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# QUANTIZATION ON CURVED MANIFOLDS

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**Abstract.** Since the early days of quantum mechanics many techniques have been developed in order to deal with manifolds with non-trivial topology. Among them two techniques have received a great attention in the literature and are shortly reviewed here as they are most geometrical in nature. These are the Kostant–Souriau geometric quantization scheme and the so called constrained quantum mechanics. A notable difference between them are the geometrical structures used in these theories. The first is based on the symplectic structure of the phase space and the second one relies on the Riemannian metric of the configurational manifold. Both approaches are illustrated in full details. Presented examples include the $n$-dimensional variants of the harmonic oscillator and the Kepler problem which are treated within geometric quantization scheme by making use of the Marsden–Weinstein reduction theorem and even a combination of both methods is applied in the study of quantum-mechanical aspects of the geodesic flows on axisymmetric ellipsoids.

## 1. Introduction

The material presented in this report is based on a revised and expanded format of lectures delivered at the third edition of Varna International Conference on *Geometry, Integrability and Quantization* held in June 14–23, 2001.

No claims whatsoever are made with regard to completeness and at many places the reader is referred to the original works given in the list of references. The principal objective is to present the available techniques for treating quantum-mechanical systems defined on topologically non-trivial manifolds. As they appear even at early stages of quantum theory we start with a brief recapitulation of the old Bohr–Sommerfeld theory. Then we proceed with Kostant–Souriau geometric quantization scheme and end up with quantum-mechanical systems constrained on surfaces in $\mathbb{R}^3$. As applications we consider the multidimensional analogs of the harmonic oscillator and Kepler problem and quantum-mechanical aspects of the free motion (geodesic flow) of particles constrained on the prolate respectively oblate ellipsoids.

## 2. Bohr–Sommerfeld Theory

It is a common consent that quantum mechanics is the most revolutionary development in modern theoretical physics. However, the great changes unified into a consistent and coherent theory in the period 1925–1928 came about only gradually during the first quarter of the century.

The old quantum theory was born in 1900 when Max Planck announced his theoretical derivation of the spectral distribution law for black-body radiation, which he had previously formulated on the basis of empirical considerations. Actually he had showed that the experimental results can be accounted for by postulating that light can not be radiated continuously, but only in whole multiples of $h\nu$, where $\nu$ is the frequency and $h$ is a new universal constant which nowadays is known as *Planck's constant.*

In 1905 Einstein suggested that the radiant energy in the process of emission of light is sent not in all but just in one direction like a particle.

The third stage in the progress of old quantum theory was initiated by Bohr. Guided by the results of Rutherford scattering experiments, the Balmer formula and the difficulties of the classical electrodynamics to explain them, he put forward in 1913 the hypothesis that the electrons do not radiate energy continuously but that they can revolve along certain discrete orbits without radiating and that just when they are jumping from one such orbit to another they will emit a quantum of light with a frequency $\nu$ given by the formula

$$h\nu = E_2 - E_1 \tag{2.1}$$

where $E_1$ and $E_2$ are the energies of the first and second orbit respectively. In addition Bohr gave a method for determining the quantized states of motion (the so called stationary states) of the hydrogen atom. This method of quantization involves a restriction of the angular momentum of circular orbits to integral multiples of the quantum $h/2\pi$ and though leading to correct energy levels was soon superseded by a more powerful method of quantization proposed independently in 1915 by Ishihara [22], Sommerfeld [56] and Wilson [62]. Namely, they have noticed that the quantization of the angular momentum (in the plane) is a result of the condition

$$\oint p_\varphi \, \mathrm{d}\varphi = n_\varphi h \quad \text{for some} \quad n_\varphi \in \mathbb{N}\,. \tag{2.2}$$

