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ONE DIMENSIONAL QUASI-EXACTLY SOLVABLE DIFFERENTIAL EQUATIONS

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Abstract. In this paper by means of similarity transformation we find some one-dimensional quasi-exactly solvable differential equations and their related Hamiltonians which appear in physical problems. We have provided also two examples with application of these differential equations.

1. Introduction

During the last decade a remarkable new class of quasi-exactly solvable spectral problems was introduced in [5]. These occupy an intermediate position between exactly solvable and unsolvable models in the sense that exact solution in an algebraic form exists only for a part of the spectrum.

In this paper we suggest a generalization of Bender-Dunne [1] approach to possible one-dimensional elliptic quasi-exactly solvable second order differential equations.

For this purpose, and with an attention to applications of elliptic potential we are motivated to obtain generalized master functions $A(x)$ that lead to elliptic quasi-exactly solvable potentials. By appropriate choice of the generalized master function $A(x)$ we obtain some one dimensional quasi-exactly solvable potentials that in all cases are functions of **Jacobi elliptic function**. These functions are periodic functions.

The paper is organized as follows: In Section 2 we show that we can generalize the usual quadratic master function to a master function of at most four order polynomials, then the most general elliptic quasi-exactly solvable differential operators related to generalized master function of degree $k = 3$ and $k = 4$ are given. Also by expanding their solutions in powers of x, we get three-term and four-term recursion relations among their coefficients, where Bender–Dunne factorization follows through imposing the quasi-exactly solvability conditions and in Section 3 we derive all one-dimensional elliptic quasi-exactly solvable differential equations for $k = 3$ and $k = 4$ and respectively the relative quantum Hamiltonian via prescription of references [3, 4]. Finally, in Section 4, as an example, we derive Lame potential from the special case of the potential which is given by the generalized master function $A(x) = 4x(1 - x)(1 - k^2x)$.

2. Quasi-Exactly Solvable Differential Equations Associated with Generalized Master Function

In the following, by generalizing master function of order up to two to polynomial of order up to k together with the non-negative weight function $W(x)$, defined on the interval (a, b) such that $\frac{1}{W(x)}\frac{\mathrm{d}}{\mathrm{d}x}(A(x)W(x))$ is a polynomial of degree at most $k - 1$, we can define the operator

$$L = -\frac{1}{W(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(A(x)W(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) + B(x) \tag{1}$$

where $B(x)$ is a polynomial of order up to $k - 2$. The interval (a, b) is chosen so that, we have $A(a)W(a) = A(b)W(b) = 0$. It is straightforward to show that the above defined operator L is a self-adjoint linear operator which maps a given polynomial of order m to another polynomial of order $m + k - 2$. Now, by an appropriate choice of $B(x)$ and weight function $W(x)$, the operator L can have an invariant subspace of polynomials of order up to n. Then by choosing the set of orthogonal polynomials $\{\phi_0, \phi_1, \ldots, \phi_n\}$ defined in the interval (a, b) with respect to the weight function $W(x)$

$$\int_a^b \phi_m(x)\phi_n(x)W(x)\,\mathrm{d}x = 0 \quad \text{for} \quad m \neq n \tag{2}$$

as a basis, the matrix elements of the operator L on this base will have the following block diagonal form

$$L_{ij} = 0 \quad \text{if} \quad \{i \leq n \text{ and } j \geq n + 1\} \quad \text{or} \quad \{i \geq n + 1 \text{ and } j \leq n\}. \tag{3}$$

Since, according to the well known theorem of orthogonal polynomials, $\phi_n(x)$ is orthogonal to any polynomial of order up to $n - 1$ and, therefore, for the matrix L we get

$$L = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} \tag{4}$$

where M is an $(n + 1) \times (n + 1)$ matrix with matrix elements

$$M_{ij} = \int_a^b W(x)\phi_1(x)L(x)\phi_j(x)\,\mathrm{d}x, \qquad i, j = 0, 1, 2, \ldots, n \tag{5}$$

and N is an infinite matrix element defined as above with $i, j \geq n + 1$.

