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QUANTUM GROUPS AND STOCHASTIC MODELS

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Abstract. The aim of this paper is to show that stochastic models provide a very good playground to enhance the utility of quantum groups. Quantum groups arise naturally and the deformation parameter has a direct physical meaning for diffusion systems where it is just the ratio of left/right probability rate. In the matrix product state approach to diffusion processes the stationary probability distribution is expressed as a matrix product state with respect to a quadratic algebra which defines a noncommutative space with a quantum group action as its symmetry. Boundary processes amount for the appearance of parameter-dependent linear terms in the algebra which leads to a reduction of the bulk symmetry.

1. Introduction

Stochastic reaction-diffusion processes are of both theoretical and experimental interest not only because they describe various mechanisms in physics and chemistry but they also provide a way of modelling phenomena like traffic flow, kinetics of biopolymerization, interface growth [11, 8, 12].

A stochastic process is described in terms of a master equation for the probability distribution $P(s_i, t)$ of a stochastic variable $s_i = 0, 1, 2, \ldots, n-1$ at a site $i = 1, 2, \ldots, L$ of a linear chain. A configuration on the lattice at a time t is determined by the set of occupation numbers $s_1, s_2, \ldots, s_L$ and a transition to another configuration s' during an infinitesimal time step $\mathrm{d}t$ is given by the probability $\Gamma(s, s')\,\mathrm{d}t$. The time evolution of the stochastic system is governed by the master equation

$$\frac{\mathrm{d}P(s,t)}{\mathrm{d}t} = \sum_{s'} \Gamma(s, s') P(s', t)$$

for the probability $P(s,t)$ of finding the configuration s at a time t. With the restriction of dynamics to changes of configuration only at two adjacent sites the rates for such changes depend only on these sites. The two-site rates $\Gamma \equiv \Gamma^{ik}_{jl}$, $i,j,k,l = 0,1,2,\ldots,n-1$ are assumed to be independent from the position in the bulk. At the boundaries, i.e., sites 1 and L additional processes can take place with single-site rates L^i_k and R^i_k. The master equations can be mapped to a Schrödinger equation in imaginary time for a quantum Hamiltonian with nearest neighbour interaction in the bulk and single site boundary terms

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t} = -HP(t), \qquad H = \sum_j H_{j,j+1} + H^{(L)} + H^{(R)}.$$

The ground state of this in general **non-hermitian Hamiltonian** corresponds to the stationary probability distribution of the stochastic dynamics. The mapping provides a connection with integrable quantum spin chains. Examples are the processes of particles hopping between lattice sites i, j with rates g_{ij} and a hard core repulsion (i.e., a site is empty or occupied by one particle). In the case of the symmetric exclusion process $g_{ij} = g_{ji}$ and the stochastic Hamiltonian it coincides with the $\mathrm{SU}(2)$ symmetric spin $1/2$ isotropic Heisenberg ferromagnet. The diffusion driven lattice gas of particles with rates $g_{i,i+1}/g_{i+1,i} = q \neq 1$ is mapped to a $\mathrm{SU}_q(2)$-symmetric XXZ chain with anisotropy $\Delta = (q+q^{-1})/2$.
The stationary probability distribution, i.e., the ground state of the quantum Hamiltonian with nearest neighbour interaction in the bulk and single site boundary terms is expressed as a product of (or a trace over) matrices that form representation of a quadratic algebra determined by the dynamics of the process. Following this matrix product ground states Anzatz Derrida *et al* [4] have considered **Asymmetric Simple Exclusion Process** (ASEP) with open boundaries of three-species diffusion-type, in which the reaction-diffusion processes

$$\Gamma^{ik}_{jl} D_i D_k = x_l D_j - x_j D_l$$

takes place with diffusion $\Gamma^{ik}_{ki} = g_{ik}$. We consider n species diffusion process on a chain with L sites with nearest-neighbour interaction with exclusion which means that a site is either occupied or empty. The empty site is referred to as a vacancy (or a hole), the rest $n-1$ species as different types of particles. On successive sites the particles i and k exchange places with probability $g_{ik}\,\mathrm{d}t$. The simplest form is the n-species symmetric exclusion process known as the lattice gas model when each particle hops between nearest-neighbour sites with a constant rate $g_{ik} = g_{ki} = g$. The diffusion-driven lattice gas of particles moving under the action of an external field is an example of the n-species asymmetric exclusion process of particles hopping in a preferred direction. The process is totally asymmetric if all jumps are in one direction only and partially asymmetric if the rates g_{ik} for moving to the left (or backward) are different from the rates g_{ki} for moving to the right. The particle number n_i of each species in the bulk is conserved $\sum_{i=0}^{n-1} n_i = L$. One distinguishes closed systems (periodic boundary conditions) and open systems with boundary processes – at site 1 (left) and at site L (right) the particle i is replaced by the particle k with probabilities $L^i_k\,\mathrm{d}t$ and $R^i_k\,\mathrm{d}t$ respectively,

$$L^i_i = -\sum_{j=0}^{n-1} L^i_j, \qquad R^i_i = -\sum_{j=0}^{n-1} R^i_j.$$

The diffusion algebra is generated by the commutators

$$g_{ik} D_i D_k - g_{ki} D_k D_i = x_k D_i - x_i D_k \tag{1}$$

where $i, k = 0, 1, \ldots, n-1$ and x_i are c-numbers $\sum_{i=0}^{n-1} x_i = 0$. This is an algebra with involution, hence hermitian D_i

$$D_i = D_i^+, \qquad g_{ik}^+ = g_{ki}, \qquad x_i = -x_i^+.$$

For the probability distribution one assumes:

- periodic boundary conditions

$$P(s_1, \ldots, s_L) = \mathrm{Tr}(D_{s_1} D_{s_2} \ldots D_{s_L})$$

- boundary processes

$$P(s_1, \ldots, s_L) = \langle w | D_{s_1} D_{s_2} \ldots D_{s_L} | v \rangle$$

  where the vectors $|v\rangle$ and $\langle w|$ are defined by

$$\langle w | (L^k_i D_k + x_i) = 0, \qquad (R^k_i D_k - x_i) | v \rangle = 0.$$

Thus to find the stationary probability distribution one has to compute traces or matrix elements with respect to the vectors $|v\rangle$ and $\langle w|$ of monomials of the form $D^{m_1}_{s_1} D^{m_2}_{s_2} \ldots D^{m_L}_{s_L}$. The problem to be solved is twofold – find a representation of the matrices D that is a solution of the quadratic algebra and match the algebraic solution with the boundary conditions.

The relations (1) allow an ordering of the elements D_k. Monomials of given order are the **Poincare–Birkhoff–Witt** (PBW) **basis** for polynomials of fixed degree as the probability distribution is due to the bulk conservation laws. We thus consider an associative algebra [7] generated by the unit e and n additional elements D_k obeying $n(n-1)/2$ relations (1). The alphabetically ordered monomials

$$D^{n_1}_{s_1} D^{n_2}_{s_2} \ldots D^{n_l}_{s_l} \tag{2}$$

where $s_1 < s_2 < \cdots < s_l$, $l \geq 1$ and $n_1, n_2, \ldots, n_l$ are non-negative integers, are a linear basis in the algebra, the PBW basis.

