\gamma = \gamma(p_0,\mathcal{G}) < \infty$. Finally, since the S_k are disjoint, we have for $C > \gamma$

$\sigma((n,e^{Cn});1,p_0) = P_{p_0}\{\ \exists$ occupied horizontal crossing on $\mathcal{G}$ of $[0,n]\times[0,e^{Cn}]\} \geq P_{p_0}\{\ \exists$ occupied horizontal crossing on $\mathcal{G}$ of some S_k with $1 \leq k \leq \frac{1}{2\Lambda_3}e^{Cn} - 1\}$

$\geq 1-(1-e^{-\gamma n})^{(2\Lambda_3)^{-1}e^{Cn}-1} \to 1 \quad (n\to\infty)$.

This proves (7.7).

Now let C be such that (7.7) holds. For the sake of argument assume that the implication in (3.36) is valid for $j = 2$. Let n be so large that $\sigma((n,\exp Cn);1,p_0) \geq \frac{1}{2}$. We can then choose $m = m(n)$ as the smallest integer $\leq \exp Cn$ for which

(7.10) $$\sigma((n,m);1,p_0) \geq \frac{1}{2} .$$

In exactly the same way as we proved (7.7) we prove

(7.11) $$\lim_{n\to\infty} \sigma^*((n,\tfrac{1}{C}\log n);2,p_0) = \lim_{k\to\infty} P\{\ \exists \text{ vacant vertical crossing of } [0,e^{Ck}]\times[0,k] \text{ on } \mathcal{G}^*\} = 1.$$

But if there exists a vacant vertical crossing r of $[0,n_1]\times[0,n_2]$ for some n_1, n_2, then there cannot be an occupied horizontal crossing r' of $[0,n_1]\times[0,n_2]$. For r and r' would have to intersect, necessarily in a vertex of $\mathcal{M}$ (see Comment 2.2 (vii)) and this vertex in $r \cap r'$ would have to be vacant as well as occupied. This is clearly impossible. Consequently

(7.12) $$\sigma((n_1,n_2);1,p_0) \leq 1-\sigma^*((n_1,n_2);2,p_0).$$

Taking $n_1 = n$, $n_2 = \frac{1}{C}\log n$ we obtian from (7.11) and (7.12) that

$$\sigma((n,\tfrac{1}{C}\log n);1,p_0) \to 0 \quad (n\to\infty).$$

Comparing this with (7.10) and using the monotonicity property of σ (Comment 3.3 (v))we see that

(7.13) $$m(n) \geq \frac{1}{C}\log n \quad \text{eventually.}$$

We now use Prop. 2.2 to prove that the inequality in (7.12) can almost be reversed. More precisely, let

$$\Lambda_4 = \lceil \Lambda_3 + \Lambda \rceil + 1$$

and assume $n_1, n_2 > 2\Lambda_4$. Then for any $p \in \mathcal{P}_\lambda$ one has

$$\sigma((n_1,n_2);1,p) + \sigma^*((n_1 + 2\Lambda_4, n_2 - 2\Lambda_4);2,p) \geq 1 \tag{7.14}$$

as well as

$$\sigma((n_1 + 2\Lambda_4, n_2 - 2\Lambda_4);2,p) + \sigma^*((n_1,n_2);1,p) \geq 1 \ . \tag{7.15}$$

We only prove (7.14). For (7.15) we only need to interchange the roles of $\mathcal{G}$ and $\mathcal{G}^*$ and of occupied and vacant. To prove (7.14) we take a self-avoiding vertical crossing r_1 on $\mathcal{M}$ of $[-\Lambda_4, -\Lambda-1] \times [0,n_2]$. Such a vertical crossing exists by Lemma A.3 and our choice of Λ_3. Similarly we take a selfavoiding vertical crossing r_3 on $\mathcal{M}$ of $[n_1 + \Lambda + 1, n_1 + \Lambda_4] \times [0,n_2]$ and horizontal crossings r_2 and r_4 of $[-\Lambda_4, n_1 + \Lambda_4] \times [\Lambda + 1, \Lambda_4]$ and $[-\Lambda_4, n_1 + \Lambda_4] \times [n_2 - \Lambda_4, n_2 - \Lambda - 1]$, respectively (see Fig. 7.1). Once again we remind the reader of the observa-

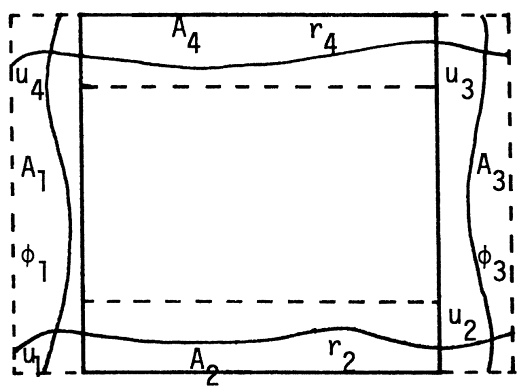

Figure 7.1 The solid rectangle is $[0,n_1] \times [0,n_2]$; the outer dashed rectangle is $[-\Lambda_4, n_1+\Lambda_4] \times [0,n_2]$; the inner dashed rectangle is $[0,n_1] \times [\Lambda_4, n_2-\Lambda_4]$.

tion at the beginning of Sect. 2.3: Since $\mathcal{M}$ is planar, and r_i is self-avoiding, the curve made up from the edges of r_i is a simple curve. r_1 contains therefore a simple curve, ϕ_1 say, inside the rectangle $[-\Lambda_4,-\Lambda-1]\times[0,n_2]$, and connecting the top and bottom edge of this rectangle. Similarly r_3 contains a simple curve ϕ_3 inside $[n_1+\Lambda+1,n_1+\Lambda_4]$, and connecting the top and bottom edge of this rectangle. Now, both r_2 and r_4 must intersect ϕ_1 as well as ϕ_3. Starting from the left endpoint of r_2 (r_4) let u_1 (u_3) be the last intersection of r_2 (r_4) with ϕ_1 and u_2 (u_4) the first intersection of r_2 (r_4) with ϕ_3. Denote the closed segment of ϕ_1 from u_4 to u_1 by A_1, the closed segment of r_2 from u_1 to u_2 by A_2, the closed segment of ϕ_3 from u_2 to u_3 by A_3, and the closed segment of r_4 from u_3 to u_4 by A_4. By construction A_1 is strictly to the left of the vertical line $x(1) = -\Lambda$ and A_3 to the right of $x(1) = n_1+\Lambda$. Similarly A_2 is below the horizontal line $x(2) = \Lambda_4$ and A_4 above $x(2) = \Lambda_4$. In addition, the A_i are simple curves. It is not hard to see from this that the composition of A_1,A_2,A_3 and A_4 is a Jordan curve, J say. Any path on $\mathcal{G}$ inside $\bar{J} = J\cup \mathrm{int}(J)$ from a vertex on A_1 to a vertex on A_3 has to contain a horizontal crossing of $[0,n_1]\times[0,n_2]$, since A_2 lies strictly above the horizontal line $x(2) = 0$ and r_4 strictly below $x(2) = n_2$. If all vertices on r in $\mathrm{int}(J)$ are occupied then r contains an occupied horizontal crossing of $[0,n_1]\times[0,n_2]$. Thus, if there does not exist an occupied horizontal crossing of $[0,n_1]\times[0,n_2]$, then no path r of the above nature can exist. By Prop. 2.2 this implies the existence of a vacant path r^* on $\mathcal{G}^*$ and inside $\bar{J}\setminus A_1\cup A_3$ with initial point on $\mathring{A}_2$ and final point on $\mathring{A}_4$. Finally, any such path r^* contains a vacant vertical crossing of $[-\Lambda_4,n_1+\Lambda_4]\times[\Lambda_4,n_2-\Lambda_4]$. For the crossing probabilities this implies

$$
\begin{aligned}
1-\sigma((n_1,n_2);1,p) &= P_p\{\text{there does not exist an occupied}\\
&\quad\text{horizontal crossing of } [0,n_1]\times[0,n_2] \text{ on } \mathcal{G}\}\\
&\le P_p\{\,\exists \text{ vacant vertical crossing of } [-\Lambda_4,n_1+\Lambda_4]\times[\Lambda_4,n_2-\Lambda_4]\\
&\quad\text{on } \mathcal{G}^*\} = \sigma^*((n_1+2\Lambda_4,n_2-2\Lambda_4);2,p)
\end{aligned}
$$

(use periodicity for the last equality). This proves (7.14).