Here $p_\varphi$ is the generalized momentum corresponding to the azimuthal coordinate $\varphi$ and the integration is over the period of $\varphi$. That is why it was postulated that the stationary states of a system with $n$ degrees of freedom $q_1, q_2, \ldots, q_n$ are selected by the conditions

$$\oint p_k \, \mathrm{d}q_k = n_k h \quad \text{for some} \quad n_k \in \mathbb{N}\,,\; k = 1, 2, \ldots, n\,. \tag{2.3}$$

Sommerfeld had applied this method for finding the spectrum of the hydrogen atom in both relativistic and non-relativistic cases, Schwarzschild [50] had explained the splitting of the spectral lines due to the external electric field and had found the spectrum of the free rotational motion of axially symmetric rigid body, while Debye [9] have used these concepts in order to find an interpretation of the so-called normal Zeeman effect.

However this theory was not free of internal controversies. At the beginning any problem under consideration have to be treated with the methods of classical mechanics and then one have to select the quantum states from the continuum set of classical motions. Besides it is not quite clear in which coordinates the quantum conditions (2.3) should be imposed. Separability of the Hamilton–Jacobi equation is some indication but could not be taken as a principle because there are situations when the separation can be performed in more than one coordinate system. Even more unpleasant feature of the Bohr–Sommerfeld theory is the failure to provide a method of calculating transition probabilities and the intensities of the spectral lines. This dissatisfaction with the old quantum mechanics leads to the appearance of the Schrödinger wave mechanics in 1926 when it became clear that the Bohr–Sommerfeld theory produces just the leading term in the semiclassical approximation.

However in many cases it is capable to provide exact results as well [53]. Such developments in the subsequent years include finding the spectra of the hamiltonian of the relative motion of two relativistic particles interacting through a Coulombian potential [34], the motion in the ring-shaped potential which appears in some axially symmetric systems in quantum chemistry [58] and that one of the free motion of the asymmetric rigid body [36].

## 3. Geometric Quantization

In almost all physically significant problems the phase spaces of the classical Hamiltonian systems are just cotangent bundles with their canonical symplectic structure [1, 29, 57]. The other source of symplectic manifolds is the algebraic geometry which gives a lot of deep information about (compact) algebraic manifolds — the symplectic structure there is given by the Kählerian form. The connection between these has been exploited during the last two centuries through the procedure which is now called reduction. Starting with the work of Jacobi (and implicitly by Kepler, Newton and Euler before him) this method has produced the most beautiful solutions of physical problems in terms of algebro-geometric entities as $\theta$-functions etc. One of the purposes of the present section is to discuss the process of reduction from noncompact to compact (algebraic) phase spaces as a method for quantization, which we think is the basic fact of the geometric quantization scheme. Compact complex (specially algebraic) manifolds involve a lot of discrete characteristics starting from continuous background. A further purpose of this section is to show how naturally these spaces appear in classical mechanics, and how the quantum mechanical picture arises when geometric quantization is applied to them. The scheme can be applied to various submanifolds in complex projective spaces. These are actually the orbit manifolds associated with concrete mechanical systems: free and coupled harmonic oscillators, the Kepler problem, the geodesic flows on spheres and projective spaces (cf. [38, 39, 41]). In all cases the quantum spectrum along with corresponding "wave" functions can be described explicitly. On the way the modification of the quantization scheme is argued and explained. In standard quantum-mechanical terminology transition to the orbit manifolds corresponds to passage from Schrödinger to Heisenberg picture in quantum mechanics. From mathematical point of view quantization of momentum mapping associated with free torus action on the symplectic manifolds nevermind how simple this picture might be contains as a special case the whole representation theory of semi-simple group and the theory of universal spaces for vector bundles.