The block-diagonal form of the operator L indicates that by diagonalizing the $(n+1)\times(n+1)$ matrix M, we can find $n+1$ eigenvalues of the operator L together with the related eigenfunctions as linear functions of orthogonal polynomials $\{\phi_0, \phi_1, \ldots, \phi_n\}$.

In order to determine the appropriate $B(x)$ and $W(x)$ for given generalized master function $A(x)$, we use the Taylor expansion of these functions

$$A(x) = \sum_{i=0}^{k} \frac{A^{(i)}(0)}{i!} x^i, \quad \text{with} \quad A^{(i)}(0) = \left.\frac{\mathrm{d}^i A(x)}{\mathrm{d}x^i}\right|_{x=0} \tag{6}$$

$$\frac{(A(x)W(x))'}{W(x)} = \sum_{i=0}^{k-1} \frac{\left(\frac{(AW)'}{W}\right)^{(i)}(0)}{i!} x^i \tag{7}$$

and, therefore,

$$\left(\frac{(AW)'}{W}\right)^{(i)}(0) = \left.\frac{\mathrm{d}^i \left(\frac{(A(x)W(x))'}{W(x)}\right)}{\mathrm{d}x^i}\right|_{x=0}$$

$$B(x) = \sum_{i=0}^{k-2} \frac{B^{(i)}(0)}{i!} x^i, \quad \text{with} \quad B^{(i)}(0) = \left.\frac{\mathrm{d}^i B(x)}{\mathrm{d}x^i}\right|_{x=0}. \tag{8}$$

Then, the existence of invariant subspace built by the polynomials of order n of the operator L leads to the following linear equations between the coefficients of the above Taylor expansions

$$-\frac{A^{(i+2)}}{(i+2)!} l(l-1) - \frac{(\frac{(AW)'}{W})^{(i+1)}}{(i+1)!} l + \frac{B^{(i)}}{i!} = 0 \tag{9}$$

where

$$\begin{cases} l = n & \text{and} \quad i = 1, 2, \ldots, k-2 \\ l = n-1 & \text{and} \quad i = 2, 3, \ldots, k-2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ l = n-k+4 & \text{and} \quad i = k-3, k-2 \\ l = n-k+3 & \text{and} \quad i = k-2. \end{cases} \tag{10}$$

The number of above equations for a given value of k is $\frac{(k-1)(k-2)}{2}$. If we are to determine only the unknown function $B(x)$ without having any further constraint on the weight function $W(x)$, then the above $\frac{(k-1)(k-2)}{2}$ equations should be satisfied with $(k-2)$ coefficients of Taylor expansion of B as the only unknowns, since $B^{(0)}$ can be absorbed in the eigenspectrum operator L. Therefore, we are left with $k-2$ unknowns to be determined, where the compatibility of equations (9) require that $k=3$ at most. On the other hand, if we add the coefficients of Taylor expansions of $A(x)$ and $\frac{(A(x)W(x))'}{W(x)}$ to our list of unknowns, (to be determined by solving equations (9)), then their compatibility conditions require that

$$3(k-1) \geq \frac{(k-1)(k-2)}{2} \tag{11}$$

or $k \leq 8$, where further investigations show that we can have at most $k = 4$, since for $k \geq 5$ the coefficients $A^{(k)}(0)$ and $\left(\frac{(A(x)W(x))'}{W(x)}\right)^{(k-1)}(0)$ will vanish. Below we summarize the above-mentioned discussion for $k = 3$ and $k = 4$, separately.

2.1. The Case $k = 3$

In this case, $B(x)$ is a second order polynomial where $B^{(1)}$ can be determined by solving equations (9)

$$B^{(1)} = \frac{n}{2}\left(\frac{A^{(3)}(0)}{3}(n-1) + \left(\frac{(AW)'}{W}\right)^{(2)}\right) \tag{12}$$

which is the only unknown in this case.

