

## HOW MANY TYPES OF SOLITON SOLUTIONS DO WE KNOW?

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**Abstract.** We discuss several ways of how one could classify the various types of soliton solutions related to NLEE that are solvable with the generalized  $n \times n$  Zakharov–Shabat system. In doing so we make use of the fundamental analytic solutions, the dressing procedure and other tools characteristic for the inverse scattering method. We propose to relate to each subalgebra  $\mathfrak{sl}(p)$ ,  $2 \leq p \leq n$  of  $\mathfrak{sl}(n)$ , a type of one-soliton solutions which have  $p - 1$  internal degrees of freedom.

### 1. Introduction

It is our impression that the question in the title has not been answered satisfactorily even for some of the best known type of soliton equations such as the  $N$ -wave equations, the multicomponent **nonlinear Schrödinger (NLS) equation** and others.

We are using the term “soliton solution” as a special solution to a given **nonlinear evolution equation (NLEE)** which is solvable by the so called **inverse scattering method (ISM)** [20, 4]. That means that the NLEE allows **Lax representation**

$$[L(\lambda), M(\lambda)] = 0 \quad (1)$$

where  $L(\lambda)$  and  $M(\lambda)$  are two linear operators. In what follows we take them to be first order matrix differential operators

$$L\psi(x, t, \lambda) \equiv i \frac{d\psi}{dx} + U(x, t, \lambda)\psi(x, t, \lambda) = 0 \quad (2)$$

$$M\psi(x, t, \lambda) \equiv i\frac{d\psi}{dt} + V(x, t, \lambda)\psi(x, t, \lambda) = \psi(x, t, \lambda)C(\lambda). \quad (3)$$

The compatibility condition (1) which must hold true identically with respect to  $\lambda$  takes the form

$$i\frac{\partial V}{\partial x} - i\frac{\partial U}{\partial t} + [U(x, t, \lambda), V(x, t, \lambda)] = 0$$

and is valid for any choice of  $C(\lambda)$ .

The one-soliton solutions are related to one or a set of several discrete eigenvalues of the Lax operator  $L$ . Therefore one first has to study the different configurations of discrete eigenvalues of  $L$ , see [12]. The next step in classifying the types of one-soliton solutions is related to the study of their internal degrees of freedom.

In order to make the problem not too difficult we will specify  $L$  to be the **generalized Zakharov–Shabat system**

$$L(\lambda)\psi(x, \lambda) \equiv i\frac{d\psi}{dx} + (q(x) - \lambda J)\psi(x, \lambda) = 0 \quad (4)$$

where we take the potential  $q(x, t)$  to be  $n \times n$  matrix-valued smooth function of  $x$  tending to zero sufficiently rapid as  $x \rightarrow \pm\infty$ . We also restrict  $J$  to be a real constant diagonal matrix with different eigenvalues. Thus, we have

$$J = \text{diag}(J_1, \dots, J_n), \quad J_1 > J_2 > \dots > J_n, \quad \text{tr } J = 0.$$

By carrying out a gauge transformation which commutes with  $J$ , we can always take  $q(x)$  to be of the form  $q(x) = [J, q'(x)]$ , i.e.,  $q_{jj} \equiv 0$ . The linear subspace in  $\mathfrak{sl}(n)$  of matrix-valued functions  $q(x) = [J, q'(x)]$  are known in the literature to be the co-adjoint orbit in  $\mathfrak{g}$  passing through  $J$ . The co-adjoint orbits can be supplied in a natural way with non-degenerate symplectic structures which make them natural choices for the phase spaces  $\mathcal{M}_J$  and Hamiltonian structures of the corresponding NLEE.

We will try to answer the question in the title first for the simplest class of Lax operators of the type (4) with real-valued  $J$ . In doing this we will be using the dressing method, one of the best known methods for constructing reflectionless potentials and soliton solutions. The choice of  $\mathcal{M}_J$  determines the number of independent matrix elements in  $q(x)$  that will satisfy the NLEE.

In Section 2 below we first outline the well known facts about the soliton types of NLEE solvable by the  $\mathfrak{sl}(2)$  Zakharov–Shabat system. In Section 3 we treat the different one-soliton solutions for the  $\mathfrak{sl}(n)$  Zakharov–Shabat systems related to the subalgebras  $\mathfrak{sl}(p)$ . Most of our results are illustrated for the  $\mathfrak{sl}(5)$  system, but it is not difficult to extend them to any  $\mathfrak{sl}(n)$  system. The structure of the eigenfunctions of  $L(\lambda)$  corresponding to the different types of solitons is outlined in Section 4. In the last Section 5 we discuss possible generalizations to other Zakharov–Shabat systems having additional symmetry properties.

## 2. Zakharov–Shabat System and $\mathfrak{sl}(2)$ Solitons

To each choice of the Lax operator  $L(\lambda)$ , i.e. for each specific  $U(x, t, \lambda)$  one can relate a class of NLEE. The best known examples of such NLEE are related to the **Zakharov–Shabat system**  $L_0(\lambda)$  with

$$U_0(x, t, \lambda) = q_0(x, t) - \lambda\sigma_3, \quad q_0(x, t) = \begin{pmatrix} 0 & q_0^+(x, t) \\ q_0^-(x, t) & 0 \end{pmatrix}.$$

The class of NLEE related to  $L_0$  are systems of equations for the functions  $q_0^\pm(x, t)$  are written in the compact form [2, 17, 10]

$$i\sigma_3 \frac{\partial q_0}{\partial t} + 2f(\Lambda_0)q_0(x, t) = 0 \quad (5)$$

where  $f(\lambda)$  is the dispersion law of the NLEE and  $\Lambda_0$  is (anyone of) the recursion operators, acting on the space  $\mathcal{M}_0$  of off-diagonal matrix-valued functions as follows

$$\Lambda_{0\pm}X \equiv \frac{i}{4} \left[ \sigma_3, \frac{dX}{dx} \right] + \frac{i}{2}q(x) \int_{\pm\infty}^x dy \operatorname{tr} (q(y), [\sigma_3, X(y)]).$$

Choosing  $f(\lambda)$  to be linear function we get a simple linear system of equations for  $q_0^\pm(x, t)$ . For  $f(\lambda) = -2\lambda^2$  the NLEE (5) reduces to the system

$$\begin{aligned} iq_{0t}^+ + q_{0xx}^+ + 2(q_0^+(x, t))^2 q_0^-(x, t) &= 0 \\ iq_{0t}^- - q_{0xx}^- - 2(q_0^-(x, t))^2 q_0^+(x, t) &= 0 \end{aligned} \quad (6)$$

and for  $f(\lambda) = 4\lambda^3$ , one gets the system

$$\begin{aligned} q_{0t}^+ + q_{0xxx}^+ + 6q_0^+(x, t)q_0^-(x, t)q_{0x}^+ &= 0 \\ q_{0t}^- + q_{0xxx}^- + 6q_0^-(x, t)q_0^+(x, t)q_{0x}^- &= 0. \end{aligned} \quad (7)$$

The main idea of the ISM is based on the possibility to analyze and solve the direct and the inverse scattering problems for  $L_0(\lambda)$ . More precisely we introduce the Jost solutions of  $L_0(\lambda)$  as  $2 \times 2$  matrix-valued solutions

$$L_0(\lambda)\psi_0(x, t, \lambda) = 0, \quad L_0(\lambda)\phi_0(x, t, \lambda) = 0$$

defined by their asymptotic behavior for  $x \rightarrow \pm\infty$  respectively

$$\begin{aligned} \psi_0(x, t, \lambda) &= |\psi_0^-, \psi_0^+|, & \lim_{x \rightarrow \infty} \psi_0(x, t, \lambda)e^{i\lambda\sigma_3 x} &= \mathbb{1} \\ \phi_0(x, t, \lambda) &= |\phi_0^+, \phi_0^-|, & \lim_{x \rightarrow -\infty} \phi_0(x, t, \lambda)e^{i\lambda\sigma_3 x} &= \mathbb{1}. \end{aligned} \quad (8)$$

The superscripts  $\pm$  in the columns of the Jost solutions refer, as we shall see below, to their analyticity properties.

The definitions (8) hold true for all  $t$  if we take  $C(\lambda) = C_0(\lambda)$  in equation (3) as follows

$$C_0(\lambda) \equiv \lim_{x \rightarrow \pm\infty} V(x, t, \lambda) = f_0(\lambda)\sigma_3.$$

Then one introduces the scattering matrix  $T_0(\lambda, t)$  by

$$T_0(\lambda, t) \equiv (\psi(x, t, \lambda))^{-1} \phi(x, t, \lambda) = \begin{pmatrix} a_0^+(\lambda) & -b_0^-(\lambda, t) \\ b_0^+(\lambda, t) & a_0^-(\lambda) \end{pmatrix}$$

which is  $x$ -independent. Its  $t$ -dependence can be derived from the Lax representation to be

$$i \frac{dT_0}{dt} + [f_0(\lambda)\sigma_3, T(\lambda, t)] = 0.$$

Thus, if  $q_0^\pm(x, t)$  satisfy the system of equations (5) we get

$$\frac{da_0^\pm(\lambda)}{dt} = 0, \quad i \frac{db_0^\pm(\lambda)}{dt} \mp 2f_0(\lambda)b_0^\pm(\lambda) = 0. \quad (9)$$

The matrix elements of  $T_0(\lambda, t)$  are not independent. They satisfy the ‘‘unitarity’’ condition  $\det T_0(\lambda) \equiv a_0^+ a_0^- + b_0^+ b_0^- = 1$ . Besides, the diagonal elements  $a_0^+$  and  $a_0^-$  allow analytic extension with respect to  $\lambda$  in the upper and lower complex  $\lambda$ -plane, respectively. In fact the minimal set of scattering data which uniquely determines both the scattering matrix and the corresponding potential  $q_0(x)$  consists of two types of variables: i) the reflection coefficients  $\rho_0^\pm(\lambda) = b_0^\pm/a_0^\pm$  defined for real  $\lambda \in \mathbb{R}$  and ii) a discrete set of scattering data including the discrete eigenvalues  $\lambda_k^\pm \in \mathbb{C}_\pm$  and the constants  $C_k^\pm$  which determine the norm of the corresponding Jost solutions [14].