It is easy now to complete the proof. For $n_2 > 2\Lambda_4$ and $m(n)-1 \ge \frac{1}{C}\log n - 1 > 2\Lambda_4$ we have (7.10) as well as

$$\sigma((n,m(n)-1);1,p_0) < \frac{1}{2}\,, \tag{7.16}$$

by virtue of the definition of $m(n)$. Then, by (7.14)

$$\sigma^*((n+2\Lambda_4, m(n)-2\Lambda_4-1);2,p_0) \geq \frac{1}{2} ,$$

and finally, by (3.36) in Condition A

$$\sigma(n+2\Lambda_4-\rho_1^*, m(n)-2\Lambda_4-1-\rho_2^*);2,p_0) \geq \delta .$$

This, together with (7.10) implies (7.5) with $\bar{m}_n = (n,m(n))$ and $\bar{\Lambda} = (2\Lambda_4-\rho_1^*,-2\Lambda_4-1-\rho_2^*)$. When (3.36) holds for $j = 1$, then one merely has to interchange the roles of the first and second coordinate. To prove (7.6) one interchanges the roles of $\mathcal{G}$ and $\mathcal{G}^*$ in the above proof. □

Lemma 7.2. *Assume that* p_0 *satisfies (7.4) and Condition A or B in Ch. 3.3. Then there exist sequences of vectors* $\{\bar{N}_\ell = (N_{\ell 1}, N_{\ell 2})\}_{\ell \geq 1}$, $\{\bar{M}_\ell = (M_{\ell 1}, M_{\ell 2})\}_{\ell \geq 1}$, *and for each integer* k *a number* $\delta_k > 0$ *such that*

(7.17) $$N_{\ell i} \to \infty \quad , M_{\ell i} \to \infty , \ i = 1,2, \text{ as } \ell \to \infty ,$$

(7.18) $$\sigma((kN_{\ell 1}, N_{\ell 2});1,p_0,\mathcal{G}_{p\ell}) \geq \delta_k > 0,$$

$$\sigma((N_{\ell 1}, kN_{\ell 2});2,p_0,\mathcal{G}_{p\ell}) \geq \delta_k > 0,$$

and

(7.19) $$\sigma^*((kM_{\ell 1}, M_{\ell 2});1,p_0,\mathcal{G}_{p\ell}) \geq \delta_k > 0,$$

$$\sigma^*((M_{\ell 1}, kM_{\ell 2});2,p_0,\mathcal{G}_{p\ell}) \geq \delta_k > 0.$$

Moreover

(7.20) $$P_{p_0}\{ \exists \text{ occupied circuit on } \mathcal{G}_{p\ell} \text{ surrounding } 0$$
$$\text{and inside the annulus } [-2N_{\ell 1}, 2N_{\ell 1}] \times [-2N_{\ell 2}, 2N_{\ell 2}] \setminus$$
$$(-N_{\ell 1}, N_{\ell 1}) \times (-N_{\ell 2}, N_{\ell 2})\} \geq \delta_4^4,$$

and

(7.21) $P_{p_0}\{\exists$ vacant circuit on $\mathcal{G}^*_{p\ell}$ surrounding 0

and inside the annulus $[-2M_{\ell 1},2M_{\ell 1}]\times[-2M_{\ell 2},2M_{\ell 2}]$

$(-M_{\ell 1},M_{\ell 1})\times(-M_{\ell 2},M_{\ell 2})\}\geq \delta_4^4$.

Proof: Again we restrict ourselves to proving (7.18) and (7.20). First assume Condition A is satisfied. By the last lemma we then have (7.5). With Λ as in (7.8) this implies, by virtue of Lemma 2.1b,

(7.22) $$\sigma((m_{n1}-2\Lambda,m_{n2}+2\Lambda);1,p_0,\mathcal{G}_{p\ell})\geq\delta \quad\text{and}$$

$$\sigma((m_{n1}+\Lambda(1)+2\Lambda,m_{n2}+\Lambda(2)-2\Lambda);2,p_0,\mathcal{G}_{p\ell})\geq\delta$$

(Basically an occupied horizontal crossing on $\mathcal{G}$ is turned into a horizontal crossing on $\mathcal{G}_{p\ell}$ by inserting central vertices of $\mathcal{G}$. These central vertices are occupied with probability one by virtue of (7.2). The resulting horizontal crossing on $\mathcal{G}_{p\ell}$ is therefore again occupied. The same argument applies to vertical crossings.) We can now apply the RSW theorem (Theorem 6.1) with $\pi = 2$, $\overline{n} = (m_{n1}-2\Lambda,m_{n2}+2\Lambda)$ and $\overline{m} = (m_{n1}+\Lambda(1)+2\Lambda,m_{n2}+\Lambda(2)-2\Lambda)$. (7.18) is then immediate from (6.9) and (6.10) and the monotonicity properties of σ (see Comment 3.3 (v)) with $\overline{N}_\ell = 5\overline{m}_\ell$ and $\delta_k = f(\delta,\delta,2,10k)$. (7.20) follows from (7.18), because one can construct a circuit from two horizontal and two vertical crossings of suitable rectangles, as explained in the proof of Cor. 6.1 at the end of Ch. 6. This proves the lemma under Condition A.

Now assume that Condition B holds. Instead of (7.22) we now obtain from (3.38) and Lemma 2.1b

$$\sigma((n_{\ell 1}-2\Lambda,n_{\ell 2}+2\Lambda;1,p_0,\mathcal{G}_{p\ell})\geq\delta \quad\text{and}$$

$$\sigma(a_1n_{\ell 1}+2\Lambda,a_2n_{\ell 2}-2\Lambda);2,p_0,\mathcal{G}_{p\ell})\geq\delta\ .$$

The Lemma again follows from the RSW theorem (this time with $\pi = 2\lceil\max\{a_1,a_2,a_1^{-1},a_2^{-1}\}\rceil$). □

Lemma 7.3. Assume $p\in\mathcal{P}_\lambda$ satisfies

$$\overline{0} \ll p \ll \overline{1}\ .$$

If for some vertex v *of* $\mathcal{G}$

(7.23) $$E_p\{\#W(v)\} < \infty ,$$

then for every vertex w *of* $\mathcal{G}^*$

(7.24) $$P_p\{\#W^*(w) = \infty\} > 0 .$$

Also

(7.25) $$E_p\{\#W^*(w)\} < \infty$$

implies

(7.26) $$P_p\{\#W(v) = \infty\} > 0.$$

Proof: We shall show that (7.23) implies

(7.27) $$P_p\{\exists \text{ an infinite vacant component on } \mathcal{G}^* \text{ inside the first quadrant}\} = 1.$$

This will imply that

$$P_p\{\#W^*(w_1) = \infty\} > 0$$

for some $w_1 \in \mathcal{G}^*$. (7.24) follows then for any w by (4.8) (with $n = \infty$). A similar proof will work for obtaining (7.26) from (7.25).

To prove (7.27) we first use (7.14) with

$$n_1 = 2^k - 2\Lambda_4 \quad , \quad n_2 = 2^{k+1} + 2\Lambda_4 .$$

We obtain

(7.28) $$1-\sigma^*((2^k,2^{k+1});2,p,\mathcal{G}) \le \sigma((2^k-2\Lambda_4,2^{k+1}+2\Lambda_4);1,p,\mathcal{G}).$$

Next we claim that (7.23) implies

(7.29) $$\sum_{k=1}^{\infty} \sigma((2^k+\Lambda_5,2^{k+1}+\Lambda_6);1,p) < \infty$$

for any Λ_5, Λ_6. This was essentially already proved in Lemma 5.4. Exactly as at the end of the proof of that lemma (cf. (5.55)) one shows that

$$\sigma((2^k+\Lambda_5, 2^{k+1}+\Lambda_6);1,p)$$

$$\leq \mu(2\Lambda+1)(2^{k+1}+\Lambda_6+2\Lambda+1) \sup_{v_0(1)\leq\Lambda} P_p\{W(v_0) \text{ contains a vertex}$$

$$\text{to the right of } x(1) = 2^k+\Lambda_5-\Lambda\}$$

$$\leq \mu(2\Lambda+1)(2^{k+1}+\Lambda_6+2\Lambda+1)$$

$$\cdot \sum_{v_0\varepsilon[0,1)\times[0,1)} P_p\{\#W(v_0) \geq \frac{1}{\Lambda}(2^k+\Lambda_5-2\Lambda)\}$$

(use periodicity for the last inequality). But (7.23) for some v implies

$$E_p\{\#W(v_0)\} < \infty$$

for all v_0 (see the Application 4.1 of the FKG inequality) and consequently

$$\sum_{v_0\varepsilon[0,1)\times[0,1)} \sum_k 2^{k+2} P_p\{\#W(v_0) \geq \frac{1}{\Lambda}(2^k+\Lambda_5-2\Lambda)\} < \infty .$$

(7.29) follows.

From the Borel-Cantelli lemma (Renyi (1970) Lemma VII.5.A), (7.28) (7.29) it now follows that

(7.30) $\quad P_p\{\exists$ vacant vertical crossing on $\mathcal{G}^*$ of $[0,2^k]\times[0,2^{k+1}]$ for all large $k\} = 1$.

In the same way one sees

(7.31) $\quad P_p\{\exists$ vacant horizontal crossing on $\mathcal{G}^*$ of $[0,2^{k+1}]\times[0,2^k]$ for all large $k\} = 1$.

Since a horizontal crossing of $[0,2^{k+1}]\times[0,2^k]$ or of $[0,2^{k+3}]\times[0,2^{k+2}]$ must intersect a vertical crossing of $[0,2^{k+1}]\times[0,2^{k+2}]$, one easily sees that if for all large k there exists a vacant horizontal crossing on $\mathcal{G}^*$ of $[0,2^{2k+1}]\times[0,2^{2k}]$ and a vacant vertical crossing on $\mathcal{G}^*$ of $[0,2^{2k+1}]\times[0,2^{2k+2}]$, then these crossings combine to an infinite vacant cluster on $\mathcal{G}^*$ in the first quadrant. Thus (7.27) follows from (7.30) and (7.31). $\square$

Remark.

(i) The above proof is taken from Smythe and Wierman (1978), Theorem 3.2. Together with parts (ii) and (iii) of Theorem 3.1 it will show that one actually has infinite occupied clusters on $\mathcal{G}$ in the first quadrant under $P_{p''}$ with $p'' >> p_0$, and infinite vacant clusters on $\mathcal{G}^*$ in the first quadrant under $P_{p'}$ with $p' << p_0$.