Proposition 1. 1. *In the case of Lie-algebra type diffusion algebras the n generators D_i, and e can be mapped to the generators J_{jk} of $\mathrm{SU}(n) \times U(1)$ and the mapping is invertible. The **Universal Enveloping Algebra** (UEA) generated by D_i is a subalgerbra of the UEA of the Lie-algebra of $\mathrm{SU}(n) \times U(1)$.*

2. *The multiparameter quantized noncommutative space can be realized equivalently as a q-deformed Heisenberg algebra of* n *oscillators depending on* $n(n-1)/2+1$ *parameters (in general on* $n(n-1)/2+n$ *parameters). The UEA of the elements* D_i *in the case of a diffusion algebra with all coefficients* x_i *on the RHS of equation* (1) *equal to zero belongs to the UEA of a multiparameter deformed Heisenberg algebra to which a consistent multiparameter* $\mathrm{SU}_q(n)$ *quantization corresponds.*

3. *In an algebra with* x*-terms on the RHS of* (1) *only then is braid associativity satisfied if out of the coefficients* x_i, x_k, x_l *corresponding to a triple* $D_iD_kD_l$ *either one coefficient* x *is zero or two coefficients* x *are zero and the rates are respectively related. The diffusion algebras in this case can be obtained by either a change of basis in the* n*-dimensional noncommutative space or by a suitable change of basis of the lower dimensional quantum space. The appearance of the nonzero linear terms in the RHS of the quantum plane relations leads to a lower dimensional noncommutative space and a reduction of the* $\mathrm{SU}_q(n)$ *invariance.*

2. Representations of the Diffusion Algebras

2.1. Lie-Algebra Types

There are two such algebras. The first type arises when all rates are equal and corresponds to the n-species symmetric exclusion process. The second algebra appears in the description in the multispecies governed by totally asymmetric exclusion process.

1. All rates equal, $g_{ij} = g_{ji} = g$.

After rescaling the generators D_i, $i = 0, 1, 2, \ldots, n-1$ by

$$D_i = \frac{x_i}{g} D_i', \qquad \sum_{i=1}^{n-1} x_i = 0$$

the commutators take the form (the primes are omitted)

$$\begin{aligned}
[D_0, D_1] &= D_0 - D_1 \\
[D_0, D_2] &= D_0 - D_2 \\
&\vdots \\
[D_{n-2}, D_{n-1}] &= D_{n-2} - D_{n-1}.
\end{aligned}$$

These algebraic relations could be solved in terms of the GL(n) Lie-algebra Weyl generators J_i^j

$$\begin{aligned}
D_0 &= -J_0^0 + J_0^1 + J_0^2 + \cdots + J_0^{n-1} \\
D_1 &= J_1^0 - J_1^1 + J_1^2 + \cdots + J_1^{n-1} \\
D_2 &= J_2^0 + J_2^1 - J_2^2 + \cdots + J_2^{n-1} \\
&\vdots \\
D_{n-1} &= J_{n-1}^0 + J_{n-1}^1 + J_{n-1}^2 + \cdots - J_{n-1}^{n-1}.
\end{aligned}$$

The conventional basis for fundamental representation of the $U(n)$ generators given by the $(e_{ij})_{ab} = \delta_{ia}\delta_{jb}$, $i, j, a, b = 0, 1, 2 \ldots, n-1$ provides the n-dimensional matrix representation of the generators D, which entries are 1 only in the first row of D_0, the second row of D_1, the third row of $D_3, \ldots$ the last row of D_{n-1} and all other entries zero. The correspondence is one-to-one since the $U(n)$ Lie-algebra generators can be expressed with the help of the transposed matrices, namely

$$J_i^j = \frac{1}{n} D_i D_j^T.$$

The Poincare–Birkhoff–Witt basis of the algebra generated by the elements D thus belongs to the basis of the universal enveloping algebra of $su(n) \oplus u(1)$ and this is the hidden symmetry algebra of a stochastic diffusion system with all rates equal.

One can show that compatibility of the algebraic solution with the boundary value problem determines a Fock representation of the diffusion algebra with a constraint for the rates which in the case $n = 2$ has the form

$$g(L_0^1 + L_1^0 + R_0^1 + R_1^0) = (L_0^1 + L_1^0)(R_0^1 + R_1^0).$$

2. For algebras with only one element $g_{ij} \neq 0$ the algebraic relations read $g_{ij} D_i D_j = x_j D_i - x_i D_j$. The symmetry reduces from SU$(n) \times U(1)$ to $U(1)^{n-1} \times U(1)$.

2.2. The Quantized Noncommutative Space of a Diffusion Model

2.2.1. Algebras with no x-Dependent Linear Terms

The algebraic relations (1) without the x-terms on the RHS define the Manin's multiparameter quantized space with the n elements D_i viewed as its co-ordinates

$$g_{ik} D_i D_k - g_{ki} D_k D_i = 0. \tag{3}$$

A representation of the quantum space is obtained by identifying the monomials (2) with the states of n oscillators

$$a_0^{+n_0} a_1^{+n_1} \ldots a_{n-1}^{+n_{n-1}} |0\rangle$$

and in this case the generators D_i correspond to n creation operators and the non-commutative space is equivalent to a multideformed oscillator algebra. It is convenient to consider the ratios

$$q_{ij} = \frac{g_{ij}}{g_{ji}}, \qquad i < j$$

as the set of $n(n-1)/2$ independent parameters and introduce further n real parameters r_i. The latter are at this stage auxiliary parameters needed for a consistent quantized phase space calculus whose realization proceeds as follows. One starts with n classical oscillators A_i and A_i^+, obeying $[A_i, A_j^+] = \delta_{ij}$ for $i, j = 0, 1, \dots, n-1$ and defines $n_k = A_k^+ A_k$. A deformation of the Heisenberg algebra is achieved through the invertible maps

$$a_i = \prod_{k>i} q_{ik}^{n_k/2} \sqrt{\frac{r_i^{n_i+1} - 1}{(r_i - 1)(n_i + 1)}} A_i, \quad a_i^+ = \prod_{k>i} q_{ik}^{-n_k/2} \sqrt{\frac{r_i^{n_i+1} - 1}{(r_i - 1)n_i}} A_i^+$$

where

$$\frac{r_i^{n_i} - 1}{r_i - 1} = a_i^+ a_i.$$

These deformed oscillators obey to the following algebraic relations

$$\begin{aligned} a_i a_i^+ - r_i a_i^+ a_i &= 1 \\ a_i^+ a_j^+ - q_{ji} a_j^+ a_i^+ &= 0 \\ a_i a_j - q_{ji} a_j a_i &= 0 \\ a_i a_j^+ - q_{ji}^{-1} a_j^+ a_i &= 0 \end{aligned}$$

with $i < j$. The deformed oscillators can be arranged in bilinear form in order to construct the $\mathrm{GL}_q(n)$ generators $J_{ij} = a_i^+ a_j$, $i, j = 0, 1, \dots, n-1$. A consistent $\mathrm{SU}_q(n)$ quantization with a corresponding R-matrix satisfying an Yang–Baxter equation is achieved with all deformation parameters q_{ij} and equal r_i, i.e.,

$$q_{ij} = q = r_i^{-1}, \qquad i < j, \qquad i, j = 0, 1, \dots, n-1.$$

A representation of the matrices D_i and D_i^T and the generators J_{ij} is provided by the action of the oscillators a_i^+ and a_i on the basis vectors $|n_i\rangle \equiv |n_0 n_1 \dots n_{n-1}\rangle$ (with $n_i = 0, 1, 2 \dots$). The symmetry of the model is $\mathrm{SU}_q(n)$.