Along with the Jost solutions we can introduce the fundamental analytic solutions (FAS) as follows

$$\chi_0^+(x, t, \lambda) = |\phi_0^+, \psi_0^+|, \quad \chi_0^-(x, t, \lambda) = |\psi_0^-, \phi_0^-|.$$

Indeed, one can prove [22, 2] that  $\chi_0^\pm(x, \lambda)e^{i\lambda\sigma_3 x}$  are analytic functions of  $\lambda$  for  $\lambda \in \mathbb{C}_\pm$ .

The functions  $a_0^\pm(\lambda) = \det \chi^\pm(x, \lambda)$  are known as the **Evans functions** [22, 3] of the system  $L_0(\lambda)$ . Their importance comes from the fact that they are  $t$ -independent (see equation (9)) and, therefore, they (or rather  $\ln a_0^\pm$ ) can be viewed as generating functionals of the (local) integrals of motion. In addition it is known that their zeroes determine the discrete eigenvalues of  $L_0(\lambda)$

$$a_0^+(\lambda_{0k}^+) = 0, \quad \lambda_{0k}^+ \in \mathbb{C}_+, \quad a_0^-(\lambda_{0k}^-) = 0, \quad \lambda_{0k}^- \in \mathbb{C}_-.$$

As a consequence of their analyticity properties one can also show that  $a_0^+(\lambda)$  can be reconstructed from the reflection coefficients,  $\rho_0^\pm = b_0^\pm/a_0^\pm$ , and the bound state

eigenvalues,  $\lambda_{0j}^\pm$ , using the dispersion relations (see [14])

$$\ln a_0^+(\lambda) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{d\mu}{\mu - \lambda} \ln(1 + \rho_0^+ \rho_0^-(\mu)) + \sum_{j=1}^n \ln \frac{\lambda - \lambda_{0j}^+}{\lambda - \lambda_{0j}^-}.$$

Another important aspect is the Lie-algebraic nature of the Lax representation. Indeed, it is natural to view  $U(x, t, \lambda)$  and  $V(x, t, \lambda)$  as elements of a simple<sup>1</sup> Lie algebra  $\mathfrak{g}$ . In what follows for simplicity we assume that  $\mathfrak{g} \simeq \mathfrak{sl}(n)$  which means that  $\text{tr } U(x, t, \lambda) = 0$  and  $\text{tr } V(x, t, \lambda) = 0$ .

After these preliminaries we can define the soliton solutions of the NLEE as the ones for which  $\rho_0^\pm(\lambda) = 0$  for all  $\lambda \in \mathbb{R}$ . Thus, the soliton solutions of the NLEE (5) are parametrized by the discrete eigenvalues and the constants  $C_{0k}^\pm$  whose  $t$ -dependence is determined from

$$\frac{d\lambda_{0k}^\pm}{dt} = 0, \quad i \frac{dC_{0k}^\pm}{dt} \mp 2f_{0k}^\pm C_{0k}^\pm = 0, \quad f_{0k}^\pm = f_0(\lambda_{0k}^\pm).$$

In fact we will analyze the various possible different types of one-soliton solutions – in our case they are determined by one *pair* of discrete eigenvalues  $\lambda_{01}^\pm \in \mathbb{C}_\pm$  and one *pair* of norming constants  $C_{01}^\pm$ . It is obvious that the Zakharov–Shabat system  $L_0(\lambda)$  is related to the only simple Lie algebra of rank one,  $\mathfrak{sl}(2)$ . Thus, for the generic NLEE (5) we get just one type of one-soliton solutions. One of the most effective ways to derive its explicit form consists in using the dressing Zakharov–Shabat method [23]. The main idea consists in introducing the dressing factor  $u(x, t, \lambda)$  which transforms the Jost solutions of  $L_0(\lambda)$  into the Jost solutions of another Zakharov–Shabat system  $L'_0(\lambda)$  but with different potential  $q'_0(x, t)$

$$\begin{aligned} \psi'_0(x, t, \lambda) &= u_0(x, t, \lambda) \psi_0(x, t, \lambda) (u_{0+}(\lambda))^{-1} \\ \phi'_0(x, t, \lambda) &= u_0(x, t, \lambda) \phi_0(x, t, \lambda) (u_{0-}(\lambda))^{-1} \\ u_{0\pm}(\lambda) &= \lim_{x \rightarrow \pm\infty} u_0(x, t, \lambda). \end{aligned}$$

Obviously the dressing factor must satisfy the linear equation

$$i \frac{du_0}{dx} + q'_0(x, t) u_0(x, t, \lambda) - u_0(x, t, \lambda) q_0(x, t) - [\lambda \sigma_3, u_0(x, t, \lambda)] = 0. \quad (10)$$

The effectiveness of the dressing method comes from the fact that equation (10) allows a solution  $u_0(x, t, \lambda)$ , whose  $\lambda$ -dependence is provided by an explicit rational function

$$u_0(x, t, \lambda) = \mathbb{1} + (c_1(\lambda) - 1) P_1(x, t).$$

<sup>1</sup>If  $\mathfrak{g}$  is not simple, then it does not allow non-degenerate bilinear form (the Killing form) and as a result the inverse scattering problem does not allow unique solution.

Here  $c_1(\lambda)$  is the Bläschke–Potapov factor, which relates the transmission coefficients of  $L_0(\lambda)$  and  $L'_0(\lambda)$

$$a_0^+ = c_1(\lambda)a_0^+(\lambda), \quad a_0'^- = \frac{a_0^+(\lambda)}{c_1(\lambda)}, \quad c_1(\lambda) = \frac{\lambda - \lambda_{01}^+}{\lambda - \lambda_{01}^-}$$

and  $P_1(x, t)$  is a projector of rank one. If we denote by  $|n_1(x, t)\rangle$  and  $\langle m_1(x, t)|$  its right and left eigenvectors

$$\begin{aligned} P_1|n_1(x, t)\rangle &= |n_1(x, t)\rangle, & |n_1(x, t)\rangle &= \begin{pmatrix} n_1^1(x, t) \\ n_1^2(x, t) \end{pmatrix} \\ \langle m_1(x, t)|P_1(x, t) &= \langle m_1(x, t)|, & \langle m_1(x, t)| &= \left( m_1^1(x, t), m_1^2(x, t) \right) \end{aligned}$$

then  $P_1$  can be written down in the form

$$P_1(x, t) = \frac{|n_1(x, t)\rangle\langle m_1(x, t)|}{\langle m_1(x, t)|n_1(x, t)\rangle}. \quad (11)$$

Obviously from equation (11) there follows that  $P_1^2 = P_1$ .

The final touch is that in order that  $u_0(x, t, \lambda)$  satisfies (10) the vectors  $|n_1(x, t)\rangle$  and  $\langle m_1(x, t)|$  must satisfy

$$\begin{aligned} i\frac{d|n_1\rangle}{dx} + U_0(x, t, \lambda_{01}^+)|n_1(x, t)\rangle &= 0 \\ i\frac{d\langle m_1|}{dx} - \langle m_1(x, t)|U_0(x, t, \lambda_{01}^-) &= 0 \end{aligned}$$

or in other words

$$|n_1(x, t)\rangle = \chi_0^+(x, \lambda_{01}^+)|n_{10}\rangle, \quad \langle m_1(x, t)| = \langle m_{10}|(\chi_0^-)^{-1}(x, t, \lambda_{01}^-)$$

where  $|n_{10}\rangle$  and  $\langle m_{10}|$  are some constant vectors and  $\chi_0^\pm(x, \lambda)$  are the fundamental analytic solutions of  $L_0(\lambda)$ .

In order to determine the corresponding one-soliton solution we need to determine the potential  $q'_0(x, t)$ . This can be done by taking the limit  $\lambda \rightarrow \infty$  in equation (10) with the result

$$q'_0(x, t) - q_0(x, t) = -(\lambda_1^+ - \lambda_1^-)[\sigma_3, P_1(x, t)]. \quad (12)$$

Therefore, if we know explicitly the Jost solutions (or the fundamental analytic solutions) of the Zakharov–Shabat system for some nontrivial potential  $q_0(x, t)$ , then by applying them to properly chosen constant vectors  $|n_{10}\rangle$  and  $\langle m_{10}|$  we can construct the eigenvectors of  $P_1(x, t)$  and as a result, obtain  $P_1(x, t)$  explicitly, see equation (11). It then remains only to insert it into equation (12) in order to obtain the corresponding potential  $q'_0(x, t)$  explicitly. It can be proved that the spectrum of  $L'_0(\lambda)$  will differ from the spectrum of  $L_0(\lambda)$  only by an additional pair of discrete eigenvalues located at  $\lambda_1^\pm \in \mathbb{C}_\pm$ .