Proof of Theorem 3.1 (i): With $\overline{M}_\ell$ as in Lemma 7.2 consider the annuli

$$U_\ell := [-2M_{\ell 1}, 2M_{\ell 1}] \times [-2M_{\ell 2}, 2M_{\ell 2}] \setminus (-M_{\ell 1}, M_{\ell 1}) \times (-M_{\ell 2}, M_{\ell 2}). \tag{7.32}$$

Without loss of generality we may assume these annuli disjoint. In this case the occurrences of occupied circuits in different U_ℓ are independent of each other. Therefore, by (7.21) and the Borel-Cantelli Lemma (Renyi (1970), Lemma VII.5.B), with P_{p_0}-probability one infinitely many U_ℓ contain a vacant circuit on $\mathcal{G}^*_{p\ell}$ surrounding the origin. If $M_{\ell 1} > \Lambda$, $M_{\ell 2} > \Lambda$, and U_ℓ contains a vacant circuit on $\mathcal{G}^*_{p\ell}$ surrounding 0, then by Lemma 2.1a

$$\begin{aligned}(7.33)\qquad & [-2M_{\ell 1}-\Lambda, 2M_{\ell 1}+\Lambda] \times [-2M_{\ell 2}-\Lambda, 2M_{\ell 2}+\Lambda] \setminus \\ & \qquad (-M_{\ell 1}+\Lambda, M_{\ell 1}-\Lambda) \times (-M_{\ell 2}+\Lambda, M_{\ell 2}-\Lambda)\end{aligned}$$

contains a vacant circuit on $\mathcal{G}^*$ surrounding 0. In fact this latter circuit must surround all of $(-M_{\ell 1}+\Lambda, M_{\ell 1}-\Lambda) \times (-M_{\ell 2}+\Lambda, M_{\ell 2}-\Lambda)$. Hence for all N

$$\begin{aligned}(7.34)\qquad & P_{p_0}\{\ \exists \text{ a vacant circuit on } \mathcal{G}^* \text{ surrounding} \\ & \qquad [-N,+N] \times [-N,+N]\} = 1.\end{aligned}$$

In the same way we obtain arbitrarily large occupied circuits on $\mathcal{G}$, and (3.45) follows. (3.43) is immediate from this, because if $v \in [-N,N] \times [-N,N]$ and $[-N,N] \times [-N,N]$ is surrounded by a vacant circuit J on $\mathcal{G}^*$, then $W(v)$ is contained in $\text{int}(J)$, and hence $\#W(v) < \infty$. In fact any path on $\mathcal{G}$ from v to the complement of $\text{int}(J)$ would have to be intersect J, necessarily in a vertex of $\mathcal{G}$ and $\mathcal{G}^*$ (see Comment 2.2 (vii)) and this vertex would have to be vacant. Thus no vertex in $\text{ext}(J)$ or on J can belong to $W(v)$. Similarly

$\#W^*(v)$ is shown to be finite with probability one.

Finally, (3.44) is a consequence of (3.43) and Lemma 7.3. $\square$

Lemma 7.4. Assume p_0 satisfies (7.4) and Condition A or B. Let δ_k, $\{\bar{N}_\ell\}$ and $\{\bar{M}_\ell\}$ be as in Lemma 7.2 so that (7.17)-(7.21) hold. Then for

$$p' << p_0 << p''$$

$$\lim_{\ell\to\infty} \tau(2\bar{M}_\ell;i,p',\mathcal{G}) = 0,\ i = 1,2, \tag{7.35}$$

and

$$\lim_{\ell\to\infty} \tau^*(2\bar{N}_\ell;i,p'',\mathcal{G}) = 0,\ i = 1,2, \tag{7.36}$$

(see (5.5) and (5.6) for τ and τ^*).

Proof: We shall only prove (with Λ as in (7.8))

$$P_{p'}\{\ \exists \text{ occupied horizontal crossing on } \mathcal{G}_{p\ell} \text{ of } [\Lambda,2M_{\ell 1}-\Lambda]\times[-\Lambda,6M_{\ell 2}+\Lambda]\} \to 0 \quad (\ell\to\infty). \tag{7.37}$$

By Lemma 2.1b $\tau(2\bar{M}_\ell;1,p',\mathcal{G}) = P_{p'}\{\ \exists$ occupied horizontal crossing on $\mathcal{G}$ of $[0,2M_{\ell 1}]\times[0,6M_{\ell 2}]\}$ is bounded by the left hand side of (7.37). Therefore (7.37) will imply (7.35) for $i = 1$. The proofs of (7.35) for $i = 2$ and of (7.36) are similar.

To prove (7.37) take Λ_3 and Λ_4 as in the proof of Lemma 7.1. Suppress the subscript ℓ for the time being. Very much as in the proof of Lemma 7.1 take self-avoiding vertical crossings r_1 and r_3 on $\mathcal{M}$ of the strips $[\Lambda,\Lambda_4-1]\times[-\Lambda_4,6M_2+\Lambda_4]$ and $[2M_1-\Lambda_4+1,2M_1-\Lambda]\times[-\Lambda_4,6M_2+\Lambda_4]$, respectively. Also we take horizontal crossings r_2 and r_4 on $\mathcal{M}$ of $[0,2M_1]\times[-\Lambda_4,-\Lambda-1]$ and $[0,2M_1]\times[6M_2+\Lambda+1,6M_2+\Lambda_4]$, respectively (see Fig. 7.2). Again ϕ_1 (ϕ_3) is a simple curve in $[\Lambda,\Lambda_4-1]\times[-\Lambda_4,6M_2+\Lambda_4]$ ($2M_1-\Lambda_4+1,2M_1-\Lambda]\times[-\Lambda_4,6M_2+\Lambda_4]$) connecting the top and bottom edge of this rectangle. Starting from the left endpoint of r_2 (r_4) let u_1 (u_4) be the last intersection of r_2 (r_4) with ϕ_1; and u_2 (u_3) the first intersection of r_2 (r_4) with ϕ_3. We denote the closed segment of ϕ_1 from u_4 to u_1 by B_1, the closed segment of ϕ_3 from u_3 to u_2 by B_2, the closed segment of r_2 from u_1 to u_2 by A and the closed segment of r_4 from u_4 to u_3

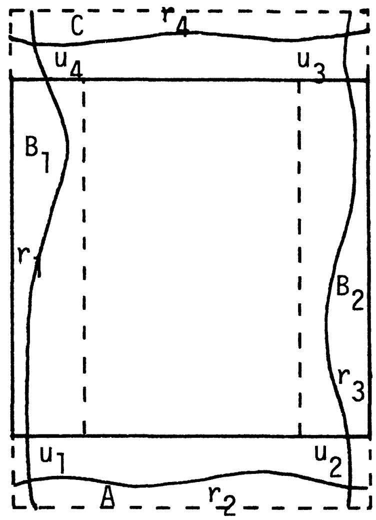

Figure 7.2 The solid rectangle is $[0,2M_1] \times [0,6M_2]$; the outer dashed rectangle is $[0,2M_1] \times [-\Lambda_4, 6M_2+\Lambda_4]$; the inner dashed rectangle is $[\Lambda_4-1,\ 2M_1-\Lambda_4+1] \times [0,6M_2]$.

by C. As in Lemma 7.1, as soon as $M_1 > 2\Lambda_4$ the composition of B_1, A, B_2 and C is a Jordan curve, which we again denote by J. Also

$$[\Lambda_4, 2M_1-\Lambda_4] \times [-\Lambda, 6M_2+\Lambda] \subset \mathrm{int}(J). \tag{7.38}$$

Any horizontal crossing $\tilde{r}$ on $\mathcal{G}_{p\ell}$ of $[\Lambda, 2M_1-\Lambda] \times [-\Lambda, 6M_2+\Lambda]$ contains some point $v = (v(1), v(2))$ in the interior of the rectangle in the left hand side of (7.38), and hence in int(J). Let ψ be the segment of $\tilde{r}$ from its last intersection with the vertical line $x(1) = \Lambda$ to the first intersection with the line $x(1) = 2M_1-\Lambda$. Then ψ starts on $x(1) = \Lambda$ to the "left of B_1" and when it reaches v it lies to the "right of B_1". Since ψ minus its endpoints lies between the horizontal lines $x(2) = -\Lambda$ and $x(2) = 6M_2+\Lambda$, ψ must intersect B_1 between its initial point and v. (Note that B_1 runs from u_1 below $x(2) = -\Lambda$ to u_4 above $x(2) = 6M_2+\Lambda$.) Similarly the piece of ψ between v and its final point must intersect B_2. The piece of $\tilde{r}$ between the last intersection before v with B_1 and the first intersection after v with B_2 is therefore a path $r = (v_0, e_1, \ldots, e_\nu, v_\nu)$ on $\mathcal{G}_{p\ell}$

with the following properties:

(7.39) $$(v_1,e_2,\dots,e_{\nu-1},v_{\nu-1}) \subset \text{int}(J),$$

(7.40) e_1 intersects J only in the point $v_0 \in B_1$,

(7.41) e_ν intersects J only in the point $v_\nu \in B_2$.

Thus if we introduce the event

$$E = \{\exists \text{ occupied path } r = (v_0,e_1,\dots,e_\nu,v_\nu) \text{ on } \mathcal{G}_{p\ell} \text{ with the properties (7.39)-(7.41)}\},$$

then the left hand side of (7.37) is bounded by $P_{p'}\{E\}$.

