Seventh International Conference on
Geometry, Integrability and Quantization
June 2–10, 2005, Varna, Bulgaria
Ivaïlo M. Mladenov and Manuel de León, Editors
SOFTEX, Sofia 2005, pp 140–153

CARTAN FORMS AND SECOND VARIATION FOR CONSTRAINED VARIATIONAL PROBLEMS

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Abstract. Using the Cartan form of first order constrained variational problems introduced earlier we define the second variation. This definition coincides in the unconstrained case with the usual one in terms of the double Lie derivative of the Lagrangian density, an expression, that in the constrained case does not work. The Hessian metric and other associated concepts introduced in this way are compared with those obtained through the Lagrange multiplier rule. The theory is illustrated with an example of isoperimetric problem.

1. Introduction

One of the most characterizing aspects of modern variational calculus on fibred manifolds has been without doubt the promotion to "basic concept" of the so called Cartan form. Essential for the intrinsic formulation of the classical Euler-Lagrange equations and a key object for Noether's theory of infinitesimal symmetries and conservation laws, it is also a fundamental element for the multi-symplectic formulation of the theory. As one can expect, it turns out to have an important role again for the second variation, where, in the unconstrained case, this formula can be obtained both from the double Lie derivative of the Lagrangian density or via the Cartan form [4, 5, 10].

In [3] the authors proposed a Cartan formulation for first order variational problems with differential constraints, where, unlike to the unconstrained case, the Cartan form lies in the second jet of the fibred manifold where the problem is given. From this object, it is possible to introduce a (third order) Euler-Lagrange operator which characterizes critical sections, a Noether theory of infinitesimal symmetries and conservation laws, and a multi-symplectic formulation for these problems. On the other hand, the famous Lagrange multiplier rule appears naturally in this new framework allowing to relate this doctrine with the treatments of the subject given up to now.

In the present work we deal with the second variation from this new point of view. The most remarkable fact is that, in contrast with the unconstrained case, when trying to define a Hessian metric at a critical section from the double Lie derivative of the Lagrangian density, the result depends on the chosen vector fields extending the corresponding tangent vectors along the section. It is not the case, however, for the double Lie derivative of the Cartan form, which allows us to define a generalization of the Hessian metric to the constrained problems under consideration.

The work is organized as follows: we make in Section 2 a brief review of [3] with the object to fix the notation, concepts and basic results. In Section 3 we introduce the Hessian metric, Jacobi operator and related concepts and study its relationship with the Lagrange multiplier rule. Finally, the theory is illustrated in Section 4 with an example of isoperimetric problem.

2. First Variation

Consider the following setup of our problem: Let $p\colon Y \to X$ be a fiber bundle over a n-dimensional manifold X, oriented by a volume element ω. If we denote by J^1Y the bundle of 1-jets $j^1_x s$ of sections $s \in \Gamma(X, Y)$ of p, any $\mathcal{C}^\infty$-function $\mathcal{L}$ on J^1Y defines (where the integral exists) a functional $\mathbb{L}$ and its differential $\delta_s\mathbb{L}$

$$
\begin{aligned}
\mathbb{L}\colon s \in \Gamma(X,Y) &\mapsto \int_X (j^1 s)^* \mathcal{L}\omega \in \mathbb{R} \\
\delta_s\mathbb{L}\colon D_s \in \Gamma^c(X, s^*VY) &\mapsto \int_X (j^1 s)^* L_{j^1 D}\mathcal{L}\omega \in \mathbb{R}
\end{aligned}
\tag{1}
$$

where $VY \to Y$ is the vector bundle of p-vertical vector fields on Y and Γ^c denotes sections with compact support.

When one considers variational problems with differential constraints, except for some few cases (existence of non-regular extremals in semi-Riemannian geometry, for example [11]), the constrained problem can be related to a problem without constraints by incorporating new independent variables (the Lagrange multipliers [2]). All critical sections of the latter project to critical sections of the constrained problem, and conversely (under some general assumptions of regularity). In general, this correspondence is not one-to-one. The relation of the second variation of both problems turns out to be more obscure.

The differential constraint will be defined by a submanifold $S \subseteq J^1Y$. A section $s \in \Gamma(X, Y)$ is said to satisfy the constraint if $j^1 s \subset S$. For convenience we will assume that the constraint submanifold S satisfies the following

Hypothesis 1. *There exists a rank k vector bundle $E \to Y$ and a bundle morphism $\Phi : J^1Y \to E$, with $\mathrm{d}\Phi$ of maximal rank on $V(J^1Y/Y)$, such that*

$$S = \Phi^{-1}(0) \subseteq J^1Y.$$

Locally, if e_α are a local basis of E and $\Phi = \phi^\alpha e_\alpha$, $\phi^\alpha \in \mathcal{C}^\infty(J^1Y)$, this condition states that $S \equiv \{\phi^\alpha = 0\}$ and $\mathrm{rk}\,(\partial\phi^\alpha/\partial y^i_\nu) = k$. Thus S is a fiber bundle over Y. Here and in the following, for local expressions we take a fibered local coordinate system (x^ν, y^i) on Y, for which $\omega = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, and denote by (x^ν, y^j, y^j_ν) the induced coordinate system on J^1Y.

In this situation, we will consider the set $\Gamma_S(X,Y)$ of admissible sections

$$\Gamma_S(X,Y) = \Big\{s \in \Gamma(X,Y)\,;\ \mathrm{im}\, j^1s \subset S\Big\} = \Big\{s \in \Gamma(X,Y)\,;\ \Phi \circ j^1s = 0\Big\}$$

and the space $T_s(\Gamma_S(X,Y))$ of admissible infinitesimal variations at an admissible section $s \in \Gamma_S(X,Y)$

$$\begin{aligned} T_s(\Gamma_S(X,Y)) &= \{D_s \in \Gamma(X, s^*VY)\,;\ j^1D_s \text{ tangential to } S\} \\ &= \{D_s \in \Gamma(X, s^*VY)\,;\ j^1D_s(\Phi) = 0\} \end{aligned}$$

where derivatives of E-valued objects are taken with respect to a given connection ∇_E on the bundle E. It is clear that, if $\{s_t\}$ is a variation of $s = s_0$ with admissible sections, then $(\mathrm{d}/\mathrm{d}t)|_{t=0}\, s_t \in T_s(\Gamma_S(X,Y))$. It is also known in the case of mechanics ($X = \mathbb{R}$) that for regular solutions of $j^1s \subset S$, any $D_s \in T_s(\Gamma_S(X,Y))$ is induced by a variation by admissible sections [7].