A pure soliton solution is obtained by assuming  $q_0(x, t) = 0$ ; as a result we have

$$\begin{aligned}
|n_1(x, t)\rangle &= e^{(-i(x\lambda_{01}^+ + f_{01}^+ t)\sigma_3)} |n_{10}\rangle \\
\langle m_1(x, t)| &= \langle m_{10}| e^{i(x\lambda_{01}^- + f_{01}^- t)\sigma_3} \\
P_1(x, t) &= \frac{1}{2 \cosh \Phi_0(x, t)} \begin{pmatrix} e^{\Phi_0(x, t)} & \kappa_2 e^{-i\Phi(x, t)} \\ \frac{1}{\kappa_2} e^{i\Phi(x, t)} & e^{-\Phi_0(x, t)} \end{pmatrix} \\
\Phi_0(x, t) &= -i(\lambda_{01}^+ - \lambda_{01}^-)x - i(f_{01}^+ - f_{01}^-)t - \ln \kappa_1 \\
\Phi(x, t) &= (\lambda_{01}^+ + \lambda_{01}^-)x + (f_{01}^+ + f_{01}^-)t
\end{aligned} \tag{13}$$

where  $f_{01}^\pm$  and the constants  $k_0$  and  $\kappa_0$  are given by

$$f_{01}^\pm = f_0(\lambda_{01}^\pm), \quad \kappa_1 = \sqrt{\frac{n_{01}^1 m_{01}^1}{n_{01}^2 m_{01}^2}}, \quad \kappa_2 = \sqrt{\frac{n_{01}^1 m_{01}^2}{n_{01}^2 m_{01}^1}}.$$

Then the corresponding one-soliton solution takes the form

$$q_0^+(x, t) = -\frac{\kappa_2(\lambda_{01}^+ - \lambda_{01}^-)e^{-i\Phi(x, t)}}{\cosh \Phi_0(x, t)}, \quad q_0^-(x, t) = \frac{\kappa_2(\lambda_{01}^+ - \lambda_{01}^-)e^{i\Phi(x, t)}}{\kappa_2 \cosh \Phi_0(x, t)}.$$

**Remark 1.** The eigenvalues  $\lambda_{01}^\pm$  are two independent complex numbers, therefore in the denominator in equation (13) we get cosh of complex argument. This function vanishes whenever its argument becomes equal to  $i(\pi/2 + p\pi)$  for some integer  $p$  and the generic solitons of (5) may have singularities for finite  $x$  and  $t$ .

One way to avoid these singularities is to impose on the Zakharov–Shabat system an involution, i.e., if we constrain the potential  $q_0(x, t)$  by

$$q_0(x, t) = q_0^\dagger(x, t), \quad \text{i.e.,} \quad q_0^+ = (q_0^-)^* = u(x, t). \tag{14}$$

Such constraint reduces the generic systems (5) to NLEE for the single function  $u(x, t)$ ; the second equation of the system becomes consequence of the first one. As a result equation (6) becomes the **NLS equation**

$$iu_t + u_{xx} + 2|u|^2 u(x, t) = 0$$

while equation (7) goes into the MKdV-type equation

$$u_t + u_{xxx} + 6|u(x, t)|^2 u_x = 0.$$

This involution imposes constraints on the scattering data – in particular we have

$$a^+(\lambda) = (a^-(\lambda^*))^*, \quad b^+(\lambda) = (b^-(\lambda^*))^*.$$

From the first relation we find that the zeroes of the functions  $a^\pm(\lambda)$  which are the eigenvalues of  $L_0(\lambda)$  must satisfy

$$\lambda_{01}^+ = (\lambda_{01}^-)^* = \mu_{01} + i\nu_{01}, \quad C_{01}^+ = (C_{01}^-)^*, \quad P_1(x, t) = P_1^\dagger(x, t).$$

So now the one-soliton solution corresponds to a pair of eigenvalues which must be mutually conjugated pairs.

As a result we find that the expression for  $P_1(x, t)$  and the one for the one-soliton solution simplify to

$$P_1(x, t) = \frac{1}{2 \cosh \Phi_{00}(x, t)} \begin{pmatrix} e^{\Phi_{00}(x, t)} & e^{-i\Phi_{01}(x, t)} \\ e^{i\Phi_{01}(x, t)} & e^{-\Phi_{00}(x, t)} \end{pmatrix}$$

$$\Phi_{00}(x, t) = 2\nu_{01}x + 2h_{01}t - \ln \left| \frac{n_{01}^1}{n_{01}^2} \right|$$

$$\Phi_{01}(x, t) = 2\mu_{01}x + 2g_{01}t - \arg n_{01}^1 + \arg n_{01}^2$$

where

$$\lambda_{01}^{\pm} = \mu_{01} \pm i\nu_{01}, \quad f_{01}^{\pm} = g_{01} \pm ih_{01}.$$

Now both functions  $\Phi_{00}(x, t)$  and  $\Phi_{01}(x, t)$  become real valued. The denominator now becomes cosh of real argument, so this soliton solution is a regular function for all  $x$  and  $t$ .

One can impose on  $q_0(x, t)$  a different involution

$$q_0(x, t) = -q_0^{\dagger}(x, t), \quad \text{i.e.,} \quad q_0^+ = -(q_0^-)^* = u(x, t).$$

However, it is well known that under this involution the Zakharov–Shabat system  $L_0(\lambda)$  becomes equivalent to an eigenvalue problem

$$\mathcal{L}\psi(x, t, \lambda) \equiv i\sigma_3 \frac{d\psi}{dx} + \sigma_3 q_0(x, t)\psi(x, t, \lambda) = \lambda\psi(x, t, \lambda)$$

where the operator  $\mathcal{L}$  is a self-adjoint one, so its spectrum must be on the real  $\lambda$ -axis. But the continuous spectrum of  $\mathcal{L}$  fills up the whole real  $\lambda$ -axis, which leaves no room for solitons.

Finally, the Zakharov–Shabat system can be restricted by a third involution, e.g.

$$q_0(x, t) = -q_0^T(x, t), \quad \text{i.e.,} \quad q_0^+ = -q_0^- = -iw_x.$$

Such involution is compatible only with those NLEE whose dispersion law is odd function  $f_0(\lambda) = -f_0(-\lambda)$ . Therefore it can not be applied to the NLS equation – applied to the MKdV equation it gives

$$w_{xt} + w_{xxxx} + 6(w_x(x, t))^2 w_{xx} = 0$$

which can be integrated ones with the result  $v = w_x$

$$v_t + v_{xxx} + 6(v(x, t))^2 v_x = 0$$

i.e., we get the MKdV equation for the real-valued function  $v(x, t)$ . It is well known also that the NLEE with dispersion law  $f(\lambda) = (2\lambda)^{-1}$  can be explicitly

derived under this reduction and comes out to be the famous **sine-Gordon equation** [1]

$$w_{xt} + \sin(2w(x, t)) = 0.$$

This second involution can be imposed together with the one in (14). The restrictions that it imposes on the scattering data are as follows

$$a_0^+(\lambda) = (a_0^-(\lambda^*))^*, \quad a_0^+(\lambda) = (a_0^-(-\lambda)).$$

Now if  $\lambda_{01}^+$  is an eigenvalue of  $L_0(\lambda)$  then  $(\lambda_{01}^+)^*$ ,  $-\lambda_{01}^+$  and  $-(\lambda_{01}^+)^*$  must also be eigenvalues. This means that we can have two configurations of eigenvalues

1. Pairs of purely imaginary eigenvalues

$$\lambda_{01}^+ = i\nu_{01} \equiv -(\lambda_{01}^+)^*, \quad \lambda_{01}^- = -i\nu_{01} \equiv -(\lambda_{01}^-)^*.$$

2. Quadruplets of complex eigenvalues

$$\begin{aligned} \lambda_{02}^+ &= \mu_{02} + i\nu_{02}, & -(\lambda_{02}^+)^* &= -\mu_{02} + i\nu_{02} \\ \lambda_{02}^- &= \mu_{02} - i\nu_{02}, & -(\lambda_{02}^-)^* &= -\mu_{02} - i\nu_{02}. \end{aligned}$$

Thus, we conclude, that the sine-Gordon and MKdV equations allow *two types* of solitons: type 1 with purely imaginary pairs of eigenvalues and type 2 each corresponding to a quadruplet of eigenvalues. Type 1 solitons are known also as **topological solitons**, or **kinks** (for details see [4]). They are parametrized by two real parameters  $\nu_{01}$  and  $|C_{01}|^+$  so they have just one degree of freedom corresponding to the uniform motion.

Type 2 solitons are known as the **breathers** and are parametrized by four real parameters:  $\mu_{02}$  and  $\nu_{02}$  and the real and imaginary parts of  $C_{02}^+$ . Therefore they have two degrees of freedom – one corresponds to the uniform motion and the second one describes the internal degree of freedom responsible for the “breathing”.

The purpose of presenting the above well-known facts in the above manner, was simply to make it clear that the structure, as well as the number of related parameters which determine what different types of solitons can exist, depend strongly on the type of, and the number of, different involutions that can be imposed on the system.