### 3.1. Dirac Programme

The modern quantum mechanics starts in the second quarter of the last century with the clear idea that the quantization is a map from the space of the classical observables (i. e. the smooth functions on the phase space $(M, \omega) \equiv (\mathbb{R}^{2n}, \, \mathrm{d}p_i \wedge \mathrm{d}q^i)$) to the self-adjoint or symmetric operators in the Hilbert space $\mathcal{H}$, $\phi \to \delta(\phi)\colon \mathcal{H} \to \mathcal{H}$ which satisfies:

1. $\delta(\phi + \psi) = \delta(\phi) + \delta(\psi)$;
2. $\delta(\lambda\phi) = \lambda\delta(\phi)\,, \quad \lambda \in \mathbb{R}$;
3. $\delta(\{\phi, \psi\}) = \mathrm{i}[\delta(\phi), \delta(\psi)]$;
4. $\delta(1) = \mathrm{Id}_{\mathcal{H}}$ is the identity operator in $\mathcal{H}$;
5. $\delta(q^i), \delta(p_i)$ are irreducible operators in $\mathcal{H}$.

The various partial realizations of the above so-called Dirac programme [10] are known as algebraic, asymptotic, deformation, geometric, group-theoretical, etc quantizations but van Hove [20] proves rigorously that this can not be done at all! However, he proves also that 1–4 has solution and that there exists an unique realization in the large for the algebra of polynomials up to second degree in the canonical coordinates $q^i, p_i$. Later on Segal [51] had transferred the above theorems to phase spaces $(T^*Q,\, \mathrm{d}\theta)$ which are cotangent bundles of some configurational manifold and finally Kostant [27] and Souriau [57] present a scheme suitable for an arbitrary symplectic manifold $(M, \omega)$.

One may also wonder how such nontrivial manifolds (two-cycles and more general surfaces) appear in real systems? The most natural situation for this to happen is the reduction procedure known since the times of Newton and Jacobi and which modern formulation is due to Marsden and Weinstein [31]. The setting of the reduction theorem is the following: if a Lie group $G$ acts in a Hamiltonian fashion on the symplectic manifold $(M, \omega)$, i. e.

$$\Phi_g : M \to M\,, \qquad \Phi_g^* \omega = \omega$$

and preserves the energy function $H$ of the Hamiltonian system $(M, \omega, H)$

$$\Phi_g^* H = H \circ \Phi_g = H\,,$$

then there exist a natural mapping called momentum

$$J\colon M \to \mathfrak{g}^* \text{ — the dual of the Lie algebra } \mathfrak{g} \text{ of } G$$

such that if $\mu$ is a fixed regular element in $\mathfrak{g}^*$ then

$$J^{-1}(\mu)/G_\mu \cong M_\mu$$

is an even-dimensional manifold and moreover there exists a two-form $\omega_\mu$ such that $(M_\mu, \omega_\mu)$ is a symplectic manifold. When applied to such reduced manifolds geometric quantization scheme produces the quantization of charge, spin and energy levels of some physical systems [38, 40].

Below we will present a short reviews of the Kostant–Souriau quantization scheme and its extension, the so called Czyz–Hess scheme which will be applied in the next sections to the multi-dimensional variants of harmonic oscillator and Kepler problem.

### 3.2. Kostant–Souriau Geometric Quantization

The non-trivial moment in the Kostant–Souriau approach is that the wave functions are not moreover functions but a sections of a line bundle $L$ over $M$, i. e. $\pi\colon L \to M$, $s\colon M \to L$ and $\pi \circ s = \mathrm{Id}_M$.

Such $L$ exists if the symplectic manifold $(M, \omega)$ is pre-quantizable [27, 57], i. e. if $[\omega/2\pi]$ is in the image of the map

$$H^2_{\text{Cech}}(M, \mathbb{Z}) \to H^2_{\text{de Rham}}(M, \mathbb{R})$$

where $[\,]$ denotes the de Rham cohomological class.
When $M$ is a compact manifold the above condition is equivalent to

$$\frac{1}{2\pi}\int\limits_{\sigma} \omega \in \mathbb{Z}\,, \quad \text{for every two-cycle } \sigma \in H_2(M, \mathbb{Z}) \tag{3.1}$$

and the quantum operator associated with $f$ acts in $\mathcal{H} \equiv \Gamma(L)$, the space of sections of the corresponding line bundle, as follows:

$$\delta(f)s = -\mathrm{i}\nabla_{X_f} s + f s\,.$$

Here $\nabla_{X_f}$ is the covariant derivative along the Hamiltonian vector field generated by the symplectic form via $i(X_f)\omega = -\,\mathrm{d}f$. Identifying the sections of $L$ with functions on $M$ (which can be done in general only locally!) the action of $\delta(f)$ in $\Gamma(L)$ can be written in the form

$$\delta(f)\varphi = (-\mathrm{i}X_f - \theta(X_f) + f)\varphi$$

where $\theta$ is some local potential one-form of the symplectic structure $\omega = \mathrm{d}\theta$.
The irreducibility of the representations which is the second stage (quantization) of the programme is achieved by introducing additionally a new structure called **polarization**. A real polarization over $M$ is a such map that juxtapose to each point $m \in M$ a real subspace $F_m \subset T_m(M)$ which is maximally isotropic integrable distribution.

**Example 3.1.** *Let $Q$ be a smooth manifold and let $T^*Q$ be its cotangent bundle. If $\{p_i, q_i\}$ are the local canonical coordinates in $T^*Q$, then an easy check shows that the vector fields*

$$X_1 = \frac{\partial}{\partial p_1}, X_2 = \frac{\partial}{\partial p_2}, \ldots, X_n = \frac{\partial}{\partial p_n}$$

*define a real polarization over $T^*Q$ which is known as vertical polarization.*

**Example 3.2.** *The two-dimensional sphere does not allows real polarization because of the non-existence of a global non-singular real vector field on $S^2$.*

This situation suggests also the generalization of the above notion. Namely, a **complex polarization** over $M$ is a map $F$ which assigns to each point $m \in M$ a subspace $F_m$ of $T^C_m(M)$ which is maximally isotropic integrable distribution, and besides the distribution $D_m = F_m \cap \bar{F}_m$ is of some fixed dimension $\kappa$ at each point $m \in M$. The polarization $F$ is called Kählerian if $F_m \cap \bar{F}_m = 0$.

For any kind of polarization $F$ the potential $\theta$ of the symplectic form $\omega$ (i. e. $\omega = \mathrm{d}\theta$) is called an adapted to it if $\theta(X) = 0$ for every $X \in F$. The quantum pre-Hilbert space is built up by the polarized sections of $L$ which definition is as follows.

Let $M, \omega, L, \nabla$ and $F$ be as defined above. The polarized sections of $L$ form the line bundle

$$L^F = \{s \in \mathrm{Sect}(L);\ \nabla_X s = 0\,, \quad \text{for all } X \in \mathfrak{X}(M, F)\}\,.$$

In order to have a true Hilbert space we need some measure (or density) which is an element of a second line bundle. This can be introduced if we consider the elements of the cotangent bundle $T^*(M)$ that vanish on $F$ and form a sub-bundle $F^\circ \subset T^*_C(M)$ which is called annihilator of $F$. By the very definition of the symplectic form we have that the map

$$v \in F \to i(v)\omega \in F^\circ$$

is an isomorphism of $F$ and $F^\circ$. This means that we can form the line bundle $K_F = \wedge^n F^\circ$ over $M$ that will be further referred as a **canonical bundle** of $F$. If $\{v_1, v_2, \ldots, v_n\}$ is a basis of $F$, then

$$K_\omega = i(v_1)\omega \wedge i(v_2)\omega \wedge \cdots \wedge i(v_n)\omega$$

is a basis in $K_F$ and for every $g \in GL(n, C)$, $(K^g)_\omega = \det g\, K_\omega$.