2.2.2. Algebras with x-Dependent Linear Terms

A. Algebras with one x-dependent linear term. These algebras are obtained from the quantum plane by a shift of the corresponding generator $D_{i_1} = D'_{i_1} + x_{i_1}/(1 - q_{ji_1})$. The rescaled matrix D_{i_1} forms a $u(1)$ algebra. The presence of one x-dependent term due to a boundary process reduces the $\mathrm{SU}_q(n)$ invariance in the bulk to $\mathrm{SU}_q(n-1) \times U(1)$ invariance.

B. Algebras with two x-dependent linear terms. A diffusion algebra with two c-numbers x always contains one (and only one) nonhomogeneous relation which upon fixing the two non zero x numbers to be x_0 and x_{n-1} (these can be any other two) is of the form

$$D_{n-1}D_0 - \frac{g_{0,n-1}}{g_{n-1,0}}D_0D_{n-1} = \frac{x_0}{g_{n-1,0}}D_{n-1} - \frac{x_{n-1}}{g_{n-1,0}}D_0. \tag{4}$$

This can be mapped to one of the quommutators of the **Heisenberg algebra**, namely

$$a_0a_0^+ - r_0a_0^+a_0 = 1$$

by a simultaneous shift of a_0 and a_0^+. Then one identifies $D_{n-1} = D_0^+$ and shifts the pair a_0^+ and a_0 to define

$$\begin{aligned} D_0 &= \frac{x_0}{g_{n-1,0}}\left(\frac{1}{1-r_0} + \frac{a_0^+}{\sqrt{1-r_0}}\right) \\ D_{n-1} &= -\frac{x_{n-1}}{g_{n-1,0}}\left(\frac{1}{1-r_0} + \frac{a_0}{\sqrt{1-r_0}}\right) \end{aligned} \tag{5}$$

which satisfies (4) with

$$r_0 = \frac{g_{0,n-1}}{g_{n-1,0}}.$$

The rest of the generators D_k, $k = 1, 2, \ldots, n-2$ are to be identified with the remaining $n-2$ creation operators a_k^+

$$D_k = a_k^+ \tag{6}$$

for k different from the fixed index $i = 0$. Thus to make the correspondence with the quantum plane one considers a quantum plane of dimension $n-1$ and identifies one of the generators of a diffusion algebra with two x-terms with a rescaled annihilation operator. The diffusion algebra is obtained from the deformed Heisenberg commutation relations of $n-1$ oscillators

$$\begin{aligned} D_{n-1}D_0 - \frac{g_{0,n-1}}{g_{n-1,0}}D_0D_{n-1} &= \frac{x_0}{g_{n-1,0}}D_{n-1} - \frac{x_{n-1}}{g_{n-1,0}}D_0 \\ D_0D_k - q_kD_kD_0 &= -\frac{x_0}{g_k}D_k \\ D_kD_{n-1} - q_kD_{n-1}D_k &= \frac{x_{n-1}}{g_k}D_k \\ D_kD_l - q_{kl}^{-1}D_lD_k &= 0 \end{aligned} \tag{7}$$

where $k = 1, 2 \ldots, n-2$, and

$$q_k = \frac{g_{k0}}{g_{0k}} = \frac{g_{n-1,k}}{g_{k,n-1}}$$
$$g_k = g_{0k} = g_{k,n-1}$$
$$g_{0k} - g_{k0} = g_{k,n-1} - g_{n-1,k} = g_{0,n-1} - g_{n-1,0}.$$

The last relation in (7) under the constraint

$$\frac{g_{kl}}{g_{lk}} = q, \qquad k < l$$

leads to an one-parameter $\mathrm{SU}_q(n-2)$ quantization with R-matrix satisfying the Yang–Baxter equation. The matrices D_0 and D_{n-1} and the unit generate the deformed UEA of $\mathrm{SU}_q(2)$. The symmetry is reduced to $\mathrm{SU}_q(n-2) \otimes \mathrm{SU}_q(2)$.

We consider as an example the $n = 2$ partially asymmetric exclusion model defined by

$$D_1 D_0 - r D_0 D_1 = x_0 D_1 - x_1 D_0$$
$$\langle w | (L_1^0 D_0 - L_0^1 D_1 + x_1) = 0$$
$$(-R_1^0 D_0 + R_0^1 D_1 - x_0) | v \rangle = 0$$

with $x_0 + x_1 = 0$, $r = g_{01}/g_{10}$ and solution

$$D_0 = \frac{x_0}{g_{10}} \left(\frac{1}{1-r} + \frac{1}{\sqrt{1-r}} \right) a^+, \qquad D_1 = -\frac{x_1}{g_{10}} \left(\frac{1}{1-r} + \frac{1}{\sqrt{1-r}} \right) a$$

where

$$a a^+ - r a^+ a = 1.$$

The explicit representation of the matrices D_0 and D_1 in the oscillator basis leads to the corresponding representation of the boundary vectors. The transition matrices for this process have the form

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -g_{01} & g_{10} & 0 \\ 0 & g_{01} & -g_{10} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{8}$$

and

$$H^L = \begin{pmatrix} -L_1^0 & L_0^1 \\ L_1^0 & -L_0^1 \end{pmatrix}, \qquad H^R = \begin{pmatrix} -R_1^0 & R_0^1 \\ R_1^0 & -R_0^1 \end{pmatrix}.$$

Consider now the R-matrix of the two-parameter standard $\mathrm{GL}_{p,q}(2)$ deformation

$$\check{R}(p,q) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \frac{1}{pq} & \frac{1}{p} & 0 \\ 0 & \frac{1}{q} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

that satisfies the quantum Yang–Baxter equation $\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23}$. Shift the R-matrix by the 4×4 unit matrix to obtain an R-matrix which at the particular point $(p, q) = (1, q)$ has the property of an intensity matrix

$$\check{R}(1, q) - 1_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q^{-1} & 1 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix $\check{R}' \equiv \check{R}(p, q) - 1_4$ satisfies the modified Yang–Baxter equation

$$\check{R}'_{12}\check{R}'_{23}\check{R}'_{12} - \check{R}'_{23}\check{R}'_{12}\check{R}'_{23} = \check{R}'_{12} - \check{R}'_{23}.$$

It is readily seen that the bulk transition rate matrix (8) of the asymmetric exclusion model is up to the factor g_{10} equivalent to the $\mathrm{SU}_q(2)$ R-matrix $\check{R}(1, q)$ with $q^{-1} = g_{01}/g_{10}$. Generalization to $n > 2$ is straightforward.