### 3. Generalized Zakharov–Shabat System and $\mathfrak{sl}(n)$ Solitons

For the sake of simplicity and clarity below, most of our discussions will be restricted to the case  $n = 5$ . However, they also could easily be reformulated for any other chosen value of  $n$ . The corresponding Lax operator  $L(\lambda)$  which is a particular case of equation (2) with

$$U(x, t, \lambda) = [J, Q(x, t)] - \lambda J, \quad J = \text{diag}(J_1, J_2, J_3, J_4, J_5)$$

$$q(x, t) = [J, Q(x, t)], \quad Q(x, t) = \begin{pmatrix} 0 & Q_{12} & Q_{13} & Q_{14} & Q_{15} \\ Q_{21} & 0 & Q_{23} & Q_{24} & Q_{25} \\ Q_{31} & Q_{32} & 0 & Q_{34} & Q_{35} \\ Q_{41} & Q_{42} & Q_{43} & 0 & Q_{45} \\ Q_{51} & Q_{52} & Q_{53} & Q_{54} & 0 \end{pmatrix}.$$

Furthermore, for definiteness we will assume that

$$\operatorname{tr} J = 0, \quad J_1 > J_2 > J_3 > 0, \quad 0 > J_4 > J_5. \quad (15)$$

The class of NLEE related to  $L(\lambda)$  are systems of equations for the functions  $Q_{jk}^\pm(x, t)$ , which may be written in the compact form [19, 16, 11, 6]

$$i \frac{\partial Q}{\partial t} + 2 \sum_{k=1}^4 \Lambda^k [H_k, Q(x, t)] = 0$$

where  $H_k$ ,  $\operatorname{tr} H_k = 0$  are constant diagonal matrices and  $f(\lambda) = \sum_{k=1}^4 \lambda^k H_k$  is the dispersion law of the NLEE. Here and below we define

$$(\operatorname{ad}_J X)_{ks} \equiv ([J, X])_{ks} = (J_k - J_s) X_{ks}, \quad (\operatorname{ad}_J^{-1} X)_{ks} = \frac{X_{ks}}{J_k - J_s}$$

for all  $X \in \mathcal{M}_J$ , i.e.,  $X_{kk} = 0$ . The operator  $\Lambda$  is (anyone of) the recursion operators  $\Lambda_\pm$ , acting on the space  $\mathcal{M}_J$  of  $5 \times 5$  off-diagonal matrix-valued functions as follows

$$\begin{aligned} \Lambda_\pm X \equiv & \operatorname{ad}_J^{-1} \left( i \frac{dX}{dx} + P_0[q(x), X(x)] \right. \\ & \left. + i \sum_{k=1}^5 [Q(x), E_{k,k}] \int_{\pm\infty}^x dy \operatorname{tr} (Q(y), X(y)), E_{k,k} \right) \end{aligned}$$

where  $P_0 \cdot$  is the projector  $\operatorname{ad}_J^{-1} \operatorname{ad}_J \cdot$ . Choosing  $H_1 = I = \operatorname{diag}(b_1, \dots, b_5)$ , so that the dispersion law  $f(\lambda) = \lambda I$  is a linear function of  $\lambda$  we get a system, generalizing the well known ***N-wave equation***

$$i[J, Q_t] - i[I, Q_x] - [[J, Q], [I, Q]] = 0 \quad (16)$$

which for  $n = 5$  contains  $N = n(n-1) = 20$  complex-valued functions  $Q_{ij}(x, t)$ . The  $M$ -operator in the Lax representation for the  $N$ -wave equation (16) is given by

$$M\psi(x, t, \lambda) \equiv i \frac{d\psi}{dt} + [I, Q(x, t)] - \lambda I \psi(x, t, \lambda) = -\lambda \psi(x, t, \lambda) I.$$

Following the idea of the ISM we have to analyze and solve the direct and the inverse scattering problems for  $L(\lambda)$ . To this end we again introduce the Jost solutions of  $L(\lambda)$  as  $5 \times 5$  matrix-valued solutions

$$L(\lambda)\psi(x, t, \lambda) = 0, \quad L(\lambda)\phi(x, t, \lambda) = 0$$

defined by their asymptotic behavior for  $x \rightarrow \pm\infty$  respectively

$$\lim_{x \rightarrow \infty} \psi(x, t, \lambda) e^{i\lambda Jx} = \mathbb{1}, \quad \lim_{x \rightarrow -\infty} \phi(x, t, \lambda) e^{i\lambda Jx} = \mathbb{1}. \quad (17)$$

The definitions (17) hold true for all  $t$ . The scattering matrix  $T(\lambda, t)$  is introduced by

$$T(\lambda, t) \equiv (\psi(x, t, \lambda))^{-1} \phi(x, t, \lambda).$$

It is  $x$ -independent and its  $t$ -dependence in the  $N$ -wave case, which follows from the Lax representation is

$$i \frac{dT}{dt} - [\lambda I, T(\lambda, t)] = 0.$$

Thus, if  $Q(x, t)$  satisfies the  $N$ -wave system (16) we get

$$\frac{dT_{kk}(\lambda)}{dt} = 0, \quad i \frac{dT_{jk}(\lambda)}{dt} - \lambda(b_j - b_k)T_{jk}(\lambda, t) = 0.$$

The set of matrix elements of  $T(\lambda, t)$  must satisfy a number of relations. Indeed, they are uniquely determined by  $Q(x, t)$ , i.e., by  $n(n-1)$  complex functions of  $x$ , so it seems natural that there should not be more than  $n(n-1)$  independent functions among  $T_{jk}(\lambda)$  for  $\lambda$  on the real axis. Of course  $T(\lambda, t)$  must satisfy the ‘‘unitarity’’ condition  $\det T(\lambda, t) = 1$ . The rest of these relations follow from the analyticity properties of certain combinations of matrix elements of  $T(\lambda, t)$ .

These analyticity properties must follow naturally from the definition of the corresponding **fundamental analytic solutions** (FAS)  $\chi^\pm(x, t, \lambda)$ . However establishing the relations between FAS and the Jost solutions is not that so simple. Indeed, only the first and the last columns of  $\psi(x, t, \lambda)$  and  $\phi(x, t, \lambda)$  allow analytic extensions in  $\lambda$  off the real axis – the other columns *do not* have analyticity properties. Nevertheless it is again possible to introduce FAS [18, 20] which are defined as follows

$$\chi^\pm(x, \lambda) = \phi(x, \lambda) \mathbb{S}^\pm(\lambda) = \psi(x, \lambda) \mathbb{T}^\mp(\lambda).$$

Here the triangular  $n \times n$  matrices  $\mathbb{S}^\pm(\lambda)$  and  $\mathbb{T}^\pm(\lambda)$  are related to the scattering matrix  $T(\lambda)$  by

$$T(\lambda) = \mathbb{T}^-(\lambda) (\mathbb{S}^+(\lambda))^{-1} = \mathbb{T}^+(\lambda) (\mathbb{S}^-(\lambda))^{-1}.$$

The matrix elements of  $\mathbb{T}^\pm(\lambda)$  and  $\mathbb{S}^\pm(\lambda)$  can be expressed in terms of the minors of  $T(\lambda)$ , see the Appendix. Here we note that their diagonal elements can be given by

$$\begin{aligned} \mathbb{S}_{jj}^+(\lambda) &= m_{j-1}^+(\lambda), & \mathbb{T}_{jj}^-(\lambda) &= m_j^+(\lambda) \\ \mathbb{T}_{jj}^+(\lambda) &= m_{n-j}^-(\lambda), & \mathbb{S}_{jj}^-(\lambda) &= m_{n+1-j}^-(\lambda) \end{aligned}$$

where  $m_0^\pm = m_n^\pm = 1$  and by  $m_k^+(\lambda)$  (respectively  $m_k^-(\lambda)$ ) we have denoted the upper (respectively lower) principal minors of  $T(\lambda)$  of order  $k$ , e.g.,

$$m_1^+(\lambda) = T_{11}(\lambda), \quad m_2^+(\lambda) = \left\{ \begin{array}{cc} 1 & 2 \\ 1 & 2 \end{array} \right\}_{T(\lambda)} \equiv T_{11}(\lambda)T_{22}(\lambda) - T_{12}(\lambda)T_{21}(\lambda)$$

$$m_3^+(\lambda) = \left\{ \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 2 & 3 \end{array} \right\}_{T(\lambda)}, \quad m_4^+(\lambda) = \left\{ \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{array} \right\}_{T(\lambda)}$$

$$m_1^-(\lambda) = T_{55}(\lambda), \quad m_2^-(\lambda) = \left\{ \begin{array}{cc} 4 & 5 \\ 4 & 5 \end{array} \right\}_{T(\lambda)} \equiv T_{44}(\lambda)T_{55}(\lambda) - T_{45}(\lambda)T_{54}(\lambda)$$

$$m_3^-(\lambda) = \left\{ \begin{array}{ccc} 3 & 4 & 5 \\ 3 & 4 & 5 \end{array} \right\}_{T(\lambda)}, \quad m_4^-(\lambda) = \left\{ \begin{array}{cccc} 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 \end{array} \right\}_{T(\lambda)}.$$

As a consequence of the analyticity of the FAS, it follows that the minors  $m_k^+(\lambda)$  (respectively  $m_k^-(\lambda)$ ) are analytic functions for  $\lambda \in \mathbb{C}_+$  (respectively for  $\lambda \in \mathbb{C}_-$ ). One can construct the kernel of the resolvent of  $L(\lambda)$  in terms of the FAS [11, 6] from which it follows that the resolvent has poles for all  $\lambda_k^\pm$  which happen to be zeroes of any of the minors  $m_k^\pm(\lambda)$ . Therefore, what we have now is that each of the minors  $m_k^\pm(\lambda)$  may be considered to be an analog of the Evans function, and thus now, there is more than one Evans function.