Let $(M, \omega)$ be a symplectic manifold and $F$ is a complex polarization on it. We will say that $M$ is a **metaplectic manifold** if there exists a line bundle $N^{1/2}$ over $M$ such that

$$N^{1/2} \otimes N^{1/2} = K_F\,.$$

One can show that $(M, \omega)$ is metaplectic if and only if the first Chern class of $K_F$ is zero modulo two and this property does not depend on the choice of $F$. In this case the group $H^1(M, \mathbb{Z}_2)$ parameterizes the set of "square roots", i. e. the set of all $N^{1/2}$ which satisfy the above condition. The sections of $N_F^{1/2}$ which are constant along $F$ are called half-forms normal to $F$. The line bundle $\tilde{Q} = L^F \otimes N_F^{1/2}$ over $M$ is called a **quantum line bundle** because its sections are considered as elements of the Hilbert space $\mathcal{H}^F$. The classical observables which can be quantized directly are those that preserve the polarization $F$, i. e. $\{f \in R^\infty(M);\ [X_f, F] \subset F\}$, where $X_f$ is defined by the equation $i(X_f)\omega = -\,\mathrm{d}f$. If $\psi = s \otimes \nu$, where $\psi \in \Gamma(\tilde{Q})$, $s \in \Gamma(L^F)$, $\nu \in \Gamma(N_F^{1/2})$ are sections of the corresponding line bundles, the associated with $f$ quantum operator acts in $\mathcal{H}^F$ as specified below:

$$\hat{f}(\psi) = (-\mathrm{i}\nabla_{X_f} + f)s \otimes \nu - \mathrm{i}s \otimes \mathcal{L}(X_f)\nu\,. \tag{3.2}$$

Identifying the sections of $L^F$ with functions on $M$ (which is possible because $L^F$ is a line bundle) the action of $\hat{f}$ in $\mathcal{H}^F$ can be written in the form

$$\hat{f}\psi = \Big( - \mathrm{i}X_f - \theta(X_f) + f\Big)\varphi \otimes \nu - \mathrm{i}\varphi \otimes \mathcal{L}(X_f)\nu \tag{3.3}$$

where $\theta$ is the potential one form of $\omega$.

Actually this explicit formula has found very few applications as most of the considerations end up with checking the consistency of the scheme relying on (3.2).

### 3.3. Czyz–Hess Scheme

After cotangent bundles and co-adjoint orbits the Kählerian manifolds form another important class of symplectic manifolds. According Darboux theorem all symplectic manifolds (of fixed dimension) are locally the same but in practice they appear with some additional geometric structure. Its presence in the setting of geometric quantization helps in many cases to answer definitely the question if the given symplectic manifold $(M, \omega)$ allows such quantization. A trivial example is provided by even-dimensional complex projective spaces. The well-known fact for these manifolds is that

$$H^2(\mathbb{P}^{2n}, \mathbb{Z}_2) \cong \mathbb{Z}_2 \,.$$

On the other hand we know that the symplectic manifold $(M, \omega)$ can be quantized if $M$ is a metaplectic manifold, i. e. $H^2(M, \mathbb{Z}_2) = 0$. So, even-dimensional complex projective spaces can not be treated in Kostant–Souriau scheme. On the other hand they appear as the orbit manifolds of the odd-dimensional harmonic oscillators which form an important class of dynamical systems. Fortunately this problem can be taken away by a slight modification of geometric quantization scheme as developed by Czyz [8] and Hess [19] and outlined bellow.

**Definition 3.1.** *Let $(M, \omega)$ be a such Kählerian manifold that the cohomology class $[q] = \dfrac{1}{2\pi}[\omega] - \dfrac{1}{2}c_1(M)$ belongs to the image of $\epsilon\colon H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$ and $q$ is positive, i. e. $q(\sigma) \geq 0$ for any positively oriented two-cycle $\sigma \in H_2(M, \mathbb{R})$. The complex line bundle $Q$ whose first Chern class $c_1(Q)$ is $q$ is called quantum bundle.*