3. Coherent State Solution

By their very origin the coherent states [10, 6, 17] are quantum states, but at the same time they are parametrized by points in the phase space of some classical system. This makes them very suitable for the study of systems where one encounters a relationship between classical and quantum descriptions. From this point of view, interacting many-particle systems with stochastic dynamics provide an appropriate playground to enhance the utility of the generalized coherent states and we can formulate

Proposition I. *The boundary vectors with respect to which one determines the stationary probability distribution of the n-species diffusion process are generalized, coherent or squeezed states of the deformed Heisenberg algebra underlying the algebraic solution of the corresponding quadratic algebra.*

3.1. Coherent States of a q-Deformed Heisenberg Algebra

We consider an associative algebra with generators a, a^+ and $q^{\pm N}$ subjected to defining relations

$$aa^+ - qa^+a = 1, \qquad q^N a^+ = qa^+ q^N, \qquad q^N a = q^{-1} a q^N$$

where $0 < q < 1$ is a real parameter and $a^+a = \frac{1-q^N}{1-q} \equiv [N]$. A Fock representation is obtained in a Hilbert space spanned by the orthonormal basis $\frac{(a^+)^n}{\sqrt{[n]!}}|0\rangle = |n\rangle$, $n = 0, 1, 2, \ldots$, and $\langle n|n'\rangle = \delta_{nn'}$

$$a|0\rangle = 0, \qquad a|n\rangle = [n]^{1/2}|n-1\rangle, \qquad a^+|n\rangle = [n+1]^{1/2}|n+1\rangle. \quad (9)$$

The Hilbert space consists of all elements $|f\rangle = \sum_{n=0}^{\infty} f_n |n\rangle$ with complex f_n and amplitudes which have a finite norm with respect to the scalar product $\langle f|f\rangle = \sum_{n=0}^{\infty} |f_n|^2$. The q-deformed oscillator algebra has a **Bargmann–Fock representation** on the Hilbert space of entire analytic functions.

Generalized or q-deformed coherent states [1] are defined as the eigenstates of the deformed annihilation operator a and are labelled by a continuous (in general complex) variable z

$$a|z\rangle = z|z\rangle, \qquad |z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]!}} |n\rangle.$$

These vectors belong to the Hilbert space for $|z|^2 < [\infty] = \frac{1}{1-q}$. The scalar product of two coherent states for different values of the parameter z is non-vanishing

$$\langle z|z'\rangle = \sum_{0}^{\infty} \frac{(\bar{z}z')^n}{[n]!} = \mathrm{e}_q^{\bar{z}z'}$$

and they can be properly normalized with the help of the q-exponent on the RHS of (3)

$$|z\rangle = \exp_q\left(-\frac{|z|^2}{2}\right) \exp_q(za^+)|0\rangle.$$

The q-deformed coherent states reduce to the conventional coherent states of a one-dimensional Heisenberg algebra in the limit $q \to 1^-$. These generalized coherent states carry the basic characteristics of the conventional ones, namely continuity and completeness (resolution of unity $I = \int |z\rangle\langle z| \exp_q(-|z|^2)\, \mathrm{d}_q^2 z$). Hence one can expand any state in the coherent states $|f\rangle = \int \mathrm{d}_q^2 |z\rangle \exp_q(-|z|^2)\langle \bar{z}|f\rangle$ and thus to obtain

$$\begin{aligned} \langle \bar{z}|a^+|f\rangle &= zf(z) \\ \langle \bar{z}|a|f\rangle &= D_q f(z) \\ \langle \bar{z}|N|f\rangle &= z\frac{\mathrm{d}}{\mathrm{d}z} f(z) \end{aligned}$$

which is the Bargmann–Fock representation of the deformed oscillators and the number operator.

3.2. Squeezed States of a Deformed Oscillator Algebra

Proposition 2. *Let a, a^+ and q^N generate a deformed Heisenberg algebra with the equivalent form of defining relations*

$$[a, a^+] = q^N, \qquad q^N a = q^{-1} a q^N, \qquad q^N a^+ = q a^+ q^N. \tag{10}$$

Then there is a two-parameter-dependent linear map to a pair of quasi-oscillators with a "quasiparticle" number operator $\mathcal{N}$

$$A = \mu a + \nu a^+, \qquad A^+ = \bar{\mu} a^+ + \bar{\nu} a. \tag{11}$$

These operators generate a deformed Heisenberg algebra

$$[A, A^+] = q^{\mathcal{N}}, \qquad q^{\mathcal{N}} A = q^{-1} A q^{\mathcal{N}}, \qquad q^{\mathcal{N}} A^+ = q A^+ q^{\mathcal{N}}$$

provided that

$$q^{\mathcal{N}} = (|\mu|^2 - |\nu|^2) q^N.$$

In the limit $q \to 1^-$ the relation between the parameters of the conventional squeezed state is recovered [16]. In the deformed "quasi-oscillator" algebra in the Fock representation space with a vacuum $|0\rangle_s$ a normalizable coherent state $|\zeta\rangle_s$ is the eigenvector of the annihilation operator A

$$|\zeta\rangle_s = \mathrm{e}_q^{-\frac{1}{2}|\zeta|^2} \mathrm{e}_q^{\zeta A^+} |0\rangle_s.$$

In order to generate a deformed squeezed state directly one needs of course to explicitly construct an operator $S_q(\mu, \nu)$, the q-analogue of the squeezed operator whose transformation of the oscillators amounts to the linear map in equation (11), $S_q a S_q^{-1} = A$. This question remains open despite the encouraging fact that the linear transformation has the proper limit $q \to 1^-$.

Proposition 3. *A squeezed state of the deformed creation and annihilation operators is a normalized solution of the eigenvalue equation*

$$(\mu a + \nu a^+)|\zeta, \mu, \nu\rangle_s = \zeta|\zeta, \mu, \nu\rangle_s = A|\zeta\rangle_s.$$

This proposition is motivated by the analogy with the non-deformed case [13, 16] and by the fact that such normalized eigenstate vectors of the written above linear combination of q-deformed oscillators appear in the solution of the boundary problem of a many-particle non-equilibrium system.

To show the effects of squeezing we consider the Hermitian quadrature operators

$$x = \frac{1}{\sqrt{2}}(a^+ + a), \qquad p = \frac{\mathrm{i}}{\sqrt{2}}(a^+ - a) \tag{12}$$

where the boson operators obey the relations of the form (10) and consequently the operators x and p satisfy the deformed canonical commutation relation $[x, p] = \mathrm{i}q^N$. The variances $(\delta x)^2 = \langle (x - \langle x \rangle)^2 \rangle$ and similarly $(\delta p)^2 = \langle (p - \langle p \rangle)^2 \rangle$ in any state obey a generalized Heisenberg–Robertson inequality of the form

$$(\delta x)^2 (\delta p)^2 \geq \frac{1}{4} |\langle [x, p] \rangle|^2. \tag{13}$$

The variances in the deformed coherent states $|z\rangle$ are equal

$$(\delta x)^2 = (\delta p)^2 = \frac{1}{2} \exp_q \left((q-1)|z|^2 \right). \tag{14}$$