We now introduce

$$N_0 = N_0(E,\omega) = \# \text{ of pivotal sites for } (E,\omega) \text{ which are vertices of } \mathcal{M}$$

and

$$p(t) = tp_0+(1-t)p',\ 0 \le t \le 1.$$

Since E is an increasing event we can apply Russo's formula (4.22) to obtain

(7.42) $$\frac{d}{dt} P_{p(t)}\{E\} \ge \inf_{v\in\mathcal{M}}\{P_{p_0}\{v \text{ is occupied}\} - P_{p'}\{v \text{ is occupied}\}\} \cdot E_{p(t)}\{N_0\} .$$

Since $p' << p_0$ and P_{p_0} and $P_{p'}$ are periodic, the constant

(7.43) $$\alpha := \inf_{v\in\mathcal{M}}\{P_{p_0}\{v \text{ is occupied}\} - P_{p'}\{v \text{ is occupied}\}\}$$

is strictly positive. We now write (7.42) as

$$(P_{p(t)}\{E\})^{-1} \frac{d}{dt} P_{p(t)}\{E\} \ge \alpha\, E_{p(t)}\{N_0|E\} \quad ,$$

and integrate over t from 0 to 1. We obtain the inequality

(7.44) $$\begin{aligned} P_{p'}\{E\} &\le P_{p_0}\{E\}\exp-\alpha \int_0^1 E_{p(t)}\{N_0|E\}dt \\ &\le \exp-\alpha \int_0^1 E_{p(t)}\{N_0|E\}dt. \end{aligned}$$

It therefore suffices to prove

(7.45) $\quad E_{p(t)}\{N_0|E\} \to \infty$ uniformly in t as $\overline{M} \to \infty$

through the sequence $\overline{M}_\ell$,

for this will imply that the left hand sides of (7.44) and (7.37) tend to 0 as $\ell \to \infty$.

We follow the lines of Kesten (1980a) to prove (7.45). (7.39)-(7.41) are just (2.23)-(2.25) in the present set up. E is the event that there exists at least one occupied path r with these properties. Proposition 2.3 (with $S = \mathbb{R}^2$) states that if E occurs, then there exists a unique minimal occupied path r satisfying (7.39)-(7.41), i.e., a path r for which the component of $\mathrm{int}(J)\setminus r$ with A in its boundary is as small as possible (see Def. 2.11 and 2.12). As in Prop. 2.3 we denote the minimal occupied path satisfying (7.39)-(7.41) by R. In Kesten (1980a) the suggestive term "lowest (occupied) left-right crossing" was used for R because there we could take for J the perimeter of a rectangle. The above comments imply

(7.46) $$E = \cup\{R = r\} ,$$

where the union is over all paths $r = (v_0,e_1,\dots,e_\nu,v_\nu)$ on $\mathcal{G}_{p\ell}$ which satisfy (7.39)-(7.41). Next we use Ex. 4.2 (iii) to find pivotal sites for E. We restrict ourselves to pivotal sites which are vertices of $\mathcal{M}$, because these are the only ones counted in N_0. A vertex v of $\mathcal{M}$ on $R \cap \mathrm{int}(J)$ which has a "vacant connection on $\mathcal{G}_{p\ell}$ to $\mathring{C}$ above r" is pivotal for E. To be more specific, for any path r on $\mathcal{G}_{p\ell}$ which satisfies (7.39)-(7.41) and vertex v of $\mathcal{M}$ on $r \cap \mathrm{int}(J)$ we shall say that v has a <u>vacant connection to $\mathring{C}$ above r</u> if there exists a vacant path $s^* = (v_0^*,e_1^*,\dots,e_\rho^*,v_\rho^*)$ on $\mathcal{G}_{p\ell}^*$ which satisfies (7.47)-(7.49) below.

(7.47) there exists an edge e^* of $\mathcal{G}_{p\ell}^*$ between v and v_0^* such that $\mathring{e}^* \subset J^+(r)$

(7.48) $$v_\rho^* \in \mathring{C}$$

(7.49) $$(v_0^*,e_1^*,\dots,v_{\rho-1}^*,e_\rho^*\setminus\{v_\rho^*\}) = s^*\setminus\{v_\rho^*\} \subset J^+(r)$$

(see Def. 2.11 for $J^+(r)$). Note that $\rho = 0$ is permitted in (7.47)-(7.49). In this case s^* reduces to the single point $\{v_0^*\}$ and (7.49) becomes vacuous. (7.47)-(7.49) are merely the conditions (4.13)-(4.15) with R replaced by r, except that in (7.47) we require e^* to belong to $\mathcal{G}^*_{p\ell}$ rather than to $\mathcal{M}_{p\ell}$, as in (4.13). The latter change does not constitute a real change from (4.13) since we assumed here that v is a vertex of $\mathcal{M}$, i.e., of $\mathcal{G}^*$ and $\mathcal{G}^*_{p\ell}$. If such a v is connected by an edge e^* to the vertex v_0^* of $\mathcal{G}^*_{p\ell}$ then e^* automatically belongs to $\mathcal{G}^*_{p\ell}$. (The vacant connections defined here correspond to the weak cut sets with respect to r of Kesten (1980a).) Ex. 4.2 (iii) now shows that any vertex v of $\mathcal{M}$ on $R \cap \text{int}(J)$ with a vacant connection to $\mathring{C}$ above R is pivotal for E. Thus

$$N_0 \geq \#\ \text{of vertices } v \text{ of } \mathcal{M} \text{ on } R \cap \text{int}(J) \text{ which have a vacant connection to } \mathring{C} \text{ above } R.$$

For the remainder of the proof we use the abbreviation

$$(7.50)\qquad N(r) = N(r,\omega) = \#\text{ of vertices of } \mathcal{M} \text{ on } r \cap \text{int}(J) \text{ which have a vacant connection to } \mathring{C} \text{ above } r.$$

Then $N_0(E,\omega) \geq N(R,\omega)$, and by virtue of (7.46)

$$(7.51)\qquad E_{p(t)}\{N_0|E\} \geq \sum_r P_{p(t)}\{R = r|E\}E_{p(t)}\{N(r)|R = r\}\ .$$

By Prop. 2.3 the event $\{R = r\}$ depends only on the occupancies of vertices in $\overline{J}^-(r)$ (note that B_i is made up from edges of $\mathcal{M}$ and a fortiori of $\mathcal{M}_{p\ell}$ here). Moreover, for any v on r the existence of a vacant connection from v to $\mathring{C}$ above r depends only on the occupancies of the vertices in $J^+(r) \cup \mathring{C}$, (see (7.48), (7.49)) which is disjoint from $\overline{J}^-(r)$. This allows us to drop the condition $R = r$ in the last expectation in the right hand side of (7.51). More precisely

$$(7.52)\qquad E_{p(t)}\{N(r)|R = r\} = \sum_{\substack{v \in r \cap \text{int}(J) \\ v \text{ a vertex of } \mathcal{M}}} P_{p(t)}\{v \text{ has a vacant connection to } \mathring{C} \text{ above } r|R = r\}$$

$$= \sum_{\substack{v \in r \cap \text{int}(J) \\ v \text{ a vertex of } \mathcal{M}}} P_{p(t)}\{v \text{ has a vacant connection to } \mathring{C} \text{ above } r\}\ .$$

Clearly

$$\{\ \exists \text{ vacant connection from } v \text{ to } \overset{\circ}{C} \text{ above } r\}$$

is a decreasing event. Lemma 4.1 shows that the $P_{p(t)}$-measure of any decreasing event is decreasing in t. It follows that the last member of (7.52) is also decreasing in t. Thus for $0 \le t \le 1$

$$E_{p(t)}\{N(r)|R = r\} \ge E_{p_0}\{N(r)\}. \tag{7.53}$$

Substituting this estimate into (7.51) and using

$$\sum_r P_{p(t)}\{R = r|E\} = 1 \qquad \text{(see (7.46))}$$

we obtain

$$E_{p(t)}\{N_0|E\} \ge \min_r E_{p_0}\{N(r)\}, \tag{7.54}$$

where the minimum is over all paths r on $\mathcal{G}_{p\ell}$ satisfying (7.39)-(7.41). Fix such an r. Let its initial point on B_1 be v_0 and its final point on B_2 be v_ν and consider the following curves on $\mathcal{M}$ (and hence on $\mathcal{M}_{p\ell}$): $\tilde{B}_1 := C$, $\tilde{A} :=$ closed segment of B_1 between u_4 and v_0, $\tilde{B}_2 := r$, $\tilde{C} :=$ closed segment of B_2 between v_ν and u_3 (see Fig. 7.3). Together these curves form a Jordan curve, which we

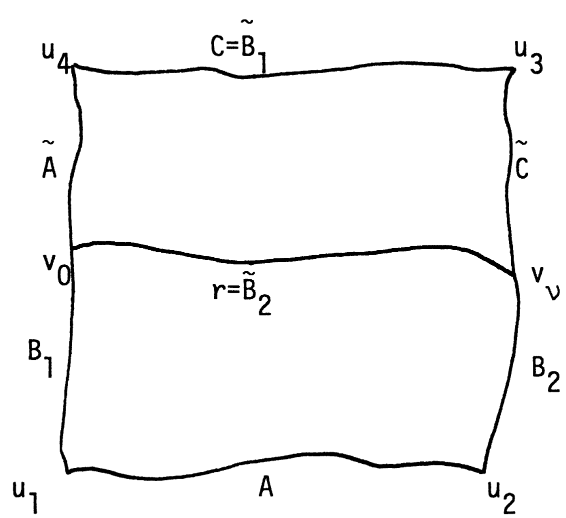

Figure 7.3

denote by $\tilde{J}$, and which is precisely $Fr(J^+(r))$. Now let $v \in r \cap int(J) \in \overset{\circ}{B}_2$ be a vertex of $\mathcal{M}$ and consider vacant paths $\tilde{s}^* = (v,e^*,v_0^*,e_1^*,\dots,e_\rho^*,v_\rho^*)$ on $\mathcal{G}^*_{p\ell}$ which satisfy

(7.55) $\tilde{s}^*$ minus its endpoints $\subset int(\tilde{J})$

and

(7.56) one endpoint of $\tilde{s}^*$ lies on $\overset{\circ}{\tilde{B}}_1$ and one endpoint of $\tilde{s}^*$ lies on $\overset{\circ}{\tilde{B}}_2$.