Definition 1. *An admissible section $s \in \Gamma_S(X,Y)$ is critical for the constrained variational problem defined by $\mathcal{L}\omega$, $S \subseteq J^1Y$ if $\delta_s\mathbb{L}$ vanishes on any admissible infinitesimal variation with compact support $D_s \in T^c_s(\Gamma_S(X,Y))$.*

In order to characterize the set of critical sections of this problem by means of a system of partial differential equations, in the case where $S = J^1Y$, we may use any of the geometrical setups leading to the corresponding Euler equations [4, 5, 6, 10]. For constrained variational problems, however, one needs geometric objects that parallel the ones used for the unconstrained case. We summarize here the main objects appearing in the theory of first variation, as given in [3]

- The structure 1-form of the jet bundle J^1Y, with values on the vertical bundle VY, whose local expression is

  $$\theta = (\mathrm{d}y^i - y^i_\nu \mathrm{d}x^\nu) \otimes (\partial/\partial y^i)$$

  and associated to any E-valued horizontal n-form $\Phi\omega$.
- The momentum form Ω_Φ, a $(n-1)$-form on J^1Y with values on the bundle $VY^* \otimes E$, and local expression

  $$\Omega_\Phi = (\partial\phi^\alpha/\partial y^i_\nu)\omega_\nu \otimes \mathrm{d}y^i \otimes e_\alpha$$

where $\omega_\nu = i_{(\partial/\partial x^\nu)}\omega$.

- The Cartan form Θ_Φ, a E-valued n-form on J^1Y defined by the equation $\Theta_\Phi = \theta\bar{\wedge}\Omega_\Phi + \Phi\omega$, where $\bar{\wedge}$ stands for the wedge product associated to the duality pairing of VY with VY^*.
- The Euler–Lagrange form $\mathbb{E}_\Phi$, a n-form on J^1Y with values on $VY^* \otimes E$ such that $\mathrm{d}\Theta_\Phi = \theta\bar{\wedge}\mathbb{E}_\Phi$. In a local coordinate system
$$\begin{aligned}\mathbb{E}_\Phi =&(\partial\phi^\alpha/\partial y^i + \bar{\gamma}^\alpha_{i\beta}\phi^\beta)\omega \otimes \mathrm{d}y^i \otimes e_\alpha - \mathrm{d}^{\nabla_E}((\partial\phi^\alpha/\partial y^i_\nu)e_\alpha) \wedge \omega_\nu \otimes \mathrm{d}y^i\\ &+ \bar{\Gamma}^j_{ki}(\partial\phi^\alpha/\partial y^j_\nu)(\mathrm{d}y^k - y^k_\mu \mathrm{d}x^\mu) \wedge \omega_\nu \otimes \mathrm{d}y^i \otimes e_\alpha\end{aligned}$$
where the connections ∇_E on E and ∇ on the bundle VY are with vanishing vertical torsion ($\bar{\Gamma}^k_{ij} = \bar{\Gamma}^k_{ji}$, where $(\partial/\partial y^i)^\nabla(\partial/\partial y^j) = \bar{\Gamma}^k_{ij}(\partial/\partial y^k)$ and $(\partial/\partial y^i)^{\nabla_E} e_\alpha = \bar{\gamma}^\beta_{i\alpha} e_\beta$). It must be noted that the structure form θ vanishes when restricted to holonomic sections: $(j^1s)^*\theta = 0$. Therefore the Euler–Lagrange form on J^1Y, when restricted to any admissible section j^1s, does not depend on the choice of the connection ∇ and allows to define the Euler–Lagrange operator $\mathcal{E}_\Phi$, VY^*-valued section on J^2Y
$$\mathcal{E}_\Phi(j^2_x s) \otimes \omega = (j^1s)^*_x \mathbb{E}_\Phi .$$

From the definition, critical sections are those for which $\int_X \langle \mathcal{E}_\mathcal{L}(j^2s), D_s\rangle\omega$ vanishes when $D_s \in T^c_s(\Gamma_S(X,Y))$. As this space is not a $C^\infty(X)$-module, we may not use the main lemma of the calculus of variations to conclude that $\langle \mathcal{E}_\mathcal{L}(j^2s), D_s\rangle$ vanishes for any $D_s \in T^c_s(\Gamma_S(X,Y))$. With a stronger hypothesis on the constraint submanifold, however, we may parameterize $T^c_s(\Gamma_S(X,Y))$ as the image of a differential operator, allowing us at the same time to generalize the Cartan form from the calculus of variations without constraints.

Hypothesis 2. *On an open subset of J^2Y, dense in S, there exists a section $N \in \Gamma(J^2Y, E^* \otimes VY)$, which is a solution of the system of linear equations*
$$\Omega_\Phi \circ N = 0, \qquad \mathcal{E}_\Phi \circ N = \mathrm{Id}_E . \tag{2}$$

Locally, Hypothesis 2 imposes the existence of solutions $N_\beta = N^j_\beta(\partial/\partial y^j) \in \Gamma(J^2Y, VY)$ for the system of linear equations
$$\frac{\partial\phi^\alpha}{\partial y^j_\nu} N^j_\beta = 0, \qquad \left(\frac{\partial\phi^\alpha}{\partial y^j} - \frac{\mathrm{d}}{\mathrm{d}x^\nu}\frac{\partial\phi^\alpha}{\partial y^j_\nu}\right) N^j_\beta = \delta^\alpha_\beta .$$

Theorem 1 ([3]). *Let $N \in \Gamma(J^2Y, E^* \otimes VY)$ be a solution of the system of equations* (2). *For every admissible section $s \in \Gamma_S(X,Y)$, the first order differential operator $P_s\colon \Gamma(X, s^*VY) \to \Gamma(X, s^*VY)$ given by formula*
$$P_s(D_s) = D_s - N_s \circ (j^1 D_s)\Phi$$

where $N_s = N(j^2 s)$, $D_s \in \Gamma(X, s^*VY)$, *is a projector of* $\Gamma(X, s^*VY)$ *onto the* $\mathbb{R}$*-subspace* $T_s(\Gamma_S(X,Y))$ *of admissible infinitesimal variations at* s. *Moreover, for* $\mathcal{E}_s \in \Gamma(X, s^*VY^*)$, $D_s \in \Gamma(X, s^*VY)$

$$\langle \mathcal{E}_s, P_s(D_s)\rangle\omega = \langle P_s^+ \mathcal{E}_s, D_s\rangle\omega + \mathrm{d}(\lambda_{\mathcal{E}_s} \circ \Omega_\Phi(s)(D_s))$$

where $P_s^+\mathcal{E}_s \otimes \omega = \mathcal{E}_s \otimes \omega + \lambda_{\mathcal{E}_s} \circ \mathcal{E}_\Phi(s) \otimes \omega - \mathrm{d}\lambda_{\mathcal{E}_s} \bar{\wedge} \Omega_\Phi(s)$ *and* $\lambda_{\mathcal{E}_s} = -\mathcal{E}_s \circ N_s \in \Gamma(J^2Y, E^*)$.