2.2. The Case $k = 4$

Again, solving the equation (9) leads to

$$B^{(1)} = \frac{n}{2}\left(\frac{A^{(3)(0)}}{3}(n-1) + \left(\frac{(AW)'}{W}\right)^{(2)}\right) \tag{13}$$

$$B^{(2)} = -\frac{A^{(4)}}{12}n(n-1) \tag{14}$$

and

$$\left(\frac{(AW)'}{W}\right)^{(3)} = -\frac{A^{(4)}}{2}(n-1). \tag{15}$$

Here, besides having a constraint over the second order polynomial $B(x)$, we have to put further constraints on the weight function $W(x)$ given in (15).

Definitely, we can determine $n+1$ eigenvalues of the operator L, simply by diagonalizing the $(n+1) \times (n+1)$ matrix M, since it is a self-adjoint operator in Hilbert space of polynomials and it has a block diagonal form given in (4).

As we are going to see at the end of this section, we can determine its eigenspectrum analytically, using some recursion relations.

2.3. Recursion Relations

Now we show that the eigenfunction of the operator L is a generating function for a new set of polynomials $P_m(E)$ where the eigenfunction equation of the operator L leads to the recursion relations between these polynomials. Quasi-exact solvable constraints (9) will lead to their factorization, that is, $P_{n+N+1}(E) = P_{n+1}(E)Q_N$ for $N \geq 0$, where the roots of polynomials $P_{n+1}(E)$ turn out to be the eigenvalues of the operator L. To achieve these results, first we expand $\psi(x)$, the eigenfunction of L, as

$$\psi(x) = \sum_{m=0}^{\infty} P_m(E)x^m \tag{16}$$

where the eigenfunction equation

$$L\psi(x) = E\psi(x) \tag{17}$$

can be expressed as

$$\begin{aligned} -A(x)\sum_{m=2}^{\infty} m(m-1)P_m(E)x^{m-2} - \frac{(AW)'}{W}\int_{m=1}^{\infty} mP_m(E)x^{m-1}\,\mathrm{d}x \\ +B(x)\sum_{m=0}^{\infty} P_m(E)x^m = E\sum_{m=0}^{\infty} P_m(E)x^m \end{aligned} \tag{18}$$

and this leads to the following recursion relations for the coefficients $P_m(E)$

$$\begin{aligned} &\left(A^{(1)}(m+1)(m+2) + \left(\frac{(AW)'}{W}\right)^{(0)}(m+2)\right)P_{m+2}(E) \\ &+\left(\frac{A^{(2)}}{2!}m(m+1) + \left(\frac{(AW)'}{W}\right)^{(1)}(m+1) + E\right)P_{m+1}(E) \\ &+\left(\frac{A^{(3)}}{3!}m(m-1) + \frac{\left(\frac{(AW)'}{W}\right)^{(2)}}{2!}m - B^{(1)}\right)P_m(E) \\ &+\left(\frac{A^{(4)}}{4!}(m-1)(m-2) + \frac{\left(\frac{(AW)'}{W}\right)^{(3)}}{3!}m - \frac{B^{(2)}}{2!}\right)P_{m-1}(E) = 0. \end{aligned} \tag{19}$$

Below we investigate recursion relations which are obtained in the cases when $k = 3$ (cubic $A(x)$) and $k = 4$ (quartic $A(x)$), separately.

Cubic A:

In this case the four-term general recursion relation reduces to the following three-term recursion relation

$$\begin{aligned}&\left(A^{(1)}(m+1)(m+2)+\left(\frac{(AW)'}{W}\right)^{(0)}(m+2)\right)P_{m+2}(E)\\&+\left(\frac{A^{(2)}}{2!}m(m+1)+\left(\frac{(AW)'}{W}\right)^{(1)}(m+1)+E\right)P_{m+1}(E)\\&+\left(\frac{A^{(3)}}{3!}m(m-1)+\frac{\left(\frac{(AW)'}{W}\right)^{(2)}}{2!}m-B^{(1)}\right)P_m(E)=0.\end{aligned} \tag{20}$$

In order to have finite eigenspectrum, that is, quasi-integrable differential equation, the above recursion relation should be truncated at some value of $m = n$, which is obviously possible by an appropriate choice of

$$B^{(1)}=\frac{n}{2}\left(\frac{A^{(3)}(0)}{3}(n-1)+\left(\frac{(AW)'}{W}\right)^{(2)}\right) \tag{21}$$

and this is in agreement with the results of previous subsection.