The deformed coherent states are thus states of equal uncertainties only. Minimum-uncertainty states are labelled by the value z for which the q-exponent in (14) has a minimum. In the limit $q \to 1^-$ for $0 < |z|^2 < \infty$ the equality of the undeformed uncertainties in the Glauber coherent states is recovered. Thus any deformed coherent state $|z, q\rangle$ for $0 \le q < 1$ and $0 < |z|^2 < \frac{1}{1-q}$ is a **Robertson intelligent state** and is sometimes referred to in the literature as a squeezed state in the sense of weak squeezing [3], i.e., $(\delta x)^2 < \frac{1}{2}$, and simultaneously $(\delta p)^2 < \frac{1}{2}$. We proceed with the discussion of the algebraic states $|\zeta, \mu, \nu, q\rangle$ which reveal stronger squeezing properties, generalizing thus the undeformed case. We show that the deformed uncertainties with respect to these states, analogously to the conventional case, are not equal. For the purpose we write the inverse of the linear map in (11)

$$\begin{aligned} a &= \frac{\bar{\mu}}{|\mu|^2 - |\nu|^2} A - \frac{\nu}{|\mu|^2 - |\nu|^2} A^+ \\ a^+ &= \frac{-\bar{\nu}}{|\mu|^2 - |\nu|^2} A + \frac{\mu}{|\mu|^2 - |\nu|^2} A^+ \end{aligned}$$

where $|\mu|^2 - |\nu|^2 \neq 0$, being the Jacobian of the linear transformation (11). Exploring the eigenvalue properties of the normalized coherent eigenstates of A

$$\begin{aligned} &\langle \zeta | A | \zeta \rangle_s = \zeta \\ &\langle \zeta | A^+ | \zeta \rangle_s = \bar{\zeta} \\ &\langle \zeta | q^{\mathcal{N}} | \zeta \rangle_s = \langle \zeta | q^{\zeta \frac{d}{d\zeta}} | \zeta \rangle_s = e_q^{(q-1)|\zeta|^2} \end{aligned}$$

we calculate the corresponding mean values. This yields a non-equality of the q-deformed uncertainties which read explicitly

$$\begin{aligned} (\delta x)^2 &= \frac{1}{2} \frac{|\mu - \nu|^2}{(|\mu|^2 - |\nu|^2)^2} \exp_q \left((q-1)|\zeta|^2 \right) \\ (\delta p)^2 &= \frac{1}{2} \frac{|\mu + \nu|^2}{(|\mu|^2 - |\nu|^2)^2} \exp_q \left((q-1)|\zeta|^2 \right). \end{aligned} \tag{15}$$

We now recall the known definition [9, 14] of a squeezed state requiring for one of the variances to be smaller than the equal uncertainties common minimal value determined by the equality (14) as the minimum in the variable z of the q-exponential function. From the analysis of the function $\frac{1}{2} \exp_q \left((q-1)|\zeta|^2 \right)$ it follows that the minimum of this function is the finite limit $\frac{1}{2} \frac{1}{(1+(1-q))_q^\infty}$ for $|\zeta|^2 \to \frac{1}{1-q}$. Hence according to the expressions (15) $|\zeta\rangle$ is a **squeezed state** if either

$$\frac{1}{2}\frac{|\mu-\nu|^2}{(|\mu|^2-|\nu|^2)^2}\exp_q\left((q-1)|\zeta|^2\right) < \frac{1}{2}\frac{1}{(1+(1-q))_q^\infty}$$

which is satisfied provided the parameters (in general complex) μ, ν of the linear transformation (11) are chosen in such a way that

$$0 < \frac{|\mu-\nu|^2}{(|\mu|^2-|\nu|^2)^2} < \frac{(1+(1-q)^2|\zeta|^2)_q^\infty}{(1+(1-q))_q^\infty} \tag{16}$$

and thus the criterion

$$(\delta x)^2 < (\delta x)^2_{\min} \tag{17}$$

holds. The ratio at the very RHS of (16) is the basic hypergeometric series ${}_1\Phi_0\left((1-q)|\zeta|^2; q, (q-1)\right)$. Alternatively from equation (15)

$$\frac{1}{2}\frac{|\mu+\nu|^2}{(|\mu|^2-|\nu|^2)^2}\exp_q\left((q-1)^2|\zeta|^2\right) < \frac{1}{2}\frac{1}{(1+(1-q))_q^\infty}$$

is satisfied if

$$0 < \frac{|\mu+\nu|^2}{(|\mu|^2-|\nu|^2)^2} < \frac{(1+(1-q)^2|\zeta|^2)_q^\infty}{(1+(1-q))_q^\infty} \tag{18}$$

which gives

$$(\delta p)^2 < (\delta p)^2_{\min}. \tag{19}$$

For $0 < q < 1$ the values $\mu = \pm\nu$ are not admissible. The inequality (17) (or (19)) together with the condition (16) (or (18)) for the parameters μ, ν, ζ, q define the eigenstates $|\zeta\rangle$ of the linear combination $\mu a + \nu a^+$ of the deformed boson operators as generalized squeezed states. In the limit $q \to 1^-$ the corresponding expressions for the x, p uncertainties with respect to the conventional harmonic oscillator squeezed states [16] are recovered.

In the deformed uncertainty relation (13) the variances of the operators x and p enter. For two Hermitian operators a third second moment, their covariance in any state, is defined by

$$\delta(xp) = \frac{1}{2}\langle xp + px\rangle - \langle x\rangle\langle p\rangle.$$

As can be readily verified the covariance $\delta(xp)$ of the quadratures (12) x and p in the deformed boson oscillator coherent state $|z\rangle$ is equal to zero. If we calculate now the x–p covariance in the deformed states $|\zeta, \mu, \nu\rangle$ we obtain

$$\delta xp = \frac{\mathrm{Im}(\mu\bar{\nu})}{|\mu|^2-|\nu|^2}|\langle[x,p]\rangle_s| = \frac{\mathrm{Im}(\mu\bar{\nu})}{(|\mu|^2-|\nu|^2)^2}\exp_q\left((q-1)|\zeta|^2\right). \tag{20}$$

As seen from (20) the x–p covariance in the deformed squeezed states for complex μ, ν is not zero. It vanishes in the particular case of real μ, ν. For Hermitian operators with a nonvanishing covariance the Robertson–Heisenberg uncertainty relation becomes the Schrödinger inequality in any state

$$(\delta x)^2(\delta p)^2 - (\delta xp)^2 \geq \frac{1}{4}|\langle [x,p]\rangle|^2. \tag{21}$$

One can further verify that the three second moments in the deformed squeezed states as given by equations (15) and (21) satisfy the equality

$$(\delta x)^2(\delta p)^2 - (\delta xp)^2 = \frac{1}{4}|\langle \zeta[x,p]\zeta\rangle|^2.$$

The q-deformed squeezed states $|\zeta,\mu,\nu\rangle$ thus minimize the Schrödinger uncertainty relation for the deformed quadratures and are, in fact, generalized Schrödinger intelligent states [14].