The condition that $\int_X \langle \mathcal{E}_\mathcal{L}(j^2s), D_s\rangle\omega$ should vanish for $D_s \in T_s^c(\Gamma_s(X,Y))$, together with the parameterization of this space by P_s and the expression of the adjoint operator P_s^+ suggests to use $s \mapsto \widetilde{\mathcal{E}}_\mathcal{L}(j^3s) = P_s^+\mathcal{E}_\mathcal{L}(j^2s)$ as Euler–Lagrange operator for the constrained problem.

In this situation, an analogous approach to the one described for problems without constraints, leads to the following

Definition 2. *Given a constrained variational problem satisfying Hypothesis* 1 *and Hypothesis* 2 *and a solution* $N \in \Gamma(J^2Y, E^* \otimes VY)$, *we shall call Cartan form of the constrained variational problem the ordinary* n*-form on* J^2Y

$$\widetilde{\Theta}_\mathcal{L} = \Theta_\mathcal{L} + \lambda_{\mathcal{E}_\mathcal{L}} \circ \Theta_\Phi$$

where $\Theta_\mathcal{L}$ *and* Θ_Φ *are the corresponding Cartan forms for* $\mathcal{L}$ *and* Φ, $\lambda_{\mathcal{E}_\mathcal{L}} = -\mathcal{E}_\mathcal{L} \circ N$ *and* $\circ$ *denotes the natural duality product.*

The definition is justified by the following expressions that relate the main objects of the theory

$$\begin{aligned} \widetilde{\Theta}_\mathcal{L} &= \theta\bar{\wedge}(\Omega_\mathcal{L} + \lambda_{\mathcal{E}_\mathcal{L}} \circ \Omega_\Phi) + \mathcal{L}\omega + \lambda_{\mathcal{E}_\mathcal{L}} \circ \Phi\omega \\ \mathrm{d}\widetilde{\Theta}_\mathcal{L} &= \theta\bar{\wedge}\widetilde{\mathbb{E}}_\mathcal{L} + \mathrm{d}\lambda_{\mathcal{E}_\mathcal{L}}\bar{\wedge}\Phi\omega, \qquad (j^2s)^*\widetilde{\mathbb{E}}_\mathcal{L} = \widetilde{\mathcal{E}}_\mathcal{L}(j^3s) \otimes \omega \end{aligned} \tag{3}$$

where $\widetilde{\mathbb{E}}_\mathcal{L} = \mathbb{E}_\mathcal{L} + \lambda_{\mathcal{E}_\mathcal{L}} \circ \mathbb{E}_\Phi - \mathrm{d}\lambda_{\mathcal{E}_\mathcal{L}}\bar{\wedge}\Omega_\Phi$ (for a more detailed description, see [3]). Using these expressions and in the same manner as for the problems without constraints, we may derive the following formula for the first variation

$$\begin{aligned} (j^1s)^*L_{j^1D}\mathcal{L}\omega &= \langle \widetilde{\mathcal{E}}_\mathcal{L}(j^3s), D_s^{\mathrm{v}}\rangle\omega + \mathrm{d}\left((j^2s)^* i_{j^2D}\widetilde{\Theta}_\mathcal{L}\right) \\ D_s^{\mathrm{v}} &= (j^1s)^*\theta(j^1D) \end{aligned} \tag{4}$$

for any admissible section $s \in \Gamma_S(X,Y)$ and any vector field $D \in \mathfrak{X}(Y)$ with $D|_{j^1s}$ tangential to S.

Moreover, we have

$$\delta_s\mathbb{L}(P_s(D_s)) = \int_X \langle \widetilde{\mathcal{E}}_\mathcal{L}(j^3s), D_s\rangle\omega = \int_X (j^2s)^* i_{j^2D_s}\,\mathrm{d}\widetilde{\Theta}_\mathcal{L}$$

for any vector field $D_s \in \Gamma^c(X, s^*VY)$ and admissible section $s \in \Gamma_S(X,Y)$.

Theorem 2 ([3]). *An admissible section $s \in \Gamma_S(X,Y)$ is critical for the constrained variational problem if and only if any of the following holds*

- $\widetilde{\mathcal{E}}_{\mathcal{L}}(j^3 s) = 0$; *Euler–Lagrange equation*
- $(j^2 s)^* i_{\bar{D}} \mathrm{d}\widetilde{\Theta}_{\mathcal{L}} = 0$, *for all* $\bar{D} \in \mathfrak{X}(J^2 Y)$ *Cartan equation.*

It must be noted that, for $S = J^1 Y$, this formalism reproduces the well-known results of unconstrained variational calculus.

The classical way to deal with constrained variational problems, generally with a formulation with integral constraints, has been the use of the Lagrange multiplier rule. Considering the Lagrangian $\widehat{\mathcal{L}} = \mathcal{L} + \lambda \circ \Phi$ on the bundle $J^1 Y \times_Y E^*$ (where λ is the natural E^*-valued function defined on $J^1 Y \times_Y E^*$), it is known that critical sections of this (unconstrained) variational problem project to critical sections of the constrained problem. In general the converse does not necessarily hold. In our setting, however

Theorem 3 ([3]). *If the constraint S satisfies the hypothesis HY1 and HY2, the mapping $\Pi\colon (s,\lambda) \in \Gamma(X, Y \times_Y E^*) \mapsto s \in \Gamma(X,Y)$ defines a bijective correspondence between the set of critical sections of the unconstrained variational problem $\widehat{\mathcal{L}} = \mathcal{L} + \lambda \circ \Phi$ on $Y \times_Y E^*$ and that of the constrained variational problem $\mathcal{L}$ with constraint $S = \Phi^{-1}(0)$*

$$\begin{gathered}\Pi\colon (s,\lambda) \in \Gamma_{\mathrm{crit}}(X, Y \times_Y E^*) \leftrightarrow s \in \Gamma_{\mathrm{crit}}(X,Y) \\ (s,\lambda) \in \Gamma_{\mathrm{crit}}(X, Y \times_Y E^*) \Leftrightarrow s \in \Gamma_S(X,Y), \quad \lambda = \lambda_{\mathcal{E}_{\mathcal{L}}}(j^2 s), \quad s \textit{ critical.}\end{gathered} \tag{5}$$