(Paths of this type correspond to the strong cut sets of Kesten (1980a).) Clearly if there exists such a vacant path, then its endpoint on $r = \tilde{B}_2$ has a vacant connection to $\overset{\circ}{\tilde{C}}$ above r. We again want to apply Prop. 2.3, this time with $\mathcal{G}_{p\ell}$ replaced by $\mathcal{G}^*_{p\ell}$ and J,A,B_1,C,B_2 replaced by $\tilde{J},\tilde{A},\tilde{B}_1,\tilde{C}$ and $\tilde{B}_2$, respectively. Analogously to Def. 2.11 we set for any path $\tilde{s}^*$ satisfying (7.55) and (7.56)

$$\tilde{J}^-(\tilde{s}^*)(\tilde{J}^+(\tilde{s}^*)) = \text{component of } int(\tilde{J})\setminus\tilde{s}^* \text{ with } \tilde{A}(\tilde{C}) \text{ in its boundary.}$$

Prop. 2.3 will give us an $\tilde{s}^*$ with minimal $\tilde{J}^-(\tilde{s}^*)$. For later estimates (see (7.64), (7.65) and their use at the end of the proof) it is important that this minimal path is not too far to the right. We shall therefore consider only vacant connections in the vertical strip

$$\chi := [\Lambda_4+1, M_1+\Lambda_4+1] \times \mathbb{R} \ .$$

We remind the reader that for any subset S of $\mathbb{R}^2$ $\tilde{s}^* \subset S$ means that all edges (and hence all vertices) of $\tilde{s}^*$ lie in S. We need to consider the event

$$F(r) := \{ \exists \text{ vacant path } \tilde{s}^* \text{ on } \mathcal{G}^*_{p\ell} \text{ which satisfies } (7.55), (7.56) \text{ and } \tilde{s}^* \subset \chi\} \ .$$

From Propositon 2.3, applied to $\mathcal{G}^*_{p\ell}$ rather than $\mathcal{G}_{p\ell}$ and with S taken as the strip χ above we conclude that when $F(r)$ occurs there is a unique $\tilde{s}^*$ with minimal $\tilde{J}^-(\tilde{s}^*)$ among the vacant paths on $\tilde{\mathcal{G}}^*_{p\ell}$ which satisfy (7.55), (7.56) and are contained in χ. We denote this path by S^*. (Intuitively, S^* is the "left-most vacant vertical crosscut" of $int(\tilde{J})$ in χ.) As we saw above S^* provides us with a vacant connection from its endpoint on $\mathring{\tilde{B}}_2 = r \setminus \{w_1,w_2\}$ to $\mathring{C}$ above r. Before we estimate the number of sites which have such a connection we estimate the probability of having at least one such site by estimating the probability that S^* exists, which equals $P_{p_0}\{F(r)\}$. Let $(u_0^*,f_1^*,\dots,f_\tau^*,u_t^*)$ be a vacant path on $\mathcal{G}_{p\ell}^*$ with

(7.57) $$u_0^* \in A,$$

(7.58) $$u_\tau^* \in \tilde{B}_1 = C,$$

(7.59) $$(f_1^* \setminus \{u_0^*\},u_1^*,\dots,u_{\tau-1}^*,f_\tau^* \setminus \{u^*\}) \subset \text{int}(J) \cap \chi .$$

Since χ is closed any such path lies entirely in χ; and since B_1 and B_2 separate A and C on J the path must intersect r. The last intersection of this path on $\mathcal{G}_{p\ell}^*$ with r - which is a path on $\mathcal{G}_{p\ell}$ - is necessarily a vertex of $\mathcal{G}_{p\ell}^*$ and of $\mathcal{G}_{p\ell}$ (see Comment 2.3(v)). Thus this last intersection of $(u_0^*,f_1^*,\dots,f_\tau^*,u_\tau^*)$ with r is one of the u_i^*, say u_σ^*, and also equals one of the v_j, but not v_0 or v_ν since the latter two lie on $B_1 \cup B_2$, hence outside χ. One now easily sees that if one takes $\tilde{s}^* = (u_\sigma^*,f_{\sigma+1}^*,\dots,f_\tau^*,u_\tau^*)$ then the requirements (7.55) and (7.56) are fulfilled. Of course this $\tilde{s}^*$ is also contained in χ to that

(7.60) $$P_{p_0}\{F(r)\} \geq P_{p_0}\{\ \exists \text{ vacant path } (u_0^*,f_1^*,\dots,f_\tau^*,u_\tau^*) \text{ on } \mathcal{G}_{p\ell}^* \text{ with the properties (7.57)-(7.59)}\}.$$

In turn it is easy to see that any vacant vertical crossing on $\mathcal{G}_{p\ell}^*$ of $[\Lambda_4+1,M_1+\Lambda_4+1] \times [-\Lambda_4,6M_2+\Lambda_4]$ contains a path with the properties (7.57)-(7.59). Indeed any such vertical crossing contains a continuous curve ψ which connects the horizontal lines $x(2) = -\Lambda_4$ (which lies below r_2) and $x(2) = 6M_2+\Lambda_4$ (which lies above r_4) (see Fig. 7.2). ψ therefore intersects r_2 in a point of A and r_4 in a point of $\tilde{B}_1$; ψ also lies in χ. Combining this observation with (7.60) we obtain

(7.61) $$\begin{aligned} P_{p_0}\{F(r)\} &\geq P_{p_0}\{\ \exists \text{ vacant vertical crossing on } \mathcal{G}_{p\ell}^* \text{ of } [\Lambda_4+1,M_1+\Lambda_4+1] \times [-\Lambda_4,6M_2+\Lambda_4]\} \\ &\geq \sigma^*((M_1,7M_2);2,p_0,\mathcal{G}_{p\ell}) \geq \delta_7 , \end{aligned}$$

as soon as $M_1, M_2 > 2\Lambda_4+1$, by virtue of (7.19). By Prop. 2.3 also

(7.62) $$P_{p_0}\{S^* \text{ exists}\} = P_{p_0}\{F(r)\} \geq \delta_7 \quad .$$

The next step in estimating $E_{p_0}\{N(r)\}$ is to write

(7.63) $$E_{p_0}\{N(r)\} \geq \sum_{\tilde{s}^*} P\{S^* = \tilde{s}^*\} E_{p_0}\{N(r) | S^* = \tilde{s}^*\}$$

$$\geq \delta_7 \min_{\tilde{s}^*} E_{p_0}\{N(r) | S^* = \tilde{s}^*\} \ ,$$

where $\tilde{s}^*$ ranges over all paths on $\mathcal{G}^*_{p\ell}$ which lie in χ and satisfy (7.55) and (7.56). For the remainder of the proof we fix a path $\tilde{s}^* = (v, e^*, \ldots, e^*_\rho, v^*_\rho)$ on $\mathcal{G}^*_{p\ell}$ which lies in χ and satisfies (7.55) and (7.56). The initial point $v = (v(1), v(2))$ lies on r and is a vertex of $\mathcal{M}$ while v^*_ρ lies on $C \cap \chi \subset \mathring{C}$. Let the annulus U_k be defined as in (7.32) and let V_k be U_k translated by $(\lfloor v(1) \rfloor, \lfloor v(2) \rfloor)$, i.e.,

$$V_k = [\lfloor v(1) \rfloor - 2M_{k1}, \lfloor v(1) \rfloor + 2M_{k1}] \times [\lfloor v(2) \rfloor - 2M_{k2}, \lfloor v(2) \rfloor + 2M_{k2}] \setminus$$

$$(\lfloor v(1) \rfloor - M_{k1}, \lfloor v(1) \rfloor + M_{k1}) \times (\lfloor v(2) \rfloor - M_{k2}, \lfloor v(2) \rfloor + M_{k2}).$$

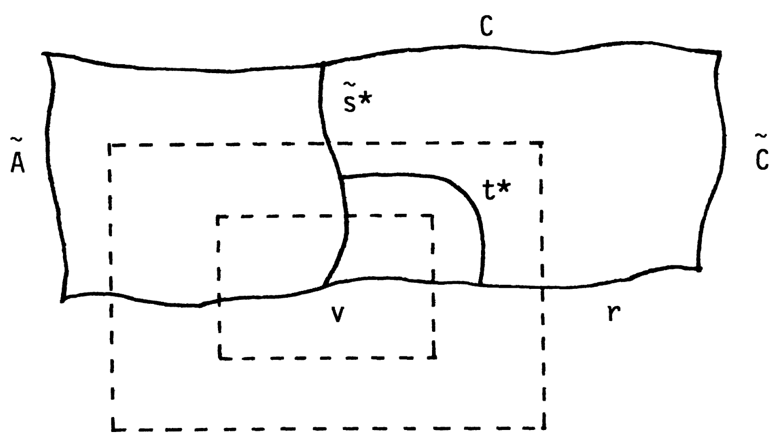

Figure 7.4 V_k is the region between the dashed rectangles.