3.3. Deformed Squeezed and Coherent State Solution of the Boundary Problem for the n-Species Process

The algebra (7) for the n-species open asymmetric exclusion process of a diffusion system coupled at both boundaries to external reservoirs of particles of fixed density is solved by the deformed oscillators (6) and the shifted deformed oscillators (5) with the following relations for the rates

$$q = \frac{g_{0,n-1}}{g_{n-1,0}}, \qquad q_{kl} = \frac{g_{kl}}{g_{lk}}, \qquad q_k = \frac{g_{k0}}{g_{0k}} = \frac{g_{n-1,k}}{g_{k,n-1}}$$

and

$$g_k = g_{0k} = g_{k,n-1}, \qquad g_{0k} - g_{k0} = g_{k,n-1} - g_{n-1,k} = g_{0,n-1} - g_{n-1,0}.$$

For the phase transition inducing boundary processes, when a particle of type k is added with a rate L_k^0 and removed with a rate L_0^k at the left end of the chain and when it is removed with a rate R_0^k and added with a rate R_k^0 at the right end of the chain, the systems of equations for the boundary vectors are reduced to the algebraic constraints

$$\langle w|L_0^k D_k = \langle w|L_k^0 D_0, \qquad R_0^k D_k|v\rangle = R_k^0 D_0|v\rangle$$

and to the pair of equations

$$\langle w|(L_{n-1}^0 D_0 - L_0^{n-1} D_{n-1}) = \langle w|, \qquad (R_0^{n-1} D_{n-1} - R_{n-1}^0 D_0)|v\rangle = |v\rangle. \tag{22}$$

Making use of the explicit solution for D_{n-1} and D_0 as shifted deformed oscillators (with $x_0 = -x_1 = 1$), we rewrite equations (22) as

$$(R_0^{n-1} a_0 - R_{n-1}^0 a_0^+)|v\rangle = \sqrt{1-q}\left(g_{n-1,0} - \frac{R_0^{n-1} - R_{n-1}^0}{1-q}\right)|v\rangle$$

$$\langle w|(L_{n-1}^0 a_0^+ - L_0^{n-1} a_0) = \langle w|\left(g_{n-1,0} - \frac{L_{n-1}^0 - L_0^{n-1}}{1-q}\right)\sqrt{1-q}.$$

The latter equations determine the boundary vectors as squeezed coherent states of the deformed boson operators a_0, a_0^+ corresponding to the eigenvalues

$$
\begin{aligned}
v &= \sqrt{1-q}\left(g_{n-1,0} - \frac{R_0^{n-1} - R_{n-1}^0}{1-q}\right) \\
w &= \sqrt{1-q}\left(g_{n-1,0} - \frac{L_{n-1}^0 - L_0^{n-1}}{1-q}\right).
\end{aligned}
\tag{23}
$$

The explicit form of these vectors is readily written, namely

$$
\langle w| = \langle n| \sum_{n=0}^{\infty} \frac{w^n}{\sqrt{[n]!}} \mathrm{e}_q^{-\frac{1}{2}vw}, \qquad |v\rangle = \mathrm{e}_q^{-\frac{1}{2}vw} \sum_{n=0}^{\infty} \frac{v^n}{\sqrt{[n]!}} |n\rangle.
$$

Thus, the left and right boundary vectors are squeezed coherent states of the shifted deformed annihilation and creation operators D_{n-1} and D_0, associated with the non-zero boundary parameters x_{n-1} and x_0, and with eigenvalues depending on the right and left boundary rates

$$
\begin{aligned}
(R_0^{n-1} a_0 - R_{n-1}^0 a_0^+)|v\rangle &= A|v\rangle = v|v\rangle \\
\langle w|(L_{n-1}^0 a_0^+ - L_0^{n-1} a) &= \langle w|A^+ = \langle w|w
\end{aligned}
$$

where the eigenvalues v and w are given by (23). The operators A and A^+ satisfy the same deformed commutation relations as a and a^+, with the only difference that they are not Hermitian-conjugate. From the inverse linear maps, with $R_0^{n-1} L_{n-1}^0 - L_0^{n-1} R_{n-1}^0 \neq 0$, we obtain

$$
\begin{aligned}
a_0 &= \frac{L_{n-1}^0}{R_0^{n-1} L_{n-1}^0 - L_0^{n-1} R_{n-1}^0} A + \frac{R_{n-1}^0}{R_0^{n-1} L_{n-1}^0 - L_0^{n-1} R_{n-1}^0} A^+ \\
a_0^+ &= \frac{R_0^{n-1}}{R_0^{n-1} L_{n-1}^0 - L_0^{n-1} R_{n-1}^0} A^+ + \frac{L_0^{n-1}}{R_0^{n-1} L_{n-1}^0 - L_0^{n-1} R_{n-1}^0} A
\end{aligned}
$$

with the help of which the mean values of the generators D_0, D_{n-1} and the rest ones D_k for $k = 1, 2, \ldots, n-2$ are readily found

$$
\begin{aligned}
\langle w|D_0|v\rangle &= \frac{1}{g_{n-1,0}(R_0^{n-1} L_{n-1}^0 - L_0^{n-1} R_{n-1}^0)} \left(\frac{1}{1-q} + \frac{R_0^{n-1} w + L_0^{n-1} v}{\sqrt{1-q}}\right) \\
\langle w|D_{n-1}|v\rangle &= \frac{1}{g_{n-1,0}(R_0^{n-1} L_{n-1}^0 - L_0^{n-1} R_{n-1}^0)} \left(\frac{1}{1-q} + \frac{R_{n-1}^0 w + L_{n-1}^0 v}{\sqrt{1-q}}\right) \\
\langle w|D_k|v\rangle &= \frac{L_k^0}{L_0^k} \langle w|D_0|v\rangle = \frac{R_k^0}{R_0^k} \langle w|D_0|v\rangle.
\end{aligned}
$$

With these expressions at hand, it is easy to calculate the expectation value of any monomial of the form $\langle w|D_{s_1} D_{s_2} \ldots Ds_L|v\rangle$ (where $D_{s_i} = D_j$ for $i =$

$1, 2, \ldots, L$, $j = 0, 1, 2, \ldots, n-1$), which enters the stationary probability distribution, the current, and the correlation functions. One first makes use of the algebra to bring all generators D_k for $k = 1, 2, \ldots, n-2$ to the very right or to the very left, which results in an expression of the expectation value as a power in D_0 and D_{n-1}. Then one writes the arbitrary power of D_0, D_{n-1} as a normally ordered product of A and A^+ to obtain, upon using the eigenvalue properties of the latter, an expression for the relevant physical quantity in terms of the probability-rate-dependent boundary eigenvalues v and w.

We note that if the boundary processes are such that there are only incoming particles of $(n-1)$th-type at the left boundary and only outgoing $(n-1)$th-type particles at the right boundary, i.e., $L_0^{n-1} = R_{n-1}^0 = 0$ in (22), then the eigenstate equations define the boundary vectors $|v\rangle$ and $\langle w|$ as q-deformed coherent states. The value $q \neq 0$ corresponds to a partially asymmetric while $q = 0$ to a totally asymmetric diffusion in the bulk of the $n-1$-type particle. The deformed oscillator coherent states defined for $0 < q < 1$ and for $q = 0$ provide a unified description of both the partially and the totally asymmetric hopping of a given type of particle.