Regarding the Cartan and Euler–Lagrange forms associated to $\widehat{\mathcal{L}}$, one easily finds

$$\Theta_{\widehat{\mathcal{L}}} = \Theta_{\mathcal{L}} + \lambda \circ \Theta_\Phi, \qquad \mathbb{E}_{\widehat{\mathcal{L}}} = \mathbb{E}_{\mathcal{L}} + \lambda \circ \mathbb{E}_\Phi - \mathrm{d}\lambda \bar{\wedge} \Omega_\Phi. \tag{6}$$

The "universal multiplier" obtained from $\lambda_{\mathcal{E}_{\mathcal{L}}} = -\mathcal{E}_{\mathcal{L}} \circ N$, allows to define the mapping

$$\varphi\colon (j^2_x s) \in J^2 Y \mapsto (j^1_x s, \lambda_{\mathcal{E}_{\mathcal{L}}}(j^2_x s)) \in J^1 Y \times E^* \tag{7}$$

which represents a key element to construct the inverse of (5). This morphism allows to relate both Cartan forms: $\widehat{\Theta} = \Theta_{\widehat{\mathcal{L}}}$ of the variational problem in the setting with constraints and $\widetilde{\Theta} = \widetilde{\Theta}_{\mathcal{L}}$ introduced in this chapter

$$\varphi^* \widehat{\Theta} = \widetilde{\Theta}. \tag{8}$$

3. Second Variation

For unconstrained variational problems (constraint submanifold $S = J^1 Y$), going one step further, the question now is: when we have a compact-supported variation $\{s_t\}$ of a critical section $s = s_0$ (thus $(\mathrm{d}/\mathrm{d}t)_{t=0} \mathbb{L}(s_t) = 0$), what happens with the second derivative $\left(\mathrm{d}^2/\mathrm{d}t^2\right)_{t=0}\mathbb{L}(s_t)$? In principle, this second derivative depends on the whole variation $\{s_t\}$, in general on $\left(\mathrm{d}^2/\mathrm{d}t^2\right)_{t=0}(s_t)$ and not only on $D_s^{\mathrm{v}} = (\mathrm{d}/\mathrm{d}t)_{t=0}(s_t)$. In analogy with the finite dimensional theory and considering $D_s^{\mathrm{v}} \in \Gamma^c(X, s^*VY)$ as the natural generalization of the "tangent vector" of the "curve" $\{s_t\}$, if s is critical, we may define (see [5, 10] for example) $\delta_s^2\mathbb{L}$ for any $D_s \in \Gamma^c(X, s^*VY)$ and formulate

Theorem 4. *For any variation $\{s_t\}$ of a critical section $s = s_0 \in \Gamma_S(X,Y)$, if $D_s \in \Gamma^c(X, s^*VY)$ is the corresponding infinitesimal variation, for any vertical extension $D \in \mathfrak{X}(Y)$ of D_s there holds*

$$\begin{aligned}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right)_{t=0}\mathbb{L}(s_t) &= \int_X (j^1s)^* L_{j^1D}L_{j^1D}\mathcal{L}\omega = \int_X (j^1s)^* L_{j^1D}L_{j^1D}\Theta_{\mathcal{L}} \\ &= \int_X (j^1s)^* i_{j^1D}L_{j^1D}\,\mathrm{d}\Theta_{\mathcal{L}} = \int_X \langle (L_{j^2D}\mathcal{E}_{\mathcal{L}})(j^2s), D_s\rangle\omega.\end{aligned} \tag{9}$$

In particular, this last expression allows to compute $\left(\mathrm{d}^2/\mathrm{d}t^2\right)_{t=0}\mathbb{L}(s_t)$ only from the value $D_s \in \Gamma(X, s^*VY)$ of D along s, regardless of the chosen extension D. These formulas lead to the definition of the Hessian bilinear form

$$\begin{aligned}&\mathbb{H}_s\colon \Gamma^c(X, s^*VY) \times \Gamma(X, s^*VY) \to \mathbb{R} \\ &\mathbb{H}_s(D_s, D_s') = \int_X (j^1s)^* i_{j^1D}L_{j^1D'}\,\mathrm{d}\Theta_{\mathcal{L}} \\ &\qquad = \int_X \langle (L_{j^2D_s'}\mathcal{E}_{\mathcal{L}})(j^2s), D_s\rangle\omega\end{aligned} \tag{10}$$

which is a well-defined symmetric bilinear form for any critical section s. This symmetric bilinear form can be related to the second variation by $\delta_s^2\mathbb{L}(D_s) = \mathbb{H}_s(D_s, D_s)$. A sufficient condition for s to be a (local) minimum is that $\mathbb{H}_s$ is positive semi-definite. A key role in the determination of this [9, 12] is played by the second order differential operator (Jacobi operator)

$$\mathcal{J}_s\colon D_s \in \Gamma(X, s^*VY) \mapsto (L_{j^2D_s}\mathcal{E}_{\mathcal{L}})(j^2s) \in \Gamma(X, s^*VY^*). \tag{11}$$

Coming back to our constrained problem, the question now is, if we have a critical section $s \in \Gamma_S(X,Y)$ for the constrained variational problem, it needs not be a critical section for the problem without constraints and the expressions (9) and (10) are not valid anymore to define the Hessian $\mathbb{H}_s$ in $T_s^c(\Gamma_S(X,Y))$. To compute this second derivative in the direction D_s in the constrained case we need to know how to extend the vector field D_s in some way respecting the constraints. However, we know that, for s admissible, $\mathbb{L}(s) = \int_X (j^2s)^*\widetilde{\Theta}_{\mathcal{L}}$. The Cartan form defines the same functional as the Lagrangian density. On the other hand, we know how to characterize critical sections using Cartan equations (Theorem 2). The key idea to pass to the second variation will then be to consider the second variation of the Cartan form and for that purpose we need