In order to understand what is going on we calculate the scattering matrix for the dressed operator  $L$ . This is easy since the Jost solutions of  $L_0$  are related to the ones of  $L$  through

$$\psi'(x, \lambda) = u(x, \lambda)\psi(x, \lambda)\hat{u}_+(\lambda), \quad \phi'(x, \lambda) = u(x, \lambda)\phi(x, \lambda)\hat{u}_-(\lambda) \quad (18)$$

$$u_\pm(\lambda) = \lim_{x \rightarrow \pm\infty} u(x, \lambda) = \mathbb{1} + (c_1(\lambda) - 1)P_\pm. \quad (19)$$

The factors  $\hat{u}_\pm(\lambda)$  in the right hand sides of equation (18) ensure that

$$\lim_{x \rightarrow \infty} \psi(x, \lambda)e^{iJ\lambda x} = \mathbb{1}, \quad \lim_{x \rightarrow -\infty} \phi(x, \lambda)e^{iJ\lambda x} = \mathbb{1}.$$

### 3.1. Generic Rank One One-Soliton Solutions

Here we describe the dressing Zakharov–Shabat method. We start with the Lax operator  $L$  with potential  $q(x)$  such that the corresponding FAS  $\chi^\pm(x, \lambda)$  are non-degenerate matrices in their regions of analyticity. Next we construct the FAS  $\chi'^\pm(x, \lambda)$  of the Lax operator  $L'(\lambda)$  with potential  $q'(x)$  whose FAS  $\chi'^\pm(x, \lambda)$  have zeroes and/or singularities at for prescribed values  $\lambda_1^\pm \in \mathbb{C}_\pm$  of  $\lambda$ . Therefore  $L'$  will have two additional discrete eigenvalues  $\lambda_1^\pm$ . The two sets of FAS are related by the dressing factor  $u(x, \lambda)$

$$L\chi^\pm(x, \lambda) = 0, \quad \chi^\pm = \phi(x, \lambda)\mathbb{S}^\pm(\lambda) = \psi(x, \lambda)\mathbb{T}^\mp(\lambda) \quad (20)$$

$$L' \chi'^{\pm}(x, \lambda) = 0, \quad \chi'^{\pm} = \phi(x, \lambda) \mathbb{S}'^{\pm}(\lambda) = \psi(x, \lambda) \mathbb{T}'^{\pm}(\lambda) \quad (21)$$

$$\chi'^{\pm}(x, \lambda) = u(x, \lambda) \chi^{\pm}(x, \lambda) u_{-}^{-1}(\lambda) \quad (22)$$

where

$$\begin{aligned} u(x, \lambda) &= \mathbb{1} + (c_1(\lambda) - 1) P_1(x), & c_1(\lambda) &= \frac{\lambda - \lambda_1^+}{\lambda - \lambda_1^-} \\ P_1(x) &= \frac{|\bar{n}_1(x)\rangle \langle \bar{m}_1(x)|}{\langle \bar{m}_1(x) | \bar{n}_1(x) \rangle}, & u_{\pm}(\lambda) &= \lim_{x \rightarrow -\infty} u(x, \lambda) \\ |\bar{n}_1(x)\rangle &= \chi^+(x, \lambda_1^+) |\bar{n}_{01}\rangle, & \langle \bar{m}_1(x) | &= \langle \bar{m}_{01} | \hat{\chi}^-(x, \lambda_1^-) \\ |\bar{n}_{01}\rangle &= \begin{pmatrix} n_{01}^1 \\ n_{01}^2 \\ \vdots \\ n_{01}^5 \end{pmatrix}, & \langle \bar{m}_{01} | &= (m_{01}^1, m_{01}^2, \dots, m_{01}^5). \end{aligned}$$

One can check, that the dressing factor defined as above satisfies identically the equation

$$i \frac{du}{dx} + q'(x)u(x, \lambda) - u(x, \lambda)q(x) - \lambda[J, u(x, \lambda)] = 0$$

where the potential  $q'(x)$  is given by

$$q'(x) = \lim_{\lambda \rightarrow \infty} \lambda \left( J - u(x, \lambda) J u^{-1}(x, \lambda) \right) = -(\lambda_1^+ - \lambda_1^-) [J, P_1(x)].$$

They are parametrized by:

1. the discrete eigenvalues  $\lambda_1^{\pm} = \mu_1 \pm i\nu_1$ ;  $\mu_1^{\pm}$  determine the soliton velocity, and  $\nu_1^{\pm}$  determine the amplitude.
2. the “polarization” vectors,  $|\bar{n}_{01}\rangle$ ,  $\langle \bar{m}_{01} |$  parametrize the internal degrees of freedom of the soliton. Note that  $P_1(x)$  is invariant under the scaling of each of these vectors. Generically each “polarization” has five components, one of which can be fixed, say to one. So each “polarization” is determined by four independent complex parameters.

We have several options that will lead to different types of solitons:

- 1) generic case when all components of  $|\bar{n}_{01}\rangle$  are non-vanishing;
  - 2) several special subcases when one (or several) of these components vanish.
- The corresponding solitons will have different structures and properties.

For the generic choice of  $|\bar{n}_{01}\rangle$  one finds

$$\lim_{x \rightarrow \pm\infty} P_1(x, t) = P_{1\pm}, \quad P_{1+} = E_{11}, \quad P_{1-} = E_{nn} \quad (23)$$

where the matrix  $E_{kj}$  has only one non-vanishing matrix element equal to 1 at position  $k, j$ , i.e.,  $(E_{kj})_{mp} = \delta_{km}\delta_{jp}$ . Therefore both the limiting values  $u_{\pm}(\lambda)$  and their inverse  $\hat{u}_{\pm}(\lambda)$  are diagonal matrices

$$u_+(\lambda) = \text{diag}(c_1(\lambda), 1, 1, \dots, 1), \quad u_-(\lambda) = \text{diag}(1, 1, \dots, 1, c_1(\lambda)). \quad (24)$$

From equations (18) we get

$$T'(\lambda) \equiv (\psi')^{-1}(x, \lambda)\phi'(x, \lambda) = u_+T(\lambda)u_-^{-1}(\lambda) \quad (25)$$

i.e., for  $n = 5$  we have

$$\begin{aligned} T'_{1j}(\lambda) &= c_1(\lambda)T_{1j}(\lambda), & j &= 1, 2, 3, 4; \\ T'_{j5}(\lambda) &= T_{j5}(\lambda)/c_1(\lambda), & j &= 2, 3, 4, 5; \\ T'_{ij}(\lambda) &= T_{ij}(\lambda), & & \text{for all other values of } i, j. \end{aligned}$$

This relation allows us to derive the interrelations between the Gauss factors of  $T(\lambda)$  and  $T'(\lambda)$ . In particular we find for the principal minors of  $T'(\lambda)$

$$m_k'^+(\lambda) = c_1(\lambda)m_k^+(\lambda), \quad m_k'^-(\lambda) = m_k^-(\lambda)/c_1(\lambda) \quad (26)$$

where  $m_k'^+(\lambda)$  (respectively  $m_k'^-(\lambda)$ ) are the upper (respectively lower) principal minors of  $T'(\lambda)$ . Since  $\chi^{\pm}(x, t, \lambda)$  are regular solutions of the Riemann–Hilbert problem then  $m_k^{\pm}(\lambda)$  have no zeroes at all, but equation (26) means *all*  $m_k^{\pm}(\lambda)$  have a simple zero at  $\lambda = \lambda_1^{\pm}$ .

The generic one-soliton solution then is obtained by taking that  $\chi^{\pm}(x, t, \lambda) = e^{-i\lambda(Jx+It)}$ . As a result we get

$$\begin{aligned} (P_1(x, t))_{ks} &= \frac{1}{k(x, t)} n_{01}^k m_{01}^s e^{-i(\lambda_1^+ R_k - \lambda_1^- R_s)} \\ k(x, t) &= \sum_{p=1}^n n_{01}^p m_{01}^p e^{-i(\lambda_1^+ - \lambda_1^-) R_p(x, t)} \\ R_k(x, t) &= J_k x + I_k t, \quad q_{ks}^{1s} = -(\lambda_1^+ - \lambda_1^-)(P_1(x, t))_{ks} \end{aligned} \quad (27)$$

i.e., in all channels we have non-trivial waves. The number of internal degrees of freedom is  $2(n-1) = 8$ . Note that the denominator  $k(x, t)$  is linear combination of exponentials with complex arguments, so it could vanish for certain values of  $x$  and  $t$ . Thus, the generic soliton (27) in this case is a singular solution.

Next we impose on  $U(x, t, \lambda)$  the involution

$$B_0 U^\dagger(x, t, \lambda^*) B_0^{-1} = U(x, t, \lambda), \quad B_0 = \text{diag}(\epsilon_1, \dots, \epsilon_n) \quad (28)$$

with  $\epsilon_j = \pm 1$ . More specifically this means that

$$B_0 q^\dagger(x, t) B_0^{-1} = q(x, t), \quad B_0 u^\dagger(x, t, \lambda^*) B_0^{-1} = u^{-1}(x, t, \lambda)$$

and

$$\lambda_1^+ = (\lambda_1^-)^* = \mu_1 + i\nu_1, \quad \langle \vec{m}_{01} | = (B_0 | \vec{n}_{01} \rangle)^\dagger.$$

Thus, only  $|\vec{n}_{01}\rangle$  is independent.

Then the one-soliton solution simplifies to

$$q_{ks}^{1s}(x, t) = -\frac{2i\nu_1(J_k - J_s)}{k_{\text{red}}(x, t)} \epsilon_s n_{01}^k (n_{01}^s)^* e^{\nu_1(R_k + R_s)} e^{-i\mu_1(R_k - R_s)}$$

$$k_{\text{red}}(x, t) = \sum_{p=1}^n \epsilon_p |n_{01}^p|^2 e^{2\nu_1 R_p(x, t)}.$$

The number of internal degrees of freedom now is  $n - 1 = 4$ . If one or more of  $\epsilon_p$  are different, then this reduced soliton may still have singularities. The singularities are absent only if all  $\epsilon_j$  are equal.

### 3.2. Non-Generic $\mathfrak{sl}(2)$ Solitons

From now on we assume that the reduction (28) with  $\epsilon_p = 1$  holds.