3.4. The Two-Species Model with Incoming and Outgoing Particles at Both Boundaries

As an example we consider the two-species partially asymmetric simple exclusion process. We simplify the notations, namely at the left boundary a particle can be added with probability $\alpha \mathrm{d}t$ and removed with probability $\gamma \mathrm{d}t$, and at the right boundary it can be removed with probability $\beta \mathrm{d}t$ and added with probability $\delta \mathrm{d}t$. The system is described by the configuration set $s_1, s_2, \ldots, s_L$ where $s_i = 0$ if a site $i = 1, 2, \ldots, L$ is empty and $s_i = 1$ if a site i is occupied by a particle. The particles hop with a probability $g_{01}\mathrm{d}t$ to the left and with a probability $g_{10}\mathrm{d}t$ to the right, where without loss of generality we can choose the right probability rate $g_{10} = 1$ and the left probability rate $g_{01} = q$. The quadratic algebra $D_1D_0 - qD_0D_1 = D_0 + D_1$ is solved by a pair of shifted deformed oscillators a, a^+ (see equation (5)). The boundary conditions have the form

$$(\beta D_1 - \delta D_0)|v\rangle = |v\rangle, \qquad \langle w|(\alpha D_0 - \gamma D_1) = \langle w|.$$

For a given configuration $(s_1, s_2, \ldots, s_L)$ the stationary probability is given by the expectation value

$$P(s) = \frac{\langle w|D_{s_1}D_{s_2}\ldots D_{s_L}|v\rangle}{Z_L}$$

where $D_{s_i} = D_1$ if a site $i = 1, 2, \ldots, L$ is occupied and $D_{s_i} = D_0$ if a site i is empty and $Z_L = \langle w|(D_0 + D_1)^L|v\rangle$ is the normalization factor to the stationary probability distribution. Within the matrix–product ansatz, one can also evaluate physical quantities such as the current J through a bond between site i and site $i+1$, the mean density $\langle s_i \rangle$ at a site i, the two-point correlation function $\langle s_i s_j \rangle$

$$
\begin{aligned}
J &= \frac{Z_{L-1}}{Z_L} \\
\langle s_i \rangle &= \frac{\langle w|(D_0+D_1)^{i-1}D_1(D_0+D_1)^{L-i}|v\rangle}{Z_L} \\
\langle s_i s_j \rangle &= \frac{\langle w|(D_0+D_1)^{i-1}D_1(D_0+D_1)^{j-i-1}D_1(D_0+D_1)^{L-j}|v\rangle}{Z_L}
\end{aligned}
$$

and higher correlation functions. In terms of the deformed boson operators the boundary conditions read:

$$
\begin{aligned}
(\beta a - \delta a^+)|v\rangle &= \sqrt{1-q}\left(1-\frac{\beta-\delta}{1-q}\right)|v\rangle \\
\langle w|(\alpha a^+ - \gamma a) &= \langle w|\left(1-\frac{\alpha-\gamma}{1-q}\right)\sqrt{1-q}.
\end{aligned}
$$

Hence, the boundary vectors $|v\rangle$ and $\langle w|$ are squeezed coherent states

$$
(\beta a - \delta a^+)|v\rangle = A|v\rangle = v|v\rangle, \qquad \langle w|(\alpha a^+ - \gamma a) = \langle w|A^+ = \langle w|w
$$

corresponding to the eigenvalues

$$
v(\beta,\delta) = \sqrt{1-q}\left(1-\frac{\beta-\delta}{1-q}\right), \qquad w(\alpha,\gamma) = \sqrt{1-q}\left(1-\frac{\alpha-\gamma}{1-q}\right).
$$

The explicit form of the (unnormalized) vectors is $\langle w| = \sum_{n=0}^{\infty} \frac{w^n(\alpha,\gamma)}{\sqrt{[n]!}}|n\rangle$, $|v\rangle = \sum_{n=0}^{\infty} \frac{v^n(\beta,\delta)}{\sqrt{[n]!}}|n\rangle$. We now use of the inverse transformation

$$
a = \frac{\alpha}{\alpha\beta-\gamma\delta}A + \frac{\delta}{\alpha\beta-\gamma\delta}A^+, \qquad a^+ = \frac{\beta}{\alpha\beta-\gamma\delta}A^+ + \frac{\gamma}{\alpha\beta-\gamma\delta}A.
$$

Hence with $\Delta = \alpha\beta - \gamma\delta \neq 0$

$$
D_0 + D_1 = \frac{2}{1-q} + \frac{\alpha+\gamma}{\Delta\sqrt{1-q}}A + \frac{\beta+\delta}{\Delta\sqrt{1-q}}A^+
$$

and the normalization factor $\langle w|(D_0+D_1)^L|v\rangle$ to the stationary probability distribution can be easily calculated in terms of the operators A and A^+. One has

$$
\begin{aligned}
(D_0+D_1)^L &= \left(\frac{2}{1-q} + \frac{\alpha+\gamma}{\Delta\sqrt{1-q}}A + \frac{\beta+\delta}{\Delta\sqrt{1-q}}A^+\right)^L \\
&= \sum_{m=0}^{L} \frac{L!}{m!(L-m)!}\frac{2^{L-m}\Delta^{-m}}{(1-q)^{L-\frac{m}{2}}}((\alpha+\gamma)A + (\beta+\delta)A^+)^m.
\end{aligned}
$$

The implementation of the eigenvalue properties of the squeezed states $\langle w|$ and $|v\rangle$ and the procedure for normal ordering in A, A^+ results in the formula

$$((\alpha+\gamma)A+(\beta+\delta)A^+)^m = \sum_{k=0}^{[m/2]} S_k^m(\alpha+\gamma)^k(\beta+\delta)^k \sum_{l=0}^{m-2k} \frac{[m-2k]!}{[l]![m-2k-l]!} \times ((\beta+\delta)A^+)^l((\alpha+\gamma)A)^{m-2k-l}.$$

One explores next the eigenvalue properties of the operators A, A^+ with respect to the vectors $|v\rangle$ and $\langle w|$ in order to find the normalization factor $\langle w|(D_0 + D_1)^L|v\rangle = Z_L$ for the stationary probability distribution

$$Z_L = \sum_{m=0}^{L} (\, L\, m\,) \frac{2^{L-m}}{(1-q)^{L-\frac{m}{2}}} \sum_{k=0}^{[m/2]} \sum_{l=0}^{m-2k} S_m^{(k)} \frac{(\alpha+\gamma)^k(\beta+\delta)^k}{(\alpha\beta-\gamma\delta)^m} \times \binom{m-2k}{l}_q ((\beta+\delta)w)^l((\alpha+\gamma)v)^{m-2k-l}.$$

Consequently one obtains directly an expression for the current J. Using the prescription of normal ordering, one can readily calculate the correlation functions and any other quantity of interest like density profiles, etc.