Lemma 1. *For any critical section $s \in \Gamma_S(X,Y)$ and vertical vector fields extending $D_s, D'_s \in T_s(\Gamma_S(X,Y))$, $D, D' \in \mathfrak{X}(Y)$ holds*

$$\begin{aligned}(j^2s)^* L_{j^2D} L_{j^2D'} \widetilde{\Theta}_{\mathcal{L}} &= (j^2s)^* i_{j^2D} L_{j^2D'} \mathrm{d}\widetilde{\Theta}_{\mathcal{L}} + \mathrm{d}((j^2s)^* i_{j^2D} L_{j^2D'} \widetilde{\Theta}_{\mathcal{L}}) \\ &= \langle (L_{j^3D'} \widetilde{\mathcal{E}}_{\mathcal{L}})(j^3s), D_s\rangle \omega + \mathrm{d}((j^2s)^* i_{j^2D} L_{j^2D'} \widetilde{\Theta}_{\mathcal{L}}).\end{aligned} \tag{12}$$

Proof: For the first equality, it suffices to consider Cartan's rule $L_{j^2D} = \mathrm{d} \circ i_{j^2D} + i_{j^2D} \circ \mathrm{d}$, then to conmute d with $L_{j^2D'}$ and $(j^2s)^*$ with d. Finally, using (3)

$$\begin{aligned}(j^2s)^* i_{j^2D} L_{j^2D'} \mathrm{d}\widetilde{\Theta}_{\mathcal{L}} &= (j^2s)^* i_{j^2D} \left(L_{j^1D'} \theta \bar{\wedge} \widetilde{\mathbb{E}}_{\mathcal{L}} + \theta \bar{\wedge} L_{j^2D'} \widetilde{\mathbb{E}}_{\mathcal{L}} \right) \\ &\quad + (j^2s)^* i_{j^2D} \left(L_{j^2D'} \mathrm{d}\lambda_{\mathcal{E}_{\mathcal{L}}} \bar{\wedge} \Phi\omega + \mathrm{d}\lambda_{\mathcal{E}_{\mathcal{L}}} \bar{\wedge} j^1D'(\Phi)\omega \right).\end{aligned}$$

Now $L_{j^1D'}\theta$ and θ vanish when restricted to any holonomic section (j^1s), $\widetilde{\mathbb{E}}_{\mathcal{L}}$ vanishes when restricted to any critical section, $\Phi\omega$ vanishes when restricted to any admissible section, and $j^1D'(\Phi)$ vanishes when restricted to j^1s because $D'_s \in T_s(\Gamma_S(X,Y))$. Therefore

$$(j^2s)^* i_{j^2D} L_{j^2D'} \mathrm{d}\widetilde{\Theta}_{\mathcal{L}} = (j^2s)^* i_{j^1D}\theta \circ L_{j^2D'} \widetilde{\mathbb{E}}_{\mathcal{L}} = \langle (L_{j^3D'} \widetilde{\mathcal{E}}_{\mathcal{L}})(j^3s), D_s\rangle \omega \tag{13}$$

where the last equality comes from the fact that the pull-back of $\widetilde{\mathbb{E}}_{\mathcal{L}}$ to J^3Y coincides with $\widetilde{\mathcal{E}}_{\mathcal{L}}$ up to a multiple of the structure form of J^3Y, which vanishes when restricted to j^3s. □

This result allows us to give the following definition

Definition 3. *For any critical section $s \in \Gamma_S(X,Y)$ the bilinear form*

$$\begin{aligned}\widetilde{\mathbb{H}}_s \colon T^c_s(\Gamma_S(X,Y)) \times T_s(\Gamma_S(X,Y)) &\to \mathbb{R} \\ \widetilde{\mathbb{H}}_s(D_s, D'_s) &= \int_X (j^2s)^* i_{j^2D} L_{j^2D'}\, \mathrm{d}\widetilde{\Theta}_{\mathcal{L}} \\ &= \int_X \langle (L_{j^3D'} \widetilde{\mathcal{E}}_{\mathcal{L}})(j^3s), D_s\rangle \omega\end{aligned} \tag{14}$$

shall be called Hessian bilinear form at s of the constrained variational problem.

Proposition 5. *At any critical section $s \in \Gamma_S(X,Y)$ the Hessian bilinear form $\widetilde{\mathbb{H}}_s$ of the constrained variational problem is a well-defined, symmetric bilinear form on $T^c_s(\Gamma_S(X,Y)) \times T_s(\Gamma_S(X,Y))$. Moreover, for any variation $\{s_t\}$ of $s = s_0$ with s_t admissible and $D^{\mathrm{v}}_s = (\mathrm{d}/\mathrm{d}t)|_{t=0}\, s_t \in T^c_s(\Gamma_S(X,Y))$*

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \mathbb{L}(s_t) = \widetilde{\mathbb{H}}_s(D^{\mathrm{v}}_s, D^{\mathrm{v}}_s).$$

Proof: Lemma 1 tells us that the definition of $\widetilde{\mathbb{H}}_s(D_s, D'_s)$ only depends on $D_s \in T_s^c(\Gamma_S(X,Y))$ and not on the choice of D. On the other hand, as s is critical,using Cartan equation (Theorem 2)

$$\begin{aligned}0 &= (j^2 s)^* i_{[j^2 D, j^2 D']} \mathrm{d}\widetilde{\Theta}_{\mathcal{L}} \\ &= (j^2 s)^* i_{j^2 D} L_{j^2 D'} \mathrm{d}\widetilde{\Theta}_{\mathcal{L}} - (j^2 s)^* i_{j^2 D'} L_{j^2 D} \mathrm{d}\widetilde{\Theta}_{\mathcal{L}} + (j^2 s)^* \mathrm{d} i_{j^2 D} i_{j^2 D'} \mathrm{d}\widetilde{\Theta}_{\mathcal{L}}.\end{aligned}$$