Using the recursion relations (20) with $B^{(1)}$ given in (21), we get a factorization of the polynomial $P_{n+N+1}(E)$ for $N \geq 0$ in terms of $P_{n+1}(E)$ as follows

$$P_{n+N+1}(E)=P_{n+1}(E)Q_N(E), \qquad N\geq 0 \tag{22}$$

where, by choosing the eigenvalue E as a root of the polynomials $P_{n+1}(E)$, all polynomials of order higher than n will vanish.

By using equations (16) we obtain the eigenfunctions $\psi_i(x)$

$$\psi_i(x)=\sum_{m=0}^{n}P_m(E_i)x^m, \qquad i=0,1,\ldots,n \tag{23}$$

where E_i are roots of the polynomial $P_{n+1}(E)$.

The above eigenfunctions are polynomials of order n, hence they have at most n roots in the interval (a, b), where, according to the well-known oscillation and comparison theorem for the second-order linear differential equation [2] these numbers order the eigenvalues according to the number of roots of corresponding eigenfunctions. Therefore, we can say that the eigenvalues thus obtained are the first $n+1$ eigenvalues of the operator L. Using the recursion relations (20), we can evaluate the polynomials $P_m(E)$ in term of $P_0(E)$, where we have chosen $P_0(E) = 1$. Following the above scheme we have evaluated the first five polynomials shown in the Appendix.

Quartic A:

Again in order to truncate the recursion relations (19) and to factorize the polynomials $P_{n+N+1}(E)$ in terms of $P_{n+1}(E)$, we should have

$$B^{(1)} = \frac{n}{2}\left(\frac{A^{(3)(0)}}{3}(n-1) + \left(\frac{(AW)'}{W}\right)^{(2)}\right) \tag{24}$$

$$\frac{B^{(2)}}{2!} = \frac{A^{(4)}}{4!}(n-1)(n-2) + \frac{(\frac{(AW)'}{W})^{(3)}}{3!}n \tag{25}$$

and

$$\frac{B^{(2)}}{2!} = \frac{A^{(4)}}{4!}n(n-1) + \frac{(\frac{(AW)'}{W})^{(3)}}{3!}(n+1). \tag{26}$$

Solving the above equations we get

$$B^{(2)} = -\frac{A^{(4)}}{12}n(n-1) \tag{27}$$

and

$$\left(\frac{(AW)'}{W}\right)^{(3)} = -\frac{A^{(4)}}{2}(n-1). \tag{28}$$

The equations (24), (27) and (28) are the same equations which are required for the reduction of the operator L to its block diagonal form.

Again the roots of the polynomial P_{n+1} will correspond to $n+1$ eigenvalues of the differential operator L with eigenfunctions which can be expressed in term of $P_m(E_i)$ for $m \leq n$, where polynomials $P_m(E)$ can be obtained from recursion relation by choosing $P_0 = 1$ and $P_{-1} = 0$.

3. Quasi-Exactly Potential Associated with Generalized Master Function

As in [3, 4], writing

$$\psi(t) = A^{1/4}(x)W^{1/2}(x)\phi(x) \tag{29}$$

by a change of the variable $\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{A(x)}$, the eigenvalue equation for the operator L reduces to the Schrödinger equation

$$H(t)\psi(t) = E\psi(t) \tag{30}$$

with the same eigenvalue E and $\psi(t)$ given in (30), in terms of eigenfunction of L, where $H(t) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} + V(t)$ is the similarity transformation of $L(x)$ defined as