3.5. The Two-Species Asymmetric Simple Exclusion Transfer Process

The model in which one considers only incoming particles at the left boundary and only outgoing particles at the right one is exactly solvable through the matrix-product states approach [4, 2]. We comment on it here just to show that the q-deformed oscillator coherent states provide the most simple and convenient approach to an unified solution of both the partially and the totally asymmetric processes. In the partially asymmetric case the probability rate of hopping to the right is $g_{01} = q$ while the left probability rate is $g_{10} = 1$. The totally asymmetric exclusion process of particles hopping to the right only is obtained for $q = 0$. At the left boundary a particle can be added with a probability αdt and it can be removed at the right boundary with a probability βdt. The quadratic algebra is generated by a unit and two generators obeying the relation:

Case A. PASEP $\quad 0 < q < 1$

$$D_1D_0 - qD_0D_1 = D_0 + D_1.$$

Case B. TASEP $\quad q = 0$

$$D_1D_0 = D_0 + D_1$$

with the same boundary conditions defining in both cases the boundary vectors $\langle w|$ and $|v\rangle$

$$\langle w|D_0 = \langle w|\frac{1}{\alpha}, \qquad D_1|v\rangle = \frac{1}{\beta}|v\rangle.$$

The algebraic solutions (with the corresponding boundary problems) for the partially and for the totally asymmetric cases are of the form of shifted deformed oscillators for a real parameter $0 < q < 1$ and for $q = 0$, respectively.

Case A

$$D_0 = \frac{1}{1-q} + \frac{a^+}{\sqrt{1-q}}, \qquad D_1 = \frac{1}{1-q} + \frac{a}{\sqrt{1-q}}.$$

To solve the boundary problem we choose the vector $|v\rangle$ to be the (unnormalized!) eigenvector of the annihilation operator a for a real value of the parameter v and the vector $\langle w|$ to be the eigenvector (unnormalized and different from the conjugated one) of the creation operator for the real parameter w

$$|v\rangle = \mathrm{e}_q^{-\frac{1}{2}vw}\mathrm{e}_q^{va^+}|0\rangle, \qquad \langle w| = \langle 0|\mathrm{e}_q^{wa}\mathrm{e}_q^{-\frac{1}{2}wv}. \tag{24}$$

The factor $\mathrm{e}_q^{-\frac{1}{2}vw}$ in (24) is due to the condition $\langle w|v\rangle = 1$, which is a convenient choice in physical applications. According to the algebraic solution, these are also eigenvectors of the shifted operators with the corresponding relations of the eigenvalues

$$\frac{1}{\alpha} = \frac{1}{1-q} + \frac{w}{\sqrt{1-q}}, \qquad \frac{1}{\beta} = \frac{1}{1-q} + \frac{v}{\sqrt{1-q}}.$$

Hence the boundary vectors $|v\rangle$ and $\langle w|$ are a subset of the coherent states of the q-deformed Heisenberg algebra, labelled by the positive real parameters $v(\alpha, q)$ and $w(\beta, q)$ defined in (24). The relation of the boundary vectors to the coherent states simplifies the calculation of the stationary probability distribution. Since, according to the algebraic solution

$$\begin{aligned}(D_0 + D_1)^L &= \left(\frac{2}{1-q} + \frac{a^+ + a}{\sqrt{1-q}}\right)^L \\ &= \sum_{m=0}^{L} \frac{L!}{m!(L-m)!}\frac{2^{L-m}}{(1-q)^{L-m}(\sqrt{1-q})^m}(a^+ + a)^m\end{aligned}$$

in order to find the expectation values with respect to the coherent states, one has normally to order the m-th power of the linear combination $a + a^+$, using $aa^+ - qa^+a = 1$. This is achieved with the help of the Stirling numbers

$$(a^+ + a)^m = \sum_{k=0}^{[m/2]} S_m^{(k)} \sum_{l=0}^{m-2k} \frac{[m-2k]!}{[l]![m-2k-l]!}(a^+)^l a^{m-2k-l} \tag{25}$$

where the q-deformed **Stirling numbers** $S_m^{(k)}$ satisfy the recurrence relation

$$S_{m+1}^{(k)} = [k]S_m^{(k)} + S_m^{(k-1)}$$

with $S_m^{(0)} = \delta_{0m}$, $S_m^{(1)} = S_m^{(m)} = 1$ and $S_m^{(m-1)} = \sum_{i=1}^{i=m-1}[i]$. For the correlation functions one also needs the expressions

$$a^k a^+ = q^k a^+ a^k + [k]a^{k-1}, \qquad a(a^+)^k = q^k (a^+)^k a + [k](a^+)^{k-1}.$$

Using these relations one can easily find the relevant physical quantities. For the normalization factor Z^L one obtains

$$\langle w|(D_0 + D_1)^L|v\rangle = \sum_{m=0}^{L} \frac{L!}{m!(L-m)!} \frac{2^{L-m}}{(1-q)^{L-\frac{m}{2}}} \times \sum_{k=0}^{[m/2]} \sum_{l=0}^{m-2k} S_m^{(k)} \frac{[m-2k]!}{[l]![m-2k-l]!} w^l v^{m-2k-l}.$$

Case B

$$D_0 = 1 + a^+_{q=0}, \qquad D_1 = 1 + a_{q=0}. \tag{26}$$

As the algebra itself, the solution (26) and the boundary vectors are also obtained as the limit $q \to 0$ of the q-dependent solution and eigenvectors where the representation of the oscillator operators in (26) is found from equations (9) with $q = 0$, namely $a^+|n\rangle = |n+1\rangle$, $a|n\rangle = |n-1\rangle$ and

$$w = \frac{1-\alpha}{\alpha}, \qquad v = \frac{1-\beta}{\beta}.$$

Hence the boundary vectors have the form

$$\langle w| = \langle n| \sum_{n=0}^{\infty} \left(\frac{1-\alpha}{\alpha}\right)^n \left(\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\alpha\beta}\right)^{1/2}$$

$$|v\rangle = \left(\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\alpha\beta}\right)^{1/2} \sum_{n=0}^{\infty} \left(\frac{1-\beta}{\beta}\right)^n |n\rangle.$$

The physical quantities of the model are readily obtained from the partially asymmetric case in the limit $q \to 0$. Equation (25) becomes simply

$$(a + a^+)^L|_{q=0} = \sum_{k=0}^{[m/2]} S_m^{(k)}|_{q=0} \sum_{l=0}^{m-2k} (a^+_{q=0})^l (a_{q=0})^{m-2k-l}$$

where now $S_{m+1}^{(k)}|_{q=0} = S_m^{(k)}|_{q=0} + S_m^{(k-1)}|_{q=0}$ and $S_m^{(m-1)}|_{q=0} = m-1$. The expression for Z_L becomes

$$\langle w|(D_0 + D_1)^L|v\rangle = \sum_{m=0}^{L} \frac{2^{l-m} L!}{m!(L-m)!} \sum_{k=0}^{[m/2]} \sum_{l=0}^{m-2k} S_m^{(k)}|_{q=0} w^l v^{m-2k-l}.$$

The coherent-state description thus provides a unified solution of the partially and fully asymmetric simple exclusion models.

4. Conclusions

To summarize we have considered a quantum group solution to the n-species stochastic diffusion process and applied the q-deformed squeezed and coherent states to obtain within the matrix-product states approach a boundary problem solution to a multiparticle (general n) open stochastic system of lattice Brownian motion. The deformed coherent states provide a unified description of both the partially and the fully asymmetric cases, the solution of the fully asymmetric one being obtained in the limit $q \to 0$ of the deformation parameter q. The discussed deformed squeezed- and coherent-state solution of the boundary problem for the n-species stochastic diffusion process is proposed as a generalization of the known examples within the matrix-product states approach.

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