Integrating and considering that D_s has compact support one gets

$$\begin{aligned}\widetilde{\mathbb{H}}_s(D_s, D'_s) &= \int_X \langle (L_{j^3 D'} \widetilde{\mathcal{E}}_{\mathcal{L}})(j^3 s), D_s \rangle \omega = \int_X (j^2 s)^* i_{j^2 D} L_{j^2 D'} \, \mathrm{d}\widetilde{\Theta}_{\mathcal{L}} \\ &= \int_X (j^2 s)^* i_{j^2 D'} L_{j^2 D} \, \mathrm{d}\widetilde{\Theta}_{\mathcal{L}} = \int_X \langle (L_{j^3 D} \widetilde{\mathcal{E}}_{\mathcal{L}})(j^3 s), D'_s \rangle \omega\end{aligned} \tag{15}$$

and thus concluding the independence on the choice of D' and the symmetry.
Finally the relation with the second variation is a direct consequence of $\mathbb{L}(s_t) = \int_X (j^2 s_t)^* \widetilde{\Theta}_{\mathcal{L}}$ for admissible sections $s_t \in \Gamma_S(X,Y)$, a standard procedure to bring the derivatives into the the integral, and (12). □

In expression (14) we may find the following differential operator, which we shall call Jacobi operator associated to the constrained variational problem and to the critical section $s \in \Gamma_S(X,Y)$

$$\begin{array}{rccl}\widetilde{\mathcal{J}}_s\colon & T_s(\Gamma_S(X,Y)) & \to & \Gamma(X, s^* VY^*) \\ & D'_s & \mapsto & (L_{j^3 D'_s} \widetilde{\mathcal{E}}_{\mathcal{L}})(j^3 s).\end{array} \tag{16}$$

When trying to determine if $\widetilde{\mathbb{H}}_s$ is positive semi-definite, we are led to consider this operator, satisfying

Proposition 6. *The Jacobi operator $\widetilde{\mathcal{J}}_s$ of the constrained variational problem $(\mathcal{L}, S)$ at any critical section $s \in \Gamma_S(X,Y)$ has the following properties:*

1. *It is the auto-adjoint operator related to the Hessian bilinear form, that is, for any $D_s \in T_s^c(\Gamma_S(X,Y))$ and $D'_s \in T_s(\Gamma_S(X,Y))$, holds $\widetilde{\mathbb{H}}_s(D_s, D'_s) = \int_X \langle D_s, \widetilde{\mathcal{J}}_s(D'_s) \rangle \omega$ and $\int_X \langle D_s, \widetilde{\mathcal{J}}_s(D'_s) \rangle \omega = \int_X \langle \widetilde{\mathcal{J}}_s(D_s), D'_s \rangle \omega$.*
2. *For any variation $\{s_t\}$ of $s = s_0$ by critical sections of the constrained variational problem, the associated infinitesimal variation $D'_s = (\mathrm{d}/\mathrm{d}t)|_{t=0}\, s_t$ satisfies $\widetilde{\mathcal{J}}_s(D'_s) = 0$.*

Proof: The first part is a direct consequence of the definition of $\widetilde{\mathcal{J}}_s$ and (15). The second part comes from the fact that $(j^3 s_t)^* \widetilde{\mathcal{E}}_{\mathcal{L}}$ vanishes if s_t is critical. □

Moreover, the radical of $\widetilde{\mathbb{H}}_s$ can be determined by means of $\widetilde{\mathcal{J}}_s$.

Definition 4. *A vector field $D'_s \in T_s(\Gamma_S(X,Y))$ is a Jacobi vector field at s for the constrained variational problem if $\widetilde{\mathbb{H}}_s(D_s, D'_s) = 0$ for any $D_s \in T_s^c(\Gamma_S(X,Y))$.*

Proposition 7. *A vector field $D'_s \in T_s(\Gamma_S(X,Y))$ is a Jacobi vector field for the constrained variational problem if and only if any of the following holds:*

- $\widetilde{\mathcal{J}}_s(D'_s) = (L_{j^3 D'_s} \widetilde{\mathcal{E}}_{\mathcal{L}})(j^3 s) = 0$ *(Linearized Euler equation)*
- $(j^2 s)^* i_{\bar{D}_s} L_{j^2 D'_s} \mathrm{d}\widetilde{\Theta}_{\mathcal{L}} = 0$*, for all* $\bar{D}_s \in \mathfrak{X}(J^2 Y)$ *defined along* $j^2 s$ *(Linearized Cartan equation).*

Proof: For the first characterization, if $\widetilde{\mathcal{J}}_s(D'_s)$ vanishes, then D'_s clearly belongs to the radical. Conversely, if D'_s belongs to the radical, then $\int_X \langle \widetilde{\mathcal{J}}_s(D'_s), D_s\rangle \omega$ vanishes for any $D_s \in T^c_s(\Gamma_S(X,Y))$. If we have an arbitrary $D_s \in \Gamma^c(X, s^*VY)$, it can be decomposed in $P_s(D_s) \in T^c_s(\Gamma_S(X,Y))$ and $D_s - P_s(D_s)$, which has the form $N_s(e)$. We shall see that $\int_X \langle \widetilde{\mathcal{J}}_s(D'_s), D_s\rangle \omega$ also vanishes if D_s has the form $N_s(e)$: At any point $j^3_x \bar{s}$ with $j^1_x \bar{s} \in S$ and for any $N_{\bar{s}}(e_{\bar{s}(x)})$ holds

$$\begin{aligned}\langle \widetilde{\mathcal{E}}_{\mathcal{L}}(j^3_x \bar{s}), N_{j^2_x \bar{s}}(e_{\bar{s}(x)})\rangle \omega = {} & \langle \mathcal{E}_{\mathcal{L}}(j^2_x \bar{s}), N_{j^2_x \bar{s}}(e_{\bar{s}(x)})\rangle \omega \\ & + \langle \lambda_{\mathcal{E}_{\mathcal{L}}}(j^2_x \bar{s}) \circ \mathcal{E}_{\Phi}(j^2_x \bar{s}), N_{j^2_x \bar{s}}(e_{\bar{s}(x)})\rangle \omega \\ & - (j^2 \bar{s})^* \mathrm{d}\lambda_{\mathcal{E}_{\mathcal{L}}} \bar{\wedge} \Omega_{\Phi}(j^1_x \bar{s}) \circ N_{j^2_x \bar{s}}(e_{\bar{s}(x)}).\end{aligned}$$