$$H(t) = A^{1/4}(x)W^{1/2}(x)L(x)A^{-1/4}(x)W^{-1/2}(x) \tag{31}$$

with

$$V(t) = -\frac{3}{16}\frac{\dot{A}^2(t)}{A^2(t)} - \frac{1}{4}\frac{\dot{W}^2(t)}{W^2(t)} + \frac{1}{4}\frac{\dot{A}(t)\dot{W}(t)}{A(t)W(t)} + \frac{1}{4}\frac{\ddot{A}(t)}{A(t)} + \frac{1}{2}\frac{\ddot{W}(t)}{W(t)} + B(t) \tag{32}$$

and

$$V(x) = \frac{\ddot{A}^2(x)}{4} - \frac{\dot{A}^2(x)}{16A(x)} - \frac{A(x)\dot{W}(x)^2}{4W^2(x)} + \frac{A(x)\ddot{W}(x)}{2W(x)} + \frac{\dot{A}(x)\dot{W}(x)}{2W(x)} + B(x).$$

It is also straightforward to show that

$$\int \phi(t)H(t)\psi(t)\,\mathrm{d}t = \int_a^b W(x)\psi(x)L(x)\psi(x)\,\mathrm{d}x. \tag{33}$$

Hence block diagonalization of L leads to block-diagonalization of H.

3.1. Elliptic Quasi-Exactly Solvable Potential

The starting point to find elliptic quasi-exactly solvable potential is generalized master function $A(x)$, as mentioned before. Therefore, the selection of master function A which leads to elliptic potential, is very important. Considering the relation $\frac{\mathrm{d}x}{dt} = \sqrt{A(x)}$, we select the master function so that x comes into the form of elliptic Jacobi functions. The weight function $W(x)$ related to the given master function $A(x)$ of order three and four can be obtained so that the polynomial $\frac{1}{W}\frac{\mathrm{d}}{\mathrm{d}x}(AW)$ to be of order two or three, respectively.
After determining B_1 and B_2 from equations (13) and (14), the function $B(x)$ can be obtained easily

$$B(x) = B_1 x + \frac{1}{2!}B_2 x^2.$$

Now, we can determine operator L and potential $V(t)$ by knowing A, W and B.

The interval (a, b) for x is chosen so that to have $A(a)W(a) = A(b)W(b) = 0$, and the interval of the parameters α, β, γ and δ such that $A(x)W(x)$ has not any singularity and also $A(a)W(a) = A(b)W(b) = 0$ and equation (28) are conserved.

We introduce the possible 24 generalized master functions $A(x)$ of order three and four in Table 1 below.

4. Example

As an example we are going to obtain the Lame potential. For this purpose we consider the generalized master function $A(x) = 4x(1-x)(1-k^2x)$, $x = \mathrm{sn}^2(t,k)$ where its corresponding differential equation L, weight function $W(x)$, polynomial B, potential V and the interval of x are given bellow.

$$\begin{aligned} W = {} & x^\alpha(1-x)^\beta(1-k^2x)^\gamma, \quad 0 \le x \le 1, \quad 0 < k < 1, \\ & \alpha > -1, \quad \beta > -1, \quad -\infty < \gamma < \infty \end{aligned}$$