We restrict ourselves to k with

(7.64) $\quad M_{k1} > 1,\ M_{k2} > 1$ and $2M_{k1} < M_{\ell 1}-2\Lambda_4-1$.

(Note the $M_{\ell 1}$ in the last expression in (7.64); we reintroduced to subscript ℓ to distinguish $\overline{M}_k$ and $\overline{M}_\ell$.) Under (7.64) v lies inside the center rectangle

$$(\lfloor v(1)\rfloor - M_{k1}, \lfloor v(1)\rfloor + M_{k1}) \times (\lfloor v(2)\rfloor - M_{k2}, \lfloor v(2)\rfloor + M_{k2})$$

of V_k. Moreover, since $\tilde{s}^* \subset \chi$, $v(1) \le M_{\ell 1}+\Lambda_4+1$ and consequently

(7.65) $\quad v(1)+2M_{k1} < 2M_{\ell 1}-\Lambda_4$.

Assume now that there exists a path $t^* = (w,g^*,w_0^*,g_1^*,\dots,g_\sigma^*,w_\sigma^*)$ on $\mathcal{G}_{p\ell}^*$ with the following properties:

(7.66) $\quad w \in V_k$, w is a vertex on $r \cap \operatorname{int}(J)$

(7.67) $\quad w_\sigma^* \in C\setminus\{v_\rho^*\}$ or w_σ^* lies on $\tilde{s}^*$,

(7.68) $\quad (g^*\setminus\{w\},w_0^*,\dots,w_{\sigma-1}^*,g_\sigma^*\setminus\{w_\sigma^*\})$

$$= t^*\setminus\{w\ ,w_\sigma^*\} \subset \tilde{J}^+(\tilde{s}^*) \cap V_k$$

and

(7.69) $\quad w_0^*,\dots,w_{\sigma-1}^*$ are vacant, and if $w \in C\setminus\{v_\rho^*\}$ then also w_σ^* is vacant.

Again we allow $\sigma = 0$, $t^* = (w,g^*,w_0^*)$, in which case (7.68) reduces to $\mathring{g}^*\subset\tilde{J}^+(\tilde{s}^*) \cap V_k$. We claim that if such a t^* exists and $S^* = \tilde{s}^*$, then w, the initial point of t^*, has a vacant connection to $\mathring{C}$ above r. This is obvious if $w_\sigma^* \in C\setminus\{v_\rho^*\}$ (see (7.47)-(7.49)) and recall that $\tilde{J}^+(\tilde{s}^*) \subset \operatorname{int}(\tilde{J}) = J^+(r)$; also $w_\sigma^* \in$ closure of $(\tilde{J}^+(\tilde{s}^*) \cap V_k)$ $\cap(C\setminus v_\rho^*)$ implies $w_\sigma^* \in \mathring{C}$ (use (7.65).) But also in the other case – when w_σ^* lies on $\tilde{s}^*$– it is easy to substantiate this claim. Indeed, if $w_\sigma^* = v_i^*$ for some $0 \le i \le \rho$ then $t_1^* := (w_0^*,g_1^*,\dots,g_\sigma^*,w_\sigma^* = v_i^*, e_{i+1}^*,\dots,e_\rho^*,v_\rho^*)$ is a path on $\mathcal{G}_{p\ell}^*$ consisting of $t^*\setminus g_1^*$ followed by a piece of $\tilde{s}^*$. It is self-avoiding since $t\setminus\{w_\sigma^*\}$ does not intersect $\tilde{s}^*$ (see (7.68)). There is an edge g^* of $\mathcal{G}_{p\ell}^*$ from w to w_0^* with $\mathring{g} \in J^+(r)$, while t_1^* ends at $v_\rho^* \in \mathring{C}$. Also $t_1^*\setminus\{v_\rho^*\} \subset J^+(r)$ by

(7.68) and (7.55). Finally t_1^* is vacant by (7.69) and the fact that $\tilde{s}^* = S^*$ is vacant whenever S^* exists. Thus indeed w has a vacant connection to $\overset{\circ}{C}$ above r. The last conceivable case with $w_\sigma^* = v$ cannot occur, since $v \notin V_k$ while w_σ^* is the endpoint of g^* or g_σ^*, hence $w_\sigma^* \in \overline{V}_k = V_k$, by (7.68). This proves our claim.

As before we may assume the U_k of (7.32) disjoint. Then the V_k are also disjoint and then distinct V_k for which there exist a t^* as above provide us with distinct vertices of $\mathcal{M}$ on r which have a vacant connection above r to $\overset{\circ}{C}$. In view of the definition (7.50) we therefore have

$$(7.70)\qquad E_{p_0}\{N(r)\,|\,S^* = \tilde{s}^*\} \geq \sum_{\substack{k \text{ satisfying} \\ (7.64)}} P_{p_0}\{\ \exists\ \text{path } t^* \text{ which satisfies (7.66)-(7.69)}\,|\,S^* = \tilde{s}^*\}\ .$$

We now complete our proof by showing

$$(7.71)\qquad P_{p_0}\{\ \exists \text{ path } t^* \text{ which satisfies (7.66)-(7.69)}\,|\,S^* = \tilde{s}^*\} \geq \delta_4^4\ ,$$

whenever (7.64) holds and $\tilde{s}^* \subset \chi$ satisfies (7.55) and (7.56). This will indeed imply (7.45) by means of (7.54), (7.63), (7.70) and the fact that the number of k which satisfy (7.64) tends to ∞ as $\ell \to \infty$. Now for the proof of (7.71). To begin with observe that we may drop the condition $S^* = \tilde{s}^*$, because the existence of a path t^* which satisfies (7.66)-(7.69) depends only on the occupancies of vertices in $\tilde{J}^+(\tilde{s}^*) \cap V_k$ or vertices on $C\setminus\{v_\rho^*\}$ which are an endpoint of some edge of $\mathcal{G}_{p\ell}^*$ with interior in $\tilde{J}^+(\tilde{s}^*)$. None of these vertices lie in $\tilde{\tilde{J}}^-(\tilde{s}^*)$. The only vertices for which this is possibly in doubt are those on $C\setminus\{v_\rho^*\}$. However, these vertices would have to be in $\tilde{\tilde{J}}^+(\tilde{s}^*)$, since they are an endpoint of an edge with interior in $\tilde{J}^+(s^*)$. But the only vertex on C in $\tilde{\tilde{J}}^+(s^*) \cap \tilde{\tilde{J}}^-(s^*)$ is v_ρ^*, the final point of $\tilde{s}^*$. On the other hand, by Prop. 2.3 the event $\{S^* = \tilde{s}^*\}$ depends only on the occupancies of vertices in $\tilde{\tilde{J}}^-(\tilde{s}^*)$. Therefore the conditional probability in (7.71) is the same as the unconditional probability. Next let c^* be a vacant circuit on $\mathcal{G}_{p\ell}^*$ surrounding the point $(\lfloor v(1) \rfloor, \lfloor v(2) \rfloor)$ and with all its edges and vertices in V_k. We want to show that if such a c^* exists, then it contains a t^* with the properties (7.66)-(7.69). This is intuitively clear from Fig. 7.4 if one takes into account that by (7.65) the right edge of V_k is on the vertical line $x(1) = \lfloor v(1) \rfloor + 2M_{k1} < 2M_{\ell 1} - \Lambda_4$, while $\tilde{C}$ is part of r_3 and hence to the right of the vertical line $x(1) = 2M_{\ell 1} - \Lambda_4$. A formal proof was given in Kesten (1980a) Lemma 3 for the case where the upper edge of V_k also lies below $C = \tilde{B}_1$ (as depicted in Fig. 7.4). Here we shall appeal to Lemma A.2. Let J_1 be $\mathrm{Fr}(\tilde{J}^+(\tilde{s}^*))$, viewed as a Jordan curve. J_1 is made up of the following four arcs: $A_{11} = \{v\}$ (i.e., consisting of the single point v only), $A_{12} = \tilde{s}^*$ followed by the piece of C from v_ρ^* to u_3 (v_ρ^* is the intersection of $\tilde{s}^*$ and C, u_3 is the intersection of C and $\tilde{C}$; see Fig. 7.3 and 7.5), $A_{13} = \tilde{C}$ and A_{14} = piece of r between v_ν and v (v_ν is the intersection of r and $\tilde{C}$, v is the intersection of r and $\tilde{s}^*$ (see Fig. 7.3-7.5). For J_2 we take c^*, viewed as a Jordan curve. Then

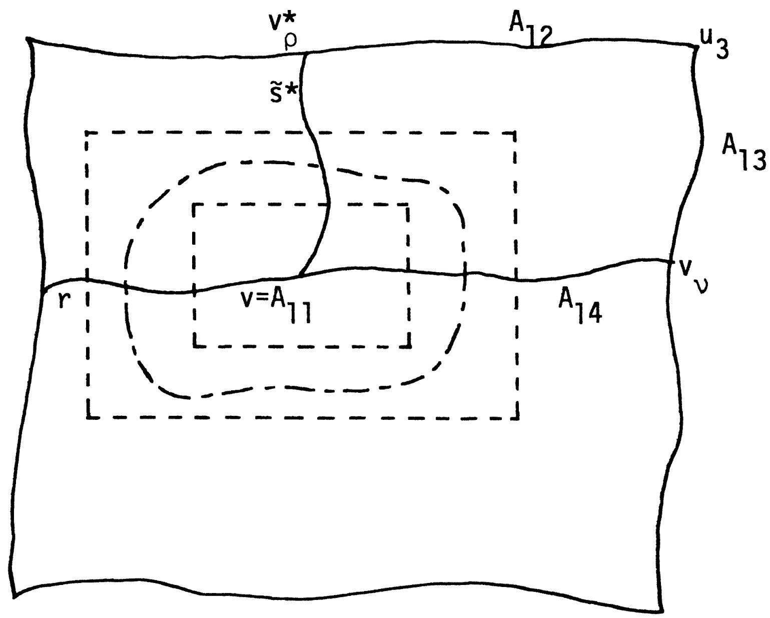

Figure 7.5 U_k is the region between the dashed rectangles.
— - — - denotes the circuit c^*.