Substituting here the expressions: $\lambda_{\mathcal{E}_{\mathcal{L}}}(j^2_x \bar{s}) = -\mathcal{E}_{\mathcal{L}}(j^2_x \bar{s}) \circ N_{j^2_x \bar{s}}$, $\mathcal{E}_{\Phi} \circ N = \mathrm{Id}_E$ and $\Omega_{\Phi} \circ N = 0$, we get: $\langle \widetilde{\mathcal{E}}_{\mathcal{L}}, N\rangle = 0$ on S.
Therefore, in

$$\langle \widetilde{\mathcal{J}}_s(D'_s), N_{j^2 s}(e)\rangle = (j^3 s)^* L_{j^3 D'} \langle \widetilde{\mathcal{E}}_{\mathcal{L}}, N(e)\rangle - (j^3 s)^* \langle \widetilde{\mathcal{E}}_{\mathcal{L}}, L_{j^3 D'} N(e)\rangle$$

the first term vanishes if $j^1 D'_s$ tangential to S and the second one vanishes if s is critical ($\widetilde{\mathcal{E}}_{\mathcal{L}}(j^3 s) = 0$). We conclude that $\langle \widetilde{\mathcal{J}}_s(D'_s), N_s(e)\rangle = 0$.
Thus if $D'_s \in T_s(\Gamma_S(X,Y))$ is in the right radical of $\widetilde{\mathbb{H}}_s$, then $\int_X \langle \widetilde{\mathcal{J}}_s(D'_s), D_s\rangle \omega$ vanishes for any $D_s \in \Gamma^c(X, s^*VY)$ with compact support. Applying now the main lemma of calculus of variations, we conclude $\widetilde{\mathcal{J}}_s(D'_s) = 0$.
The second characterization can be derived from the first one and (13), which holds also for arbitrary $\bar{D}_s$ and $D'_s \in T_s(\Gamma_S(X,Y))$. □

Using the Lagrange multiplier rule, we know there is a bijective mapping (5), induced by φ defined in (7), relating the critical sections of both problems. Considering one of these sections $s \in \Gamma_{\mathrm{crit}}(X,Y)$, besides the Jacobi differential operator (16), we have the Jacobi operator (11) associated to $\widehat{\mathcal{L}}$ and $(s, \lambda_{\mathcal{E}_{\mathcal{L}}}(s))$ from the unconstrained theory

$$\widehat{\mathcal{J}}_{(s,\lambda)} \colon \Gamma(X, s^*VY \times s^*E^*) \to \Gamma(X, s^*VY^* \times s^*E).$$

Theorem 8. *For any critical section s and admissible infinitesimal variations $D_s \in T^c_s(\Gamma_S(X,Y))$, $D'_s \in T_s(\Gamma_S(X,Y))$, there holds*

$$\widetilde{\mathcal{J}}_s(D'_s) = \varphi^* \left(\widehat{\mathcal{J}}_{(s,\lambda_s)}(\varphi_* D'_s) \right) \in \Gamma(X, s^*VY) \tag{17}$$

$$\widetilde{\mathbb{H}}_s(D_s, D'_s) = \widehat{\mathbb{H}}_{(s,\lambda_s)}(\varphi_* D_s, \varphi_* D'_s) \tag{18}$$

where $\lambda_s := \lambda_{\mathcal{E}_\mathcal{L}}(j^2 s) \in \Gamma(X, s^* E^*)$ *and* $(\varphi_* D_s)(x) := \varphi_*(j^2 D_s(x))$.

Proof: From (13) (both in the constrained and not constrained case) we have for any (D_s, K_s) and (D'_s, K'_s) in $\Gamma(X, s^*VY \times s^*E^*) = \Gamma(X, s^*VY) \oplus \Gamma(X, s^*E^*)$

$$\begin{aligned} \langle \widetilde{\mathcal{J}}_s(D'_s), D_s\rangle \omega &= (j^2 s)^* i_{j^2 D_s} L_{j^2 D'_s} \mathrm{d}\widetilde{\Theta} \\ \langle \widehat{\mathcal{J}}_{(s,\lambda_s)}(D'_s, K'_s), (D_s, K_s)\rangle \omega &= (j^1(s,\lambda_s))^* i_{j^1(D_s,K_s)} L_{j^1(D'_s,K'_s)} \mathrm{d}\widehat{\Theta}. \end{aligned} \tag{19}$$

Using here the independence on the chosen extensions and the identity (8), we may conclude (17). From this and the definitions (10), (15) one obtains (18). □

A linearized version of Theorem 3 is then

Theorem 9. *For any critical section* (s, λ) *of the variational problem* $\widehat{\mathcal{L}} = \mathcal{L} + \lambda \circ \Phi$, *the mapping*

$$\begin{array}{rccc} \Pi_*: & \Gamma(X, s^*VY \oplus s^*E^*) & \mapsto & \Gamma(X, s^*VY) \\ & (D_s, K_s) & \to & D_s \end{array}$$

defines a bijective correspondence between the space of Jacobi vector fields at (s, λ) *for the Lagrangian* $\widehat{\mathcal{L}} = \mathcal{L} + \lambda \circ \Phi$ *and the space of Jacobi vector fields at* s *of the constrained variational problem defined by the Lagrangian* $\mathcal{L}$ *and the constraint* $S = \Phi^{-1}(0)$

$$\begin{aligned} \widehat{\mathcal{J}}_{(s,\lambda)}(D'_s, K'_s) = 0 \Leftrightarrow D'_s \in T_s(\Gamma_S(X, Y)), \quad \widetilde{\mathcal{J}}_s(D'_s) = 0 \\ (D'_s, K'_s) = \varphi_* D'_s. \end{aligned} \tag{20}$$

Proof: From (17), if the right hand side in (20) holds, we conclude that for any $D_s \in \Gamma(X, s^*VY)$, $\langle \widehat{\mathcal{J}}_{(s,\lambda)}(D'_s, K'_s), \varphi_* D_s\rangle = 0$. An arbitrary (D_s, K_s) differs from $\varphi_* D_s$ only in a component of the form $(0, K_s)$. For these elements, from (19) and $\widehat{\Theta} = \Theta_\mathcal{L} + \lambda \circ \Theta_\Phi$, one may compute

$$\langle \widehat{\mathcal{J}}_{(s,\lambda_s)}(D'_s, K'_s), (0, K_s)\rangle = \langle (j^1 D'_s)\Phi, K_s\rangle. \tag{21}$$

Thus, if D'_s is admissible, we conclude $\widehat{\mathcal{J}}_{(s,\lambda_s)} = 0$ when applied to any (D_s, K_s). Conversely, if the left hand side in (20) holds, from (21) follows $(j^1 D'_s)\Phi = 0$ and D'_s is admissible.