Here  $|\vec{n}_{01}\rangle$  has only two non-vanishing components. We consider here three examples with  $n = 5$  and three different choices for the polarization vectors

$$\text{a) } |\vec{n}_{01}\rangle = \begin{pmatrix} n_{01}^1 \\ 0 \\ 0 \\ 0 \\ n_{01}^5 \end{pmatrix}; \quad \text{b) } |\vec{n}_{01}\rangle = \begin{pmatrix} 0 \\ n_{01}^2 \\ 0 \\ n_{01}^4 \\ 0 \end{pmatrix}; \quad \text{c) } |\vec{n}_{01}\rangle = \begin{pmatrix} n_{01}^1 \\ n_{01}^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (29)$$

In all these cases the corresponding one-soliton solutions  $q'(x, t)$  are given by similar analytic expressions, each having only two non-vanishing matrix elements

$$q_{jk}(x, t) = (q_{jk}(x, t))^*$$

$$= -\frac{i\nu_1(J_j - J_k) e^{i(\arg(n_{01}^j) - \arg(n_{01}^k))} e^{-i\mu_1(J_j - J_k)(x + w_{jk}t)}}{\cosh[\nu_1(J_j - J_k)(x + w_{jk}t) + \ln |n_{01}^j| - \ln |n_{01}^k|]}$$

where we remind that  $w_{jk} = (I_j - I_k)/(J_j - J_k)$ ,  $j < k$ . For the case a) we have  $j = 1$ ,  $k = 5$ ; in case b)  $j = 2$ ,  $k = 4$  and in case c)  $j = 1$  and  $k = 2$ .

The  $\mathfrak{sl}(2)$  soliton is very much like the NLS soliton (apart from the  $t$ -dependence); the NLS soliton has only one internal degree of freedom.

The different choices for the polarization vector result in different asymptotics for the projector  $P_1(x, t)$ :

$$\text{a) } \lim_{x \rightarrow \infty} P_1(x, t) = E_{11}, \quad \lim_{x \rightarrow -\infty} P_1(x, t) = E_{55}$$

$$\text{b) } \lim_{x \rightarrow \infty} P_1(x, t) = E_{22}, \quad \lim_{x \rightarrow -\infty} P_1(x, t) = E_{44}$$

$$\text{c) } \lim_{x \rightarrow \infty} P_1(x, t) = E_{11}, \quad \lim_{x \rightarrow -\infty} P_1(x, t) = E_{22}.$$

In case a) the results for the limits of  $P_1(x, t)$  and for  $u_{\pm}(\lambda)$  are the same as for the generic case, see equations (23), (24). As a consequence, such  $\mathfrak{sl}(2)$  solitons requires the vanishing of *all* Evans functions  $m_k^{\pm}(\lambda)$  for  $\lambda = \lambda_1^{\pm}$ , see equation (26). In case b) from equation (25) and from the Appendix we get that such  $\mathfrak{sl}(2)$  soliton provides for the vanishing of  $m_2^{\pm}(\lambda)$  and  $m_3^{\pm}(\lambda)$

$$\begin{aligned} m_2^+(\lambda) &= c_1(\lambda)m_2^+(\lambda), & m_3^+(\lambda) &= c_1(\lambda)m_3^+(\lambda) \\ m_2^-(\lambda) &= m_2^-(\lambda)/c_1(\lambda), & m_3^-(\lambda) &= m_3^-(\lambda)/c_1(\lambda) \end{aligned}$$

whereas  $m_1^{\pm}(\lambda) = m_1^{\pm}(\lambda)$  and  $m_4^{\pm}(\lambda) = m_4^{\pm}(\lambda)$  remain regular and do not have zeros at  $\lambda = \lambda_1^{\pm}$ .

Likewise in case c) we get that only  $m_1^+(\lambda)$  and  $m_4^-(\lambda)$  acquire zeroes

$$m_1^+(\lambda) = c_1(\lambda)m_1^+(\lambda), \quad m_4^-(\lambda) = m_4^-(\lambda)/c_1(\lambda)$$

and all the other Evans functions  $m_j^+(\lambda)$  with  $j = 2, 3, 4$ , and  $m_p^-(\lambda)$  with  $p = 1, 2, 3$  do not have zeroes.

### 3.3. Non-generic $\mathfrak{sl}(3)$ -solitons

Here  $|\vec{n}_{01}\rangle$  has three non-vanishing components. We consider three examples of such polarization vectors

$$\text{a) } |\vec{n}_{01}\rangle = \begin{pmatrix} n_{01}^1 \\ 0 \\ n_{01}^3 \\ 0 \\ n_{01}^5 \end{pmatrix}, \quad \text{b) } |\vec{n}_{01}\rangle = \begin{pmatrix} 0 \\ n_{01}^2 \\ n_{01}^3 \\ n_{01}^4 \\ 0 \end{pmatrix}, \quad \text{c) } |\vec{n}_{01}\rangle = \begin{pmatrix} n_{01}^1 \\ n_{01}^2 \\ n_{01}^3 \\ 0 \\ 0 \end{pmatrix}. \quad (30)$$

Therefore the  $\mathfrak{sl}(3)$ -solitons have two internal degrees of freedom.

The asymptotics of the projector  $P_1(x, t)$  read as follows:

$$\begin{aligned} \text{a) } \lim_{x \rightarrow \infty} P_1(x, t) &= E_{11}, & \lim_{x \rightarrow -\infty} P_1(x, t) &= E_{55} \\ \text{b) } \lim_{x \rightarrow \infty} P_1(x, t) &= E_{22}, & \lim_{x \rightarrow -\infty} P_1(x, t) &= E_{44} \\ \text{c) } \lim_{x \rightarrow \infty} P_1(x, t) &= E_{11}, & \lim_{x \rightarrow -\infty} P_1(x, t) &= E_{33}. \end{aligned} \quad (31)$$

Note that cases a) and b) in equation (31) coincide with the corresponding cases in equation (29). Therefore, the set of Evans functions that acquire zeroes will be the same as for the corresponding  $\mathfrak{sl}(2)$  solitons. In case c) of equation (31) we have

$$\begin{aligned} m_1^+(\lambda) &= c_1(\lambda)m_1^+(\lambda), & m_2^+(\lambda) &= c_1(\lambda)m_2^+(\lambda) \\ m_4^-(\lambda) &= m_4^-(\lambda)/c_1(\lambda), & m_3^-(\lambda) &= m_3^-(\lambda)/c_1(\lambda) \end{aligned}$$

whereas the remaining Evans functions  $m_j^+(\lambda)$  with  $j = 3, 4$ , and  $m_p^-(\lambda)$  with  $p = 1, 2$  remain regular.

In case a) the corresponding one-soliton solutions acquire the form

$$\begin{aligned} \text{a) } q^{1s}(x, t) &= \begin{pmatrix} 0 & 0 & q_{13} & 0 & q_{15} \\ 0 & 0 & 0 & 0 & 0 \\ q_{13}^* & 0 & 0 & 0 & q_{35} \\ 0 & 0 & 0 & 0 & 0 \\ q_{15}^* & 0 & q_{35}^* & 0 & 0 \end{pmatrix}, & \text{b) } q^{1s}(x, t) &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{23} & q_{24} & 0 \\ 0 & q_{23}^* & 0 & q_{34} & 0 \\ 0 & q_{24}^* & q_{34}^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \text{c) } q^{1s}(x, t) &= \begin{pmatrix} 0 & q_{12} & q_{23} & 0 & 0 \\ q_{12}^* & 0 & q_{23} & 0 & 0 \\ q_{13}^* & q_{23}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

where the matrix elements  $q_{ks}(x, t)$  are given by

$$\begin{aligned} q_{ks}(x, t) &= (q_{sk}(x, t))^* \\ &= \frac{i\nu_1(J_k - J_s)\epsilon_s e^{\nu_1(\tilde{J}_k + \tilde{J}_s)(x + \tilde{v}_{ks}t)} n_{01}^k (n_{01}^s)^* e^{-i\mu_1(J_k - J_s)(x + w_{15}t)}}{|n_{01}^1|^2 e^{2\nu_1(\tilde{J}_1 x + \tilde{I}_1 t)} + |n_{01}^3|^2 e^{2\nu_1(\tilde{J}_3 x + \tilde{I}_3 t)} + |n_{01}^5|^2 e^{2\nu_1(\tilde{J}_5 x + \tilde{I}_5 t)}} \end{aligned}$$

and

$$\tilde{J}_k = J_k - (J_1 + J_3 + J_5)/3, \quad \tilde{I}_k = I_k - (I_1 + I_3 + I_5)/3, \quad \tilde{v}_{ks} = \frac{\tilde{J}_k + \tilde{J}_s}{\tilde{I}_k + \tilde{I}_s}.$$

This soliton has two internal degrees of freedom and is regular.

Obviously it is by now clear how one can write down more complicated solitons like  $\mathfrak{sl}(4)$  which would be characterized by polarization vectors of the form:

$$\text{a) } |\vec{n}_{01}\rangle = \begin{pmatrix} n_{01}^1 \\ n_{01}^2 \\ n_{01}^3 \\ n_{01}^4 \\ 0 \end{pmatrix}, \quad \text{b) } |\vec{n}_{01}\rangle = \begin{pmatrix} n_{01}^1 \\ n_{01}^2 \\ n_{01}^3 \\ 0 \\ n_{01}^5 \end{pmatrix}, \quad \dots$$

The  $\mathfrak{sl}(4)$ -solitons will have three internal degrees of freedom.

We note here that due to our choice of  $J$  in (15),  $\mathfrak{sl}(4)$ -solitons cannot give rise to generalized eigenfunctions.

#### 4. Eigenfunctions and Eigensubspaces

The structure of these eigensubspaces and the corresponding solitons becomes more complicated with the growth of  $n$ .