$$B = 4nk^2(n + 2 + \alpha + \beta + \gamma)x$$

Cubic A	x	Quartic A	x
$4x(1-x)(1-k^2x)$	$\mathrm{sn}^2(t,k)$	$(x^2-k^2)(x^2-1)$	$\frac{\mathrm{dn}(t,k)}{\mathrm{cn}(t,k)}$
$4x(1+x)(1+(1-k^2)x)$	$\frac{\mathrm{sn}^2(t,k)}{\mathrm{cn}^2(t,k)}$	$(1+x^2)(1-k^2+x^2)$	$\frac{\mathrm{cn}(t,k)}{\mathrm{sn}(t,k)}$
$4x(1+k^2x)(1+(k^2-1)x)$	$\frac{\mathrm{sn}^2(t,k)}{\mathrm{dn}^2(t,k)}$	$(x^2-1)(1-k^2-x^2)$	$\mathrm{dn}(t,k)$
$4x(x-1)(x-k^2)$	$\frac{1}{\mathrm{sn}^2(t,k)}$	$(x^2-1)(x^2-k^2)$	$\frac{1}{\mathrm{sn}(t,k)}$
$4x(x-1)((1-k^2)x+k^2)$	$\frac{1}{\mathrm{cn}^2(t,k)}$	$(1-x^2)((1-k^2)x^2-1)$	$\frac{1}{\mathrm{dn}(t,k)}$
$4x(1-x)(1-k^2+k^2x)$	$\mathrm{cn}^2(t,k)$	$(x^2-1)((1-k^2)x^2+k^2)$	$\frac{1}{\mathrm{cn}(t,k)}$
$4x(1-x)(k^2-1+x)$	$\mathrm{dn}^2(t,k)$	$(1-x^2)(1-k^2x^2)$	$\mathrm{sn}(t,k)$
$4x(x-1)((k^2-1)x+1)$	$\frac{1}{\mathrm{dn}^2(t,k)}$	$(1-x^2)(1-k^2+k^2x^2)$	$\mathrm{cn}(t,k)$
$4x(1+x)(1-k^2+x)$	$\frac{\mathrm{cn}^2(t,k)}{\mathrm{sn}^2(t,k)}$	$(k^2+x^2)(k^2-1+x^2)$	$\frac{\mathrm{dn}(t,k)}{\mathrm{sn}(t,k)}$
$4x(k^2x-1)(x-1)$	$\frac{\mathrm{cn}^2(t,k)}{\mathrm{dn}^2(t,k)}$	$(k^2+x^2)(k^2-1+x^2)$	$\frac{\mathrm{dn}(t,k)}{\mathrm{sn}(t,k)}$
$4x(x-k^2)(x-1)$	$\frac{\mathrm{dn}^2(t,k)}{\mathrm{cn}^2(t,k)}$	$(1+k^2x^2)(1-(1-k^2)x^2)$	$\frac{\mathrm{sn}(t,k)}{\mathrm{dn}(t,k)}$
$4x(k^2+x)(x+k^2-1)$	$\frac{\mathrm{dn}^2(t,k)}{\mathrm{sn}^2(t,k)}$	$(1+x^2)(1+(1-k^2)x^2)$	$\frac{\mathrm{sn}(t,k)}{\mathrm{cn}(t,k)}$

Table 1. Cubic and Quartic Master Functions

$$
\begin{aligned}
L = & -4x(1-x)(1-k^2x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - [4k^2(3+\alpha+\beta+\gamma)x^2 + (-8k^2-8-4\alpha k^2 \\
& -4\alpha-4\beta-4\gamma k^2)x + 4 + 4\alpha]\frac{\mathrm{d}}{\mathrm{d}x} + 4nk^2(n+2+\alpha+\beta+\gamma)x
\end{aligned}
$$

$$
\begin{aligned}
V = & \frac{1}{4(1-k^2)}\Bigg((3C_4+C_3)\mathrm{cn}^2(t,k) - C_4\mathrm{cn}^4(t,k) + \frac{C_4\mathrm{dn}^4(t,k)}{k^4} \\
& -\left(\frac{3C_4}{k^4}+\frac{C_3}{k^2}\right)\mathrm{dn}^2(t,k) - \left(\frac{C_4}{k^4}+\frac{C_3}{k^2}+C_2+k^2C_1+k^4C_0\right)\frac{1}{\mathrm{dn}^2(t,k)} \\
& +\frac{C_0}{4\mathrm{sn}^2(t,k)} + \frac{3(1+k^2)C_4}{4k^4} + \frac{C_3}{2k^2}\Bigg)
\end{aligned}
$$