In this case, (17) shows that $(D'_s, \bar{K}'_s) = \varphi_*(j^2 D'_s)$ is a Jacobi vector field. The difference between this Jacobi field and (D'_s, K'_s) would then be a Jacobi vector field $(0, \bar{K}_s)$. We see, using the explicit expression in (6) that $\bar{K}_s$ vanishes

$$\begin{aligned} (0, \bar{K}_s) \text{ Jacobi } &\Leftrightarrow 0 = (j(s,\lambda_s))^* L_{j(0,\bar{K}_s)}(\mathbb{E}_{\mathcal{L}\omega} + \lambda \circ \mathbb{E}_{\Phi\omega} - \mathrm{d}\lambda \bar{\wedge} \Omega_{\Phi\omega}) \\ &\Leftrightarrow 0 = (j(s,\lambda_s))^*(\bar{K}_s(\lambda) \circ \mathbb{E}_{\Phi\omega} - \mathrm{d}\bar{K}_s(\lambda) \bar{\wedge} \Omega_{\Phi\omega}) \end{aligned}$$

and that the resulting object is a section of s^*VY. Composing with N, we get

$$(0, \bar{K}_s) \text{ Jacobi} \Rightarrow (j(s, \lambda_s))^*(\bar{K}_s(\lambda)) = 0 \Rightarrow \bar{K}_s = 0$$

thus concluding our proof. □

4. Example

To illustrate the theory, consider the simple example where $X = [0, 1] \subset \mathbb{R}_t$, and $Y = X \times M$ where M is the half-plane $\mathbb{R}_x^+ \times \mathbb{R}_y$. Any section $s \in \Gamma(X, Y)$ can be identified with a parametrized curve $\sigma = (x(t), y(t)) \colon X \to M$. The volume enclosed by the surface of revolution generated by this curve is $\int_0^1 \pi x^2 \dot{y}\, \mathrm{d}t$. The differential constraint we shall impose is that the parametrization of the curve is the element of enclosed volume: $S = \phi^{-1}(0)$ with $\phi \colon J^1Y = X \times TM \to \mathbb{R}$ defined to be $\phi = \pi x^2 \dot{y} - 1$. The lateral area of the surface is given as $\mathbb{L}(\sigma)$ for the Lagrangian $\mathcal{L} = 2\pi x \sqrt{\dot{x}^2 + \dot{y}^2}$. We shall study its first variation and second variation at certain critical sections. For a more detailed study, see [1, 8].

The system (2) in this case has a solution

$$\left.\begin{aligned} \Omega_\Phi &= \pi x^2 \mathrm{d}y \\ \mathcal{E}_\Phi &= 2\pi x(\dot{y}\mathrm{d}x - \dot{x}\mathrm{d}y) \end{aligned}\right\} \Rightarrow N = \frac{1}{2\pi x \dot{y}} \frac{\partial}{\partial x}$$

which leads to

$$\lambda_{\mathcal{E}_\mathcal{L}} = -\frac{1}{x}\left(\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} + x \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}^3} \right)$$

that is, $\lambda_{\mathcal{E}_\mathcal{L}}(j_t^2 s)$ is precisely the mean curvature of the surface generated by σ

$$\widetilde{\mathcal{E}}_\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{x}\left(\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} + x \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}^3} \right)\right) \pi x^2\, \mathrm{d}y.$$

Such a curve is critical for the constrained variational problem if and only if the mean curvature of its revolution surface is constant. A simple example of solution is the cylinder with radius $R = a$ and height $h = 1/\pi a^2$: $\sigma = (x(t), y(t)) = (a, (1/\pi a^2)t + b)$.

Going one step further, to the second variation, if we compute the Jacobi operator at this solution, one gets

$$D_\sigma = X(t)\frac{\partial}{\partial x} + Y(t)\frac{\partial}{\partial y} \quad \Rightarrow \quad \widetilde{\mathcal{J}}_\sigma(D_\sigma) = \left(-\pi \dot{X}(t) - a^6 \pi^3 \dddot{X}(t) \right) \mathrm{d}y.$$

So, Jacobi vector fields along the cylinder are

$$D_\sigma = X(t)\frac{\partial}{\partial x} + Y(t)\frac{\partial}{\partial y}, \qquad \begin{pmatrix} X(t) = \alpha \sin \dfrac{t - t_0}{\pi a^3} - \beta \dfrac{\pi a^3}{2} \\ Y(t) = 2\alpha \cos \dfrac{t - t_0}{\pi a^3} + \beta t + \gamma \end{pmatrix}.$$

Integrating by parts for $D_\sigma \in T^c_\sigma(\Gamma_S(X,Y))$, one can express $\widetilde{\mathbb{H}}_\sigma(D_\sigma, D_\sigma)$ as

$$\widetilde{\mathbb{H}}_\sigma(D_\sigma, D_\sigma) = \int_0^1 \frac{\pi^2 a^3}{2}\left(\pi^2 a^6 \ddot{Y}(t)^2 - \dot{Y}(t)^2\right) \mathrm{d}t.$$

It turns out that not all the cylinders have the same behavior with respect to the second variation. For cylinders with $h/R < \pi$, we may define the function $w(t) = \pi a^3 \tan((t-1/2)/\pi a^3)$, so that any $D_\sigma \in T_\sigma(\Gamma_S(X,Y))$ satisfies

$$\widetilde{\mathbb{H}}_\sigma(D_\sigma, D_\sigma) = \int_0^1 \frac{\pi^2 a^3}{2}\left(\pi a^3 \ddot{Y}(t) + \frac{1}{\pi a^3} w(t)\dot{Y}(t)\right)^2 \mathrm{d}t - \frac{\pi^2 a^3}{2}\left[w(t)\dot{Y}(t)^2\right]_0^1.$$

As the integrand is positive, for variations with compact support, $\widetilde{\mathbb{H}}_\sigma(D_\sigma, D_\sigma) > 0$. The Hessian bilinear form is positive semi-definite.

As we can see, in the case $h/R > 2\pi$ the vector field $D_\sigma = -\pi^2 a^3 \cos 2\pi t \frac{\partial}{\partial x} + \sin 2\pi t \frac{\partial}{\partial y}$ in $T^c_\sigma(\Gamma_S(X,Y))$ satisfies $\widetilde{\mathbb{H}}_\sigma(D_\sigma, D_\sigma) = \pi^4 a^6 (4\pi^4 a^6 - 1) < 0$. In this case, the cylinder can be deformed in this direction reducing its lateral area.

Acknowledgements

This work has been partially supported by the Spanish Ministerio de Educación y Ciencia, project number MTM2004-01683 and Junta de Castilla y León, project number SA067/04.

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