under (7.64) c^* also surrounds v, i.e., $A_{11} = \{v\} \subset \mathrm{int}(J_2)$, and $A_{13} = \tilde{C} \subset \mathrm{ext}(J_2)$, since by the above $\tilde{C}$ lies outside the exterior boundary of V_k whenever (7.64) and hence (7.65) holds. Therefore, by Lemma A.2 c^* contains an arc, t^* say, with one endpoint each on

$\mathring{A}_{12} \subset \tilde{s}^* \cup \mathring{C}$ and $\mathring{A}_{14} \subset r$, and such that $\tilde{t}^*$ minus its endpoints is contained in $\mathrm{int}(J_1) = \tilde{J}^+(\tilde{s}^*)$. If $t^* = (w, g^*, w_0^*, \dots, g_\sigma^*, w_\sigma^*)$ with w the endpoint on r and w_σ^* the endpoint on $\mathring{A}_{12}$, then t^* satisfies (7.66)-(7.69) (recall that $c^* \subset V_k$ and that c^* is vacant; also $w \in \mathring{A}_{14} \subset \mathrm{int}(J)$). Thus the existence of c^* implies the existence of t^* as desired. Consequently,

$$\begin{aligned}(7.72) \qquad & P_{p_0}\{\exists \text{ path } t^* \text{ which satisfies (7.66)-(7.69)} \mid S^* = \tilde{s}^*\} \\ & = P_{p_0}\{\exists \text{ path } t^* \text{ which satisfies (7.66)-(7.69)}\} \\ & \geq P_{p_0}\{\exists \text{ vacant circuit on } \mathcal{G}^*_{p\ell} \text{ surrounding} \\ & \qquad (\lfloor v(1) \rfloor, \lfloor v(2) \rfloor) \text{ and inside } V_k\}.\end{aligned}$$

Since V_k is just the translate by $(\lfloor v(1) \rfloor, \lfloor v(2) \rfloor)$ of U_k, the last member of (7.72) equals the left hand side of (7.21) (with ℓ replaced by k), by virtue of periodicity. Thus (7.71) follows from (7.72) and (7.21). The proof is complete. □

<u>Remark.</u>

(ii) In Ch. 10 it will be necessary to have an estimate for the conditional distribution of $N(r)$, given $\{R = r\}$ under $P_{p(t)}$, instead of just for the conditional expectation of $N(r)$. This estimate was already given in Kesten (1980a), Steps (i) and (ii) in the proof of Prop. 1. We shall want to restrict ourselves in Ch. 10 to counting only vertices with vacant connections in

$$\Gamma = \Gamma_\ell := [\tfrac{1}{2} M_{\ell 1}, \tfrac{3}{2} M_{\ell 1}] \times \mathbb{R} \ .$$

More precisely let $N_\Gamma(r)$ be the number of vertices v of $\mathcal{G}_{p\ell}$ on r for which there exists a vacant path $s^* = (v_0^*, e_1^*, \dots, e_\rho^*, v_\rho^*)$ on $\mathcal{G}^*_{p\ell}$ which satisfies the following properties

(7.73) there exists an edge e^* of $\mathcal{M}_{p\ell}$ between v and v_0^* such that $\mathring{e}^* \subset J^+(r) \cap \Gamma$,

$$(7.74) \qquad v_\rho^* \in \mathring{C}$$

$$(7.75) \qquad (v_0^*, e_1^*, \dots, e_{\rho-1}^* \setminus \{v_\rho^*\}) = s^* \setminus \{v_\rho^*\} \subset J^+(r) \cap \Gamma \ .$$

(Note that we dropped the restriction $v \in \mathcal{M}$, and therefore only require e^* is an edge of $\mathcal{M}_{p\ell}$ in (7.73)). Let r be as in (7.39)-(7.41) and

$$\Gamma' = [\tfrac{3}{4} M_{\ell 1}, \tfrac{5}{4} M_{\ell 1}] \times \mathbb{R} \quad .$$

The desired estimate is that for ℓ greater than some $\ell_0(m)$

$$(7.76) \qquad P_{p(t)}\{N_\Gamma(r) \geq m | R = r\}$$
$$\geq \frac{1}{2} P_{p_0}\{ \exists \text{ at least one } v \text{ on } r \text{ and a vacant path } s^* \text{ which satisfy (7.73)-(7.75) with } \Gamma \text{ replaced by } \Gamma'\} ,$$

uniformly in r, $0 \leq t \leq 1$. We briefly indicate the trivial changes necessary in the proof of Lemma 7.4 to obtain (7.76). Instead of (7.52) and (7.53) we have

$$P_{p(t)}\{N_\Gamma(r) \geq m | R = r\} = P_{p(t)}\{N_\Gamma(r) \geq m\} \geq P_{p_0}\{N_\Gamma(r) \geq m\} .$$

Also, for fixed r we again consider vacant paths $\tilde{s}^*$ satisfying (7.55) and (7.56) with the right hand side of (7.55) replaced by $\text{int}(\tilde{J}) \cap \Gamma'$. Again S^* will be the left-most of all these paths. Then there exists at least one v on r and a vacant path s^* which satisfies (7.73)-(7.75) with Γ replaced by Γ' whenever S^* exists. Instead of (7.63) we get

$$P_{p_0}\{N_\Gamma(r) \geq m\}$$
$$\geq P_{p_0}\{S^* \text{ exists}\} \cdot \min_{\tilde{s}^*} P_{p_0}\{N_\Gamma(r) \geq m | S^* = s^*\}$$

Now if $\tilde{s}^* \subset \Gamma'$ then its endpoint v on r lies in Γ'. Thus, if we strengthen (7.64) to

$$(7.77) \qquad M_{k1} > 1,\ M_{k2} > 1 \text{ and } 2M_{k1} < \frac{1}{4} M_{\ell 1} - 1 ,$$

then the whole annulus V_k lies inside Γ. Instead of (7.70) we therefore obtain

(7.78) $\quad P_{p_0}\{N_\Gamma(r) \geq m \mid S^* = s^*\} \geq P_{p_0}\{\#$ of k satisfying (7.77) for which there exists a path t^* which satisfies (7.66)-(7.69) at least $m \mid S^* = \tilde{s}^*\}$

As in the proof of (7.72) we may drop the condition $S^* = s^*$ and replace "path t which satisfies (7.66)-(7.69)" by "vacant circuit in V_k" in (7.78). In other words the right hand side of (7.78) is at least

(7.79) $\quad P_{p_0}\{\#$ of k satisfying (7.77) for which V_k contains a vacant circuit on $\mathcal{G}^*_{p\ell}$ surrounding $\lfloor v \rfloor$ is at least $m\}$.

However, the P_{p_0}-probability that any fixed V_k contains a vacant circuit on $\mathcal{G}^*_{p\ell}$ surrounding $\lfloor v \rfloor$ is at least δ_4^4 (cf. (7.21) and Lemma 4.1) and the different V_k are disjoint. Vacant circuits in different V_k occur therefore independent of each other. Moreover, as $\ell \to \infty$ the number of k which satisfies (7.77) also tends to ∞. If we call this number ν, then the number of k satisfying (7.77) for which V_k contains a vacant circuit on $\mathcal{G}^*_{p\ell}$ surrounding $\lfloor v \rfloor$ has just a binomial distribution corresponding to ν trials with success probability $\geq \delta_4^4$. Clearly the probability that such a variable is $\geq m$ tends to 1 as $\nu \to \infty$. Thus (7.79) is at least 1/2 for all large ℓ. This implies (7.76). ///

Proof of Theorem 3.1 (ii) and (iii). It suffices to prove part (ii), since part (iii) then follows by interchanging the roles of $\mathcal{G}$, p and "occupied" with those of $\mathcal{G}^*$, $\bar{1}$-p and "vacant", respectively.

(3.46) follows from (7.35), Theorem 5.1 and Lemma 7.3. To see this note that (7.35) implies (5.10) with $\bar{N} = 2\bar{M}_\ell$, ℓ large and $p = p'$. Thus by (5.11)

(7.80) $$P_{p'}\{\#W(v) = \infty\} = 0,$$

which is the first relation in (3.46). It is also immediate from (5.11) that (3.48) holds. To obtain the second relation in (3.46) pick a $p \in \mathcal{P}_\lambda$ such that $p' \ll p \ll p_0$. Then automatically $\bar{0} \ll p \ll \bar{1}$ and by the above (applied to p instead of p') also

$$E_p\{\#W(v)\} < \infty .$$

Lemma 7.3 now shows

$$P_p\{\#W^*(v) = \infty\} > 0.$$

But $\{\#W^*(v) = \infty\}$ is a decreasing event so that Lemma 4.1 implies

$$P_{p'}\{\#W^*(v) = \infty\} \geq P_p\{\#W^*(v) = \infty\} > 0 .$$

The second relation in (3.46) follows. Finally, we have from Lemma 4.1 and (7.34)

(7.81) $P_{p'}\{\exists$ a vacant circuit on $\mathcal{G}^*$ surrounding

$[N,N]\times[-N,N]\} \geq P_{p_0}\{\exists$ vacant circuit on $\mathcal{G}^*$ surrounding

$[-N,N]\times[-N,N] = 1$ for all N.