If the pre-quantum $L^F$ and the half-form $N_F^{1/2}$ line bundles which appear in Kostant–Souriau theory (cf. [27, 57, 54, 16]) exist then there exists also the bundle $\tilde{Q} = L^F \otimes N_F^{1/2}$ so that $c_1(\tilde{Q}) = c_1(Q)$ and therefore $\tilde{Q}$ and $Q$ are equivalent. Among symplectic manifolds the Kählerian ones are those which possess canonical anti-holomorphic polarization that makes identification of quantum states with holomorphic sections quite natural. Now, fixing a positive harmonic representative $\eta \in c_1(Q)$ and connection $\nabla$ which curvature is $-2\pi\mathrm{i}\eta$ we are in position to define also and $\nabla$-invariant hermitian structure $h_\eta(\,,\,)$ on $Q$. We recall that, the curvature of the hermitian metric $h_\eta$ on the bundle $Q$ satisfies

$$\frac{\mathrm{i}}{2\pi}\partial\bar{\partial}\log h_\eta \simeq \frac{\omega}{2\pi} - \frac{1}{2}c_1(M)\,.$$

The space of holomorphic sections $H^0((M,Q)$ of $Q$ can be converted into Hilbertian space $\mathcal{H}$ if we introduce the scalar product

$$\langle s,t\rangle = \int\limits_M h_\eta(s,t)\Omega_\eta\,, \quad \omega_\eta = 2\pi\eta\,,\ s,t \in \Gamma(M,Q)\,,\ n = \frac{1}{2}\dim M \qquad (3.4)$$

and where $\Omega_\eta := \dfrac{(-1)^{n(n-1)/2}}{n!}\omega_\eta\wedge\omega_\eta\wedge\cdots\wedge\omega_\eta$ is the natural volume form on $M$. If our manifold $M$ is simply-connected the hermitian structure is defined up to a positive factor and $\mathcal{H}$ is defined up to an isomorphism which depends on the choice of the connection $\nabla$. The representations are build up following the prequantization recipe in which $(L,\omega)$ is exchanged for $(Q,\omega_\eta)$ i. e. to the classical observable (i. e. a function $f$ on the phase space), there corresponds a quantum operator

$$\delta(f) \in \operatorname{End} H^0(M,Q)\,, \qquad \delta(f)s \equiv (-\mathrm{i}\nabla_{X_f} + f)s$$

where $s \in H^0(M,Q)$, and now the vector field $X_f$ is defined via $\omega_\eta$, i. e. $i(X_f)\omega_\eta = -\mathrm{d}f$. The only problem here is that $\omega_\eta$ is not always non-degenerated. More detailed exposition can be found in Czyz [8] and Hess [19]. Having in mind the applications that follow it seems appropriate to introduce some general notion that will help the mathematically minded reader to create his own picture.

Assuming a presence of complex structure we will say that the manifold $M$ allows geometric quantization if there exists a differential two-form $\omega$ such that $(M,\omega)$ meets the condition for existing of quantum bundle. The Kählerian form $\chi \in H^2(M,\mathbb{Z})$ on $M$ in that case is called a Hodge structure. When $M$ is a compact the existence of Hodge structure is equivalent to the possibility for embedding $M$ holomorphically in some complex projective space and in the latter case $M$ is an algebraic manifold, i. e. it can be described as the common locus of a finite system of homogeneous complex polynomials (see Griffiths and Harris [15]).

**Theorem 3.1.** *The compact complex manifold $M$ allows geometric quantization if and only if on it there exist a Hodge structure. The Chern class $q$ of any quantum bundle over $M$ satisfies the condition $q \in H^{1,1}(M, \mathbb{Z})$ and the form $\eta$ is of type $(1, 1)$.*

The complex manifold on which acts a group of complex (holomorphic) transformation is called $\mathbb{C}$-**homogeneous**. If every point $m \in M$ is an isolated fixed point of involutive holomorphic transformation $\sigma_m \colon M \to M$, then $M$ is called complex symmetric space or simply $\mathbb{C}$-space. Any compact $\mathbb{C}$-space $M$ can be presented as Cartesian product $M_1 \times M_2 \times \cdots \times M_k$ in which every factor coincides with some of the manifolds in the Cartan list:

1. $\mathbb{U}(p+q)/\mathbb{U}(p) \times \mathbb{U}(q)\,, \quad p, q \geq 1;$
2. $\mathbb{SO}(2p)/\mathbb{U}(p)\,, \quad p > 1;$
3. $\mathbb{S}\mathrm{p}(p)/\mathbb{U}(p)\,, \quad p \geq 1;$
4. $\mathbb{SO}(p+2)/\mathbb{SO}(p) \times \mathbb{SO}(2)\,, \quad p \geq 1,\ p \neq 2;$
5. $\mathbb{E}_6/\mathbb{S}\mathrm{pin}(10) \times \mathbb{U}(1);$
6. $\mathbb{E}_7/\mathbb{E}_6 \times \mathbb{U}(1).$

It is interesting to note that any of the Cartan classical domains listed above satisfies the condition $\dim H^2(M_i, \mathbb{Z}) = 1$ (cf. Borel [5]) and this facilitates their geometric quantization. From the viewpoint of the general representation theory this means that only the fully symmetric representations (single row Young tableaux) of the corresponding group can be recovered. It should be mentioned also that these manifolds appear quite naturally in physics — the first three classes are connected with various formulations of time-dependent Hartee–Fock theory, the fourth consists of the manifolds behind the Kepler problem (in dimensions different from three) and the remaining ones arise in modern quantum field theory models.

The generalization of this situation (inspired by the need of more quantum numbers) is immediate. Let us take $M$ which is a compact complex $\mathbb{C}$-homogeneous manifold of non-zero Eulerian characteristic $\chi(M)$ and consequently simply-connected. Under above conditions the complex transformations of $M$ generate a real compact semi-simple Lie group $G$ which acts transitively. In this case $M$ is homeomorphic to the factor-space $G/K$ where $K$ is a Lie subgroup of the same rank as that of $G$. The set of the generators of $H^2(M, \mathbb{Z})$ is in one-to-one correspondence with the simple roots of $G$ and its irreducible representations have geometric description provided by the Borel–Weil–Bott construction (see Serre [52] and Bott [6]). So, the mathematical part is relatively clear and the non-trivial moment is the extraction of such manifolds.

### 3.4. Multidimensional Harmonic Oscillator

**Definition 3.2.** *The Hamiltonian system* $(M, \Omega, H)$, *where*

$$\begin{aligned} M &= T^*\mathbb{R}^n = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n\} \\ \Omega &= \sum_{k=1}^{n} \mathrm{d}y_k \wedge \mathrm{d}x_k = \mathrm{d}y \wedge \mathrm{d}x \end{aligned} \tag{3.5}$$

*and*

$$H = \frac{1}{2}\sum_{k=1}^{n}(y_k^2 + x_k^2) = \frac{1}{2}(|y|^2 + |x|^2)$$

*is known as* $n$*-dimensional* ***harmonic oscillator****.*

In order to be coherent with the title we will suppose that $n \geq 2$. The Hamiltonian vector field $X_H$ of this dynamical system is

$$X_H = \sum_{k=1}^{n}(y_k \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial y_k}),$$

and the corresponding Hamiltonian equations of motion are

$$\frac{\mathrm{d}x_k}{\mathrm{d}t} = y_k\,, \qquad \frac{\mathrm{d}y_k}{\mathrm{d}t} = -x_k\,. \tag{3.6}$$