In what follows we start with the generic case and split the “polarization” vector into two parts

$$|\vec{n}_{01}\rangle = |\vec{p}_{01}\rangle + |\vec{d}_{01}\rangle, \quad |\vec{p}_{01}\rangle = \begin{pmatrix} n_{01}^1 \\ n_{01}^2 \\ n_{01}^3 \\ 0 \\ 0 \end{pmatrix}, \quad |\vec{d}_{01}\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ n_{01}^4 \\ n_{01}^5 \end{pmatrix} \quad (32)$$

and, therefore,

$$|\vec{n}_1\rangle = |\vec{p}_1\rangle + |\vec{d}_1\rangle, \quad |\vec{p}_1\rangle = \chi^+(x, t, \lambda_1^+) |\vec{p}_{01}\rangle, \quad |\vec{d}_1\rangle = \chi^+(x, t, \lambda_1^+) |\vec{d}_{01}\rangle.$$

This splitting is compatible with equation (15) and has the advantage: if  $\chi^+(x, t, \lambda_1^+) = e^{-i\lambda_1^+ Jx}$  then  $|\vec{p}_1\rangle$  increases exponentially for  $x \rightarrow \infty$  and decreases exponentially for  $x \rightarrow -\infty$ ;  $|\vec{d}_1\rangle$  decreases exponentially for  $x \rightarrow \infty$  and increases exponentially for  $x \rightarrow -\infty$ , see also the lemma below.

What we will prove below is that one can take a special linear combination of the columns of  $\chi^+(x, t, \lambda_1^+)$  which decreases exponentially for both  $x \rightarrow \infty$  and  $x \rightarrow -\infty$ . Doing this we will use the fact that

$$\begin{aligned} \chi^+(x, t, \lambda_1^+) |\vec{n}_{01}\rangle &\equiv (\mathbb{1} - P_1(x, t)) \chi^+(x, t, \lambda_1^+) |\vec{n}_{01}\rangle \\ &= (\mathbb{1} - P_1(x, t)) |\vec{n}_1(x, t)\rangle = 0. \end{aligned} \quad (33)$$

**Lemma 1.** *The eigenfunctions of  $L$  provided by*

$$\mathbf{f}^+(x, t) = \chi^+(x, t, \lambda_1^+) |\vec{p}_{01}\rangle = -\chi^+(x, t, \lambda_1^+) |\vec{d}_{01}\rangle \quad (34)$$

*decrease exponentially for both  $x \rightarrow \infty$  and  $x \rightarrow -\infty$ .*

**Proof:** From equations (33) and (32) there follows that both expressions for  $\mathbf{f}^+(x, t)$  coincide, so we can use each of them to our advantage, see equation (34). We will use also the fact that  $\mathbb{1} - P_1(x, t)$  is a bounded function of both  $x$  and  $t$ .

We start with

$$\begin{aligned} \lim_{x \rightarrow \infty} \mathbf{f}^+(x, t) &= \lim_{x \rightarrow \infty} \chi'^+(x, t, \lambda_1^+) |\vec{d}_{01}\rangle \\ &= (\mathbb{1} - P_{1+}) \lim_{x \rightarrow \infty} e^{-i\lambda_1^+ (Jx + It)} \mathbb{T}^-(\lambda_1^+) |\vec{d}_{01}\rangle \end{aligned}$$

where  $\mathbb{T}^-(\lambda_1^+)$  is the lower triangular matrix introduced in equation (35). If the potential is on finite support or is reflectionless then  $\mathbb{T}^-(\lambda)$  is rational function well defined for  $\lambda = \lambda_1^+$ . If the potential is generic then  $\mathbb{T}^-(\lambda)$  does not allow analytic continuation off the real axis. Nevertheless  $\mathbb{T}^-(\lambda_1^+)$  can be understood as lower triangular constant matrix (generalizing the constant  $C_{01}^+$  of the NLS case).

Being lower triangular  $\mathbb{T}^-(\lambda_1^+)$  maps  $|\vec{d}_{01}\rangle$  onto  $|\vec{d}'_{01}\rangle = \mathbb{T}_0^-(\lambda_1^+)|\vec{d}_{01}\rangle$  which is again of the form (32), i.e., its first three components vanish. Therefore,

$$\lim_{x \rightarrow \infty} e^{\nu_1 a x} \mathbf{f}^+(x, t) = \lim_{x \rightarrow \infty} (\mathbb{1} - P_{1+}) e^{\nu_1 a x} \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{-i\lambda_1^+(J_4 x + I_4 t)} n_{01}^{4'} \\ e^{-i\lambda_1^+(J_5 x + I_5 t)} n_{01}^{5'} \end{pmatrix} = 0$$

for any constant  $a > 0$  such that  $a + J_4 < 0$ .

Likewise we can calculate the limit for  $x \rightarrow -\infty$

$$\begin{aligned} \lim_{x \rightarrow -\infty} \mathbf{f}^+(x, t) &= - \lim_{x \rightarrow -\infty} \chi'^+(x, t, \lambda_1^+) |\vec{p}_{01}\rangle \\ &= -(\mathbb{1} - P_{1+}) \lim_{x \rightarrow \infty} e^{-i\lambda_1^+(Jx + It)} \mathbb{S}^+(\lambda_1^+) |\vec{p}_{01}\rangle. \end{aligned}$$

The upper triangular matrix  $\mathbb{S}^+(\lambda_1^+)$  is treated analogously as  $\mathbb{T}^-(\lambda_1^+)$ . In the generic case it is just an upper triangular constant matrix which maps  $|\vec{p}_{01}\rangle$  onto  $|\vec{p}'_{01}\rangle = \mathbb{S}^+(\lambda_1^+)|\vec{p}_{01}\rangle$  whose last two components vanish. Therefore,

$$\lim_{x \rightarrow -\infty} e^{\nu_1 b x} \mathbf{f}^+(x, t) = \lim_{x \rightarrow -\infty} e^{\nu_1 b x} (\mathbb{1} - P_{1-}) \begin{pmatrix} e^{-i\lambda_1^+(J_1 x + I_1 t)} n_{01}^{1'} \\ e^{-i\lambda_1^+(J_2 x + I_2 t)} n_{01}^{2'} \\ e^{-i\lambda_1^+(J_3 x + I_3 t)} n_{01}^{3'} \\ 0 \\ 0 \end{pmatrix} = 0$$

for any constant  $b < 0$  such that  $J_3 + b > 0$ .

The lemma is proved.  $\square$

For the choices a) and b) of  $|\vec{n}_{01}\rangle$  in equation (29) we define the square integrable discrete eigenfunctions using the splitting (32) and equation (34).

**Remark 2.** The choice c) for  $|\vec{n}_{01}\rangle$  does not allow the splitting (32). In this case we can introduce only generalized discrete eigenfunctions,  $\mathbf{f}_{\text{gen}}(x, t)$ , which are not square integrable. But upon multiplying by the exponential factor  $e^{-\nu_1 c_1 x}$  with  $c_1 = (J_1 + J_2)/2$ , we can obtain square integrable functions  $\mathbf{f}(x, t) = \mathbf{f}_{\text{gen}}(x, t) e^{-\nu_1 c_1 x}$ . See also the discussion in the next subsection.

The generalized eigenfunctions come up in situations when the splitting (32) is not possible, i.e., when either  $|\vec{p}_{01}\rangle$  or  $|\vec{d}_{01}\rangle$  vanish. Let us construct the generalized eigenfunction for the polarization vector  $|n_{01}\rangle$  of case c) in equation (30). Let  $(J_1 + J_2 + J_3)/3 = a'$ ; then  $J'_1 = J_1 - a'$ ,  $J'_2 = J_2 - a'$  and  $J'_3 = J_3 - a'$  are such that  $J'_1 > J'_2 > J'_3$  and  $J'_1 + J'_2 + J'_3 = 0$ . Let us assume for definiteness that

$J'_1 > J'_2 > 0$  and  $0 > J'_3$ . Then we can split  $|n_{01}\rangle$  into

$$|n_{01}\rangle = |p'_{01}\rangle + |d'_{01}\rangle, \quad |p'_{01}\rangle = \begin{pmatrix} n_{01}^1 \\ n_{01}^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |d'_{01}\rangle = \begin{pmatrix} 0 \\ 0 \\ n_{01}^3 \\ 0 \\ 0 \end{pmatrix}$$

and define

$$\mathbf{f}^{+'}(x, t) = \chi^+(x, t, \lambda_1^+) |p'_{01}\rangle = -\chi^+(x, t, \lambda_1^+) |d'_{01}\rangle.$$

Obviously  $\mathbf{f}^{+'}(x, t)$  is an eigenfunction of the dressed operator  $L$  corresponding to the eigenvalue  $\lambda_1^+$ .

Then we can prove the following

**Lemma 2.** *The eigenfunction  $\mathbf{f}^{+'}(x, t)$  is such that  $e^{\nu_1 a' x} \mathbf{f}^{+'}(x, t)$  decreases exponentially for both  $x \rightarrow \pm\infty$ .*

**Proof:** The proof is similar to the one of Lemma 1 and we omit it.  $\square$

Since the polarization vector  $|\vec{n}_{01}\rangle$  in case c) of equation (30) does not allow the splitting (32) the corresponding discrete eigenfunction will not be square integrable, so it will give rise to a generalized eigenfunction.

## 5. Discussion and Further Studies

Here we shall outline some further topics which could be studied and which could lead to a deeper understanding of these soliton properties.