If v_1 and v_2 are two vertices of $\mathcal{G}^*$ in $[-N,N]\times[-N,N]$ and $\#W^*(v_1) = \#W^*(v_2) = \infty$, then there exist vacant paths, π_1^* and π_2^* say, on $\mathcal{G}^*$ from v_1 to ∞ and from v_2 to ∞, respectively. Both these paths have to intersect any vacant circuit c^* on $\mathcal{G}^*$ which surrounds $[-N,N]\times[-N,N]$ (and hence v_1 and v_2). The intersection of π_i^* and c^* does not have to be a vertex of $\mathcal{G}^*$, but nevertheless π_i^* and c^* must contain a pair of neighboring vertices on $\mathcal{G}^*$, as explained in Comment 2.2 (vii). Consequently c^* has to belong to $W^*(v_i)$ for $i = 1,2$. Thus, if a vacant circuit c^* as above exists, then $W^*(v_1)$ and $W^*(v_2)$ have c^* in common and $W^*(v_1) = W^*(v_2)$. Thus (7.81) shows $W^*(v_1) = W^*(v_2)$ whenever $\#W^*(v_1) = \#W^*(v_2) = \infty$ so that there is at most one infinite vacant cluster on $\mathcal{G}^*$. The fact that there actually exists an infinite vacant cluster follows from Birkhoff's ergodic theorem (Walters (1982) Theorem 1.14) since for fixed $v \in \mathcal{G}^*$

$$\frac{1}{n}\sum_{k=0}^{n} I[\#W^*(v+k\xi_1) = \infty] \to P_{p'}\{\#W^*(v) = \infty\} > 0 \quad \text{a.e. } [P_{p'}]$$

(compare Harris (1960), Lemma 5.1 and Lemma 3.1). (3.47) is immediate from these considerations. $\square$

<u>Proof of Theorem 3.2</u>. Let $p_1 \in \mathcal{P}_\lambda$ be such that the set in the right hand side of (3.56) is nonempty, so that t_0 in (3.56) is well defined. Assume further that $\overline{0} \ll p_0 = t_0 p_1 \ll \overline{1}$. We shall now give an indirect proof of

(7.82) $$\liminf_{n\to\infty} \sigma^*((d_2n,d_3n);1,p_0) > 0$$

and

(7.83) $$\liminf_{n\to\infty} \sigma^*((c_2n,c_3n);2,p_0) > 0 \quad .$$

Assume for the sake of argument that (7.82) fails. Then there exists a sequence $n_\ell \to \infty$ such that

$$\lim_{\ell\to\infty} \sigma^*((d_2n_\ell,d_3n_\ell);1,p_0) = 0$$

and by (3.55) also

$$\lim_{\ell\to\infty} \sigma^*((n_\ell,d_1n_\ell);2,p_0) = 0.$$

If this is the case, then we see from (7.14) and (7.15) that

(7.84) $$\lim_{\ell\to\infty} \sigma((n_\ell-2\Lambda_4,d_1n_\ell+2\Lambda_4);1,p_0) = 1$$

as well as

$$\lim_{\ell\to\infty} \sigma((d_2n_\ell+2\Lambda_4,d_3n_\ell-2\Lambda_4);2,p_0) = 1.$$

By virtue of Lemma 2.1b the probabilities of occupied horizontal and vertical crossings on $\mathcal{G}_{p\ell}$ of suitable rectangles also tend to 1. More precisely, the existence of an occupied horizontal crossing on $\mathcal{G}$ of $[0,n-2\Lambda_4]\times[0,d_1n+2\Lambda_4]$ implies the existence of an occupied horizontal crossing on $\mathcal{G}_{p\ell}$ of $[\Lambda,n-2\Lambda_4-\Lambda]\times[-\Lambda,d_1n+2\Lambda_4+\Lambda]$. Therefore (7.84) implies

$$\lim_{\ell\to\infty} \sigma((n_\ell-2\Lambda_4-2\Lambda-1,d_1n_\ell+2\Lambda_4+2\Lambda+1);1,p_0,\mathcal{G}_{p\ell}) = 1.$$

Similarly

$$\lim_{\ell\to\infty} \sigma((d_2n_\ell+2\Lambda_4+2\Lambda+1,d_3n_\ell-2\Lambda_4-2\Lambda-1);2,p_0,\mathcal{G}_{p\ell}) = 1.$$

By virtue of the RSW theorem (Theorem 6.1) (and Comment 3.3 (v)) there must exist a π (depending on d_1,d_2,d_3 only) such that for all k

$$\lim_{\ell\to\infty} \sigma((kn_\ell,(\pi+3)n_\ell);1,p_0,\mathcal{G}_{p\ell}) = \lim_{\ell\to\infty} \sigma((\pi+3)n_\ell,kn_\ell);2,p_0,\mathcal{G}_{p\ell}) = 1.$$

By Lemma 2.1a we can now go back to $\mathcal{G}$ to obtain

$$(7.85)\qquad \lim_{\ell\to\infty} \sigma((kn_\ell,(\pi+4)n_\ell);1,p_0,\mathcal{G}) = \lim_{\ell\to\infty} \sigma(((\pi+4)n_\ell,kn_\ell);2,p_0,\mathcal{G}) = 1.$$

Finally, by (7.12) we have for any integers m_1, m_2

$$(7.86)\qquad \sigma((m_1,m_2);1,p_0)+\sigma^*((m_1,m_2);2,p_0) \le 1$$

and by interchanging the horizontal and vertical direction also

$$(7.87)\qquad \sigma((m_1,m_2);2,p_0)+\sigma^*((m_1,m_2);1,p_0) \le 1.$$

(7.85)-(7.87) show that

$$\lim_{\ell\to\infty} \sigma^*((kn_\ell,(\pi+4)n_\ell);2,p_0) = \lim_{\ell\to\infty} \sigma^*(((\pi+4)n_\ell,kn_\ell);1,p_0) = 0.$$

If we take $N_\ell = (\pi+4)n_\ell$ and $k = 3\pi+12$ this implies

$$(7.88)\qquad \tau^*((N_\ell,N_\ell);i,p_0,\mathcal{G}) = \tau((N_\ell,N_\ell);i,\overline{1}-p_0,\mathcal{G}^*) \to 0$$

as $\ell \to \infty$ for $i = 1,2$. Indeed $(3\pi+12)n_\ell = 3N_\ell$ so that

$$\tau^*((N_\ell,N_\ell);1,p_0,\mathcal{G}) = P_{p_0}\{\ \exists \text{ vacant horizontal crossing on } \mathcal{G}^* \text{ of } [0,N_\ell]\times[0,3N_\ell]\} = \sigma^*((\pi+4)n_\ell,(3\pi+12)n_\ell);1,p_0),$$

and similarly for the vertical direction. In particular (7.88) allows us to pick on N with

$$\tau((N,N);i,\overline{1}-p_0,\mathcal{G}^*) \le \frac{1}{2}\kappa(2),\ i = 1,2,$$

where $\kappa(2)$ is defined by (5.9). By continuity we can then also find a $0 < t_1 < t_0$ such that $\overline{0} << t_1p_1 << \overline{1}$ and

$$(7.89)\qquad \tau^*((N,N);i,t_1p_1,\mathcal{G}) = \tau^*((N,N);i,\overline{1}-t_1p_1,\mathcal{G}^*) \le \kappa(2).$$

This, however, contradicts the definition of t_0 via Theorem 5.1. Indeed (7.89) and Theorem 5.1 applied to $\mathcal{G}^*$ show

$$(7.90)\qquad P_{t_1p_1}\{\#W^*(v) \ge n\} \le C_1e^{-C_2n},\ n \ge 0$$

for any vertex v of $\mathcal{G}^*$, which implies

$$\lim_{n\to\infty} \sigma^*((n+2\Lambda_4, a_1n-2\Lambda_4); t_1p_1, 2) = 0,$$

by the same argument as at the end of the proof of Lemma 5.4 (especially (5.55)). Together with (7.14) this finally gives

$$\lim_{n\to\infty} \sigma((n, a_1n); t_1p_1, 1) = 1,$$

contradicting (3.56) since $t_1 < t_0$. It follows that (7.82) must hold and (7.83) is proved in the same way. (3.39) for some choice of δ, $\overline{m}_\ell$ is now immediate from (7.82) and (7.83). (Note however, that the b_1, b_2 for which (3.39) holds are not the b_1, b_2 of (3.53).)

Interchanging the role of $\mathcal{G}$ and $\mathcal{G}^*$ one proves in the same way that

$$\liminf_{n\to\infty} \sigma((b_2n, b_3n); 1, p_0) = 0$$

implies for some $t_2 > t_0$ with $\overline{0} << t_2p_1 << \overline{1}$

$$\lim_{n\to\infty} \sigma((n, a_1n); 1, t_2p_1) = \lim_{n\to\infty} \sigma((n, b_1n); 2, t_2p_1) = 0.$$

Again this contradicts (3.56), since $t_2 > t_0$. Hence

$$\liminf_{n\to\infty} \sigma((b_2n, b_3n); 1, p_0) > 0$$

and similarly

$$\liminf_{n\to\infty} \sigma((a_2n, a_3n); 2, p_0) > 0.$$

Thus also (3.38) holds, i.e., Condition B is fulfilled for p_0. $\square$