$$\begin{aligned}
C_1 &= -8\beta - 8\alpha k^2 - 8\gamma k^2 - 8\alpha - 4k^2 - 8\alpha\beta - 8\alpha^2 k^2 - 8\alpha\gamma k^2 - 8\alpha^2 - 4 \\
C_2 &= 32\alpha k^2 + 24\beta k^2 + 24\gamma k^2 + 26k^2 + 4\beta^2 + 4\alpha^2 + 8\alpha\beta + 4k^4 + 8\alpha \\
&\quad + 16nk^2\gamma + 8\beta\gamma k^2 + 16\alpha\beta k^2 + 8\gamma k^4 + 32nk^2 + 4\alpha^2 k^4 + 16\alpha^2 k^2 \\
&\quad + 8\alpha k^4 + 4\gamma^2 k^4 + 16n^2 k^2 + 8\beta + 4 + 16\alpha\gamma k^2 + 16nk^2\beta + 16nk^2\alpha \\
&\quad + 8\alpha\gamma k^4 \\
C_3 &= -4k^2(6\alpha k^2 + 4\beta k^2 + 4\alpha n + 4\beta n + 6\gamma k^2 + 4\gamma + 5k^2 + 2k^2\gamma^2 + 4\gamma n \\
&\quad + 2\beta^2 + 2\alpha^2 + 4\alpha\beta + 6\alpha + 4nk^2\gamma + 4n^2 + 2\beta\gamma k^2 + 2\alpha\beta k^2 + 2\beta\gamma \\
&\quad + 8n + 8nk^2 + 2\alpha^2 k^2 + 4n^2 k^2 + 6\beta + 2\alpha\gamma + 5 + 4\alpha\gamma k^2 + 4nk^2\beta \\
&\quad + 4nk^2\alpha) \\
C_4 &= k^4(2\gamma + 5 + 4n + 2\beta + 2\alpha)(2\gamma + 3 + 4n + 2\beta + 2\alpha) \\
C_0 &= 4\alpha^2 - 1.
\end{aligned}$$

Let us restrict ourselves to the case in which the parameters α, β, γ are

$$\alpha = \beta = \gamma = -\frac{1}{2}. \tag{34}$$

The relative potential of the generalized master function $A(x)$ reduces to

$$V(x) = 2n(2n+1)k^2x^2 \tag{35}$$

which is exactly the **Lame potential**.

Below we obtain the low laying eigenvalues and eigenstates for this potential. In order to find the eigenvalues and eigenstates for $n = 1$, first we obtain from $P_2 = 0$ the eigenvalues E_1 and E_2

$$\begin{aligned}
P_2 &= \frac{E^2}{24} - \frac{(k^2+1)E}{6} + \frac{k^2}{2} \\
E_1 &= 2k^2 + 2 - 2\sqrt{k^4 - k^2 + 1} \\
E_2 &= 2k^2 + 2 - 2\sqrt{k^4 - k^2 + 1}.
\end{aligned}$$

Now from $\psi_i(x) = \sum_{m=0}^{n} P_m(E_i)x^m$, we can obtain the eigenstates ψ_1 and ψ_2 as given below

$$\begin{aligned}
\psi_1(x) &= 1 + 2\left(k^2 + 1 + \sqrt{k^4 - k^2 + 1}\right)x^2 \\
\psi_2(x) &= 1 + 2\left(k^2 + 1 - \sqrt{k^4 - k^2 + 1}\right)x^2.
\end{aligned}$$

Similarly for $n = 2$ with $P_3 = 0$ we obtain E_1, E_2, E_3 and relative eigenstates as

$$\begin{aligned}
P_3 &= -\frac{1}{720}E^3 + \frac{(1+k^2)E^2}{36} - \frac{k^2(4k^2+21)E}{45} + \frac{8k^2(k^2+1)}{9} \\
E_1 &= -\frac{20}{3} - \frac{20}{3}k^2
\end{aligned}$$