The first obvious remark is that  $\mathfrak{sl}(n)$  contains as subalgebras also  $\mathfrak{so}(p)$  and  $\mathfrak{sp}(p)$  subalgebras. So it will be interesting to specify the conditions under which  $L(\lambda)$  has solitons of type  $\mathfrak{so}(p)$  or  $\mathfrak{sp}(p)$ .

Second remark of the same nature is that one can start with  $L(\lambda)$  related to  $\mathfrak{so}(n)$  or  $\mathfrak{sp}(n)$  algebras; such generalized Zakharov–Shabat systems allow one to solve special types of  $N$ -wave systems whose soliton solutions have not yet been classified. Such systems, due to the additional symmetry, have a richer structure.

The explicit form of the corresponding  $N$ -wave system related to these algebras has been reported in [5, 21, 12], see also [8, 15]. What could be done is to analyze the structure of its soliton solutions [8, 15] which are more involved due to the additional orthogonal symmetry involved. However, this symmetry complicates the construction of the dressing factors. Nevertheless, interesting new types of integrable cubic interactions could be obtained.

Also, even more complicated types of solitons will be related to projectors of higher rank. The projectors  $P_1(x, t)$ , which we used above, were all of rank one. The rank two projector  $P_2$  can be defined as

$$P_2(x, t) = \sum_{ks}^2 |\vec{n}_k\rangle \hat{M}_{ks}(x, t) \langle \vec{n}_s^\dagger|, \quad M_{sk}(x, t) = \langle \vec{n}_s^\dagger | \vec{n}_k \rangle, \quad \hat{M} \equiv M^{-1}.$$

Now each soliton will be parametrized by two polarization vectors – the corresponding eigensubspace will be two-dimensional too. Among the various types of rank two one-soliton solutions, there will be various possible configurations for the two polarization vectors.

It is known in general how the machinery, well understood for the AKNS system such as Wronskian relations, expansions over “squared solutions,” etc. can be generalized also for these types of systems. The dressing method, after some modifications, can also be applied, leading to the derivation of their soliton solutions.

An interesting problem is the study of how the different possible reductions (see e.g. [8]) of these systems will influence the number of one-soliton types.

Soliton interactions for the various different types of solitons of these systems also present interesting problems. From the results known for the  $N$ -wave systems [19, 16] it is known that new effects in soliton interaction, such as soliton decay and soliton fusion may arise.

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### Appendix A. Gauss Decompositions

Here we list the explicit expressions for the matrix elements of the Gauss factors  $\mathbb{S}^\pm(\lambda)$ ,  $\mathbb{T}^\pm(\lambda)$  in equations (20), (21) as polynomials of the matrix elements of  $T_{ks}(\lambda)$ . The results are

$$\mathbb{T}^-(\lambda) = \begin{pmatrix} \left\{ \begin{matrix} 1 \\ 1 \end{matrix} \right\} & 0 & 0 & 0 & 0 \\ \left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\} & \left\{ \begin{matrix} 1, 2 \\ 1, 2 \end{matrix} \right\} & 0 & 0 & 0 \\ \left\{ \begin{matrix} 3 \\ 1 \end{matrix} \right\} & \left\{ \begin{matrix} 1, 3 \\ 1, 2 \end{matrix} \right\} & \left\{ \begin{matrix} 1, 2, 3 \\ 1, 2, 3 \end{matrix} \right\} & 0 & 0 \\ \left\{ \begin{matrix} 4 \\ 1 \end{matrix} \right\} & \left\{ \begin{matrix} 1, 4 \\ 1, 2, 4 \end{matrix} \right\} & \left\{ \begin{matrix} 1, 2, 4 \\ 1, 2, 3, 4 \end{matrix} \right\} & 0 & 0 \\ \left\{ \begin{matrix} 5 \\ 1 \end{matrix} \right\} & \left\{ \begin{matrix} 1, 5 \\ 1, 2, 3, 4 \end{matrix} \right\} & \left\{ \begin{matrix} 1, 2, 3, 4 \\ 1, 2, 3, 5 \end{matrix} \right\} & 0 & 0 \\ \left\{ \begin{matrix} 1 \\ 1 \end{matrix} \right\} & \left\{ \begin{matrix} 1, 2 \\ 1, 2, 4 \end{matrix} \right\} & \left\{ \begin{matrix} 1, 2, 4 \\ 1, 2, 3, 4 \end{matrix} \right\} & 1 & 1 \end{pmatrix} \quad (35)$$

$$\mathbb{S}^+(\lambda) = \begin{pmatrix} 1 & -\begin{Bmatrix} 1 \\ 2 \end{Bmatrix} & \begin{Bmatrix} 1, 2 \\ 2, 3 \end{Bmatrix} & -\begin{Bmatrix} 1, 2, 3 \\ 2, 3, 4 \end{Bmatrix} & \begin{Bmatrix} 1, 2, 3, 4 \\ 2, 3, 4, 5 \end{Bmatrix} \\ 0 & \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} & -\begin{Bmatrix} 1, 2 \\ 1, 3 \end{Bmatrix} & \begin{Bmatrix} 1, 2, 3 \\ 1, 3, 4 \end{Bmatrix} & -\begin{Bmatrix} 1, 2, 3, 4 \\ 1, 3, 4, 5 \end{Bmatrix} \\ 0 & 0 & \begin{Bmatrix} 1, 2 \\ 1, 2 \end{Bmatrix} & -\begin{Bmatrix} 1, 2, 3 \\ 1, 2, 4 \end{Bmatrix} & \begin{Bmatrix} 1, 2, 3, 4 \\ 1, 2, 4, 5 \end{Bmatrix} \\ 0 & 0 & 0 & \begin{Bmatrix} 1, 2, 3 \\ 1, 2, 3 \end{Bmatrix} & -\begin{Bmatrix} 1, 2, 3, 4 \\ 1, 2, 3, 5 \end{Bmatrix} \\ 0 & 0 & 0 & 0 & \begin{Bmatrix} 1, 2, 3, 4 \\ 1, 2, 3, 4 \end{Bmatrix} \end{pmatrix} \quad (36)$$

$$\mathbb{T}^+(\lambda) = \begin{pmatrix} 1 & \begin{Bmatrix} 1, 3, 4, 5 \\ 2, 3, 4, 5 \end{Bmatrix} & \begin{Bmatrix} 1, 4, 5 \\ 3, 4, 5 \end{Bmatrix} & \begin{Bmatrix} 1, 5 \\ 4, 5 \end{Bmatrix} & \begin{Bmatrix} 1 \\ 5 \end{Bmatrix} \\ 0 & \begin{Bmatrix} 2, 3, 4, 5 \\ 2, 3, 4, 5 \end{Bmatrix} & \begin{Bmatrix} 2, 4, 5 \\ 3, 4, 5 \end{Bmatrix} & \begin{Bmatrix} 2, 5 \\ 4, 5 \end{Bmatrix} & \begin{Bmatrix} 2 \\ 5 \end{Bmatrix} \\ 0 & 0 & \begin{Bmatrix} 3, 4, 5 \\ 3, 4, 5 \end{Bmatrix} & \begin{Bmatrix} 3, 5 \\ 4, 5 \end{Bmatrix} & \begin{Bmatrix} 3 \\ 5 \end{Bmatrix} \\ 0 & 0 & 0 & \begin{Bmatrix} 4, 5 \\ 4, 5 \end{Bmatrix} & \begin{Bmatrix} 2 \\ 5 \end{Bmatrix} \\ 0 & 0 & 0 & 0 & \begin{Bmatrix} 5 \\ 5 \end{Bmatrix} \end{pmatrix} \quad (37)$$

$$\mathbb{S}^-(\lambda) = \begin{pmatrix} \begin{Bmatrix} 2, 3, 4, 5 \\ 2, 3, 4, 5 \end{Bmatrix} & 0 & 0 & 0 & 0 \\ -\begin{Bmatrix} 2, 3, 4, 5 \\ 1, 3, 4, 5 \end{Bmatrix} & \begin{Bmatrix} 3, 4, 5 \\ 3, 4, 5 \end{Bmatrix} & 0 & 0 & 0 \\ \begin{Bmatrix} 2, 3, 4, 5 \\ 1, 2, 4, 5 \end{Bmatrix} & -\begin{Bmatrix} 3, 4, 5 \\ 2, 4, 5 \end{Bmatrix} & \begin{Bmatrix} 4, 5 \\ 4, 5 \end{Bmatrix} & 0 & 0 \\ -\begin{Bmatrix} 2, 3, 4, 5 \\ 1, 2, 3, 5 \end{Bmatrix} & \begin{Bmatrix} 3, 4, 5 \\ 2, 3, 5 \end{Bmatrix} & -\begin{Bmatrix} 4, 5 \\ 3, 5 \end{Bmatrix} & \begin{Bmatrix} 5 \\ 5 \end{Bmatrix} & 0 \\ \begin{Bmatrix} 2, 3, 4, 5 \\ 1, 2, 3, 4 \end{Bmatrix} & -\begin{Bmatrix} 3, 4, 5 \\ 2, 3, 4 \end{Bmatrix} & \begin{Bmatrix} 4, 5 \\ 3, 4 \end{Bmatrix} & -\begin{Bmatrix} 5 \\ 4 \end{Bmatrix} & 1 \end{pmatrix}. \quad (38)$$

By  $\left\{ \begin{smallmatrix} i_1, \dots, i_k \\ j_1, \dots, j_k \end{smallmatrix} \right\}$  above we denote the minor of  $T(\lambda)$  formed by the rows  $i_1, \dots, i_k$  and the columns  $j_1, \dots, j_k$ . The diagonal elements of the  $\mathbb{S}^\pm$  and  $\mathbb{T}^\pm$  are given by the principal minors of  $T(\lambda)$ .

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