$$
\begin{aligned}
E_2 &= \frac{10}{3} + \frac{10}{3}k^2 + 2\sqrt{9k^4 - 4k^2 + 9} \\
E_3 &= \frac{10}{3} + \frac{10}{3}k^2 - 2\sqrt{9k^4 - 4k^2 + 9} \\
\psi_1(x) &= 1 + \frac{10}{3}(1+k^2)x^2 + \frac{1}{27}(80k^4 + 205k^2 + 80)x^4 \\
\psi_2(x) &= -\frac{2}{3} - \frac{5}{3}k^2 - \sqrt{9k^4 - 4k^2 + 9}x^2 \\
&\quad + \frac{1}{27}\left(6\sqrt{9k^4 - 4k^2 + 9}(1+k^2) + 38k^4 + 22k^2 + 38\right)x^4 \\
\psi_3(x) &= -\frac{2}{3} - \frac{5}{3}k^2 - \sqrt{9k^4 - 4k^2 + 9}x^2 \\
&\quad + \frac{1}{27}\left(-6\sqrt{9k^4 - 4k^2 + 9}(1+k^2) + 38k^4 + 22k^2 + 38\right)x^4.
\end{aligned}
$$

Appendix: The First Four Polynomials $P_n(E)$ for $k = 3$

To abbreviate, we set $F^{(i)} = \left(\frac{AW'}{W}\right)^{(i)}$.

$$
\begin{aligned}
P_0 &= 1 \\
P_1 &= -\frac{E}{F^0} \\
P_2 &= \frac{1}{2}\frac{B^1F^0 + EF^1 + E^2}{F^0(A^1 + F^0)} \\
P_3 &= -(2EB^{(1)}A^{(1)} + A^{(2)}E^2 + 2F^{(1)}B^{(0)}F^{(0)} \\
&\quad + A^{(2)}B^{(1)}F^{(0)} + A^{(2)}EF^{(1)} + 3EB^{(1)}F^{(0)} + E^3 \\
&\quad + 2EF^{(1)^2} + 3F^{(1)}E^2 - EF^{(2)}A^{(1)} - EF^{(2)}F^{(0)}) \\
&\quad /(6F^{(0)}2A^{(1)^2} + 3A^{(1)}F^{(0)} + F^{(0)^2}) \\
P_4 &= (-A^{(3)}EF^{(1)}F^{(0)} + 4A^{(2)}E^3 + 6F^{(1)}E^3 + 6EF^{(1)^3} + 11F^{(1)^2}E^2 \\
&\quad + 3B^{(1)^2}F^{(0)^2} - 2A^{(3)}E^2A^{(1)} + 3A^{(2)^2}B^{(1)}F^{(0)} + 3A^{(2)^2}EF^{(1)} \\
&\quad + 6F^{(1)^2}B^{(1)}F^{(0)} + 8E^2B^{(1)}A^{(1)} + 6E^2B^{(1)}F^{(0)} - 7E^2F^{(2)}A^{(1)} \\
&\quad - 4E^2F^{(2)}F^{(0)} + 9A^{(2)}EF^{(1)^2} + 13A^{(2)}F^{(1)}E^2 - 3F^{(2)}B^{(1)}F^{(0)^2} \\
&\quad + 6A^{(1)}B^{(1)^2}F^{(0)} - 2A^{(3)}EF^{(1)}A^{(1)} + 6A^{(2)}EB^{(1)}A^{(1)} \\
&\quad + 9A^{(2)}F^{(1)}B^{(1)}F^{(0)} + 10A^{(2)}EB^{(1)}F^{(0)} - 3A^{(2)}EF^{(2)}A^{(1)}
\end{aligned}
$$

$$
\begin{aligned}
&- 3A^{(2)}EF^{(2)}F^{(0)} + 12F^{(1)}EB^{(1)}A^{(1)} + 14F^{(1)}EB^{(1)}F^{(0)} \\
&- 9F^{(1)}EF^{(2)}A^{(1)} - 6F^{(1)}EF^{(2)}F^{(0)} - 6A^{(1)}F^{(2)}B^{(1)}F^{(0)} \\
&- 2A^{(1)}A^{(3)}B^{(1)}F^{(0)} - A^3B^1F^{(0)^2} - A^{(3)}E^2F^{(0)} + 3A^{(2)^2}E^2 + E^4) \\
&/(24F^{(0)}(6A^{(1)^3} + 11A^{1^2}F^{(0)} + 6A^{(1)}F^{(0)^2} + F^{(0)^3})).
\end{aligned}
$$

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