Introducing $z_k = (x_k - \mathrm{i}y_k)/\sqrt{2}$ we can identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$. In these complex coordinates the Hamiltonian $H$, the symplectic form $\Omega$ and the Hamiltonian equations of motion can be rewritten respectively as

$$\begin{aligned} H &= \sum_{k=1}^{n} z_k \bar{z}_k = \sum_{k=1}^{n} |z_k|^2\,, \\ \Omega &= \mathrm{i}\sum_{k=1}^{n} \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k = \mathrm{i}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} \end{aligned} \tag{3.7}$$

and

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \mathrm{i}z\,.$$

The solution which satisfies the Cauchy data $z_{(t=0)} = z_0 = (z_{01}, z_{02}, \ldots, z_{0n})$ is:

$$z(t) = \mathrm{e}^{\mathrm{i}t} z_0\,.$$

Any positive real number $E$ is a regular value of $H$. The submanifold

$$M_E = H^{-1}(E) = \{z \in \mathbb{C}^n\,;\ |z|^2 = E\}$$

is a $(2n-1)$-dimensional sphere $S^{2n-1}$. If $z_0$ is a point on this sphere, the trajectory of the system through $z_0$ is the circle $\lambda z_0$ $\{\lambda \in \mathbb{C};\ |\lambda| = 1\}$. The orbit manifold of energy $E$ is the factor-space of $M_E$ with respect to the following equivalence relation: the points $z$ and $z'$ on $S^{2n-1}$ are equivalent, if there exists a complex number $\lambda$ of unit module ($|\lambda| = 1$), such that $z' = \lambda z$.

Let us consider also an equivalency relation among points in $\mathbb{C}^n \backslash \{0\}$. We will say that $z$ and $z'$ in this space are equivalent if there exists a non-zero complex number $\lambda$ such that $z' = \lambda z$. The equivalence classes of the second relation are punctured complex planes, i. e. the vector subspaces of $\mathbb{C}^n$ of complex dimension one with the origin removed. The equivalence classes of the first relation are simply the sections of the sphere $M_E$ with these planes. This allows the manifold of orbits $\mathcal{O}(E)$ of energy $E$ to be identified with the factor-space of $\mathbb{C}^n \backslash \{0\}$ under the second equivalence relation, i. e. with the complex projective space $\mathbb{P}^{n-1}$. In the same time we see that the Hopf fibration is a part of the classical mechanics. Now we turn to its coordinate description. For any integer $k$, $1 \leq k \leq n$, let $U_k$ be an open subset of $\mathbb{P}^{n-1}$ that consists of the equivalence classes having as representatives the points $(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \backslash \{0\}$, for which $z_k \neq 0$. This open subset is covered by a local chart defined by the $(n-1)$ complex coordinate functions $t_j = z_j / z_k$, $1 \leq j \leq n$, $j \neq k$. The corresponding $2n - 2$ real coordinates are just the real and imaginary parts of $t_j$. As the union of the open subsets $\{U_k\}_{k=1}^n$ is $\mathbb{P}^{n-1}$, we have a manifold with an atlas for which the transition functions are holomorphic. Our next task will be to find the symplectic form $\Omega_E$ over $\mathcal{O}(E)$, for which $\pi_E$ is the canonical projection from the reduction theorem (see Marsden and Weinstein [31]). We will do the computations in the chart $(U_n, \varphi_n)$ using the section $s_E$ of the Hopf bundle $\pi_E \colon M_E \to \mathbb{P}^{n-1}$

$$s_E(t_1, t_2, \ldots, t_{n-1}) = (z_1, z_2, \ldots, z_{n-1}, z_n)\,,$$

where

$$z_k = z_n t_k, 1 \leq k \leq n-1, z_n = \sqrt{E}(1 + \sum_{j=1}^{n-1} |t_j|^2)^{-1/2} = \sqrt{E}(1 + |t|^2)^{-1/2}\,.$$

We will get the explicit expression of $\Omega_E$ by pulling back $\Omega$ by $s_E$ from $S^{2n-1}$ in $\mathbb{C}^n$,

$$\Omega_E = s_E^* \Omega_{|S^{2n-1}}\,.$$

Doing this way we obtain:

$$\Omega_E = \mathrm{i}E \frac{(1 + |t|^2) \sum_{k=1}^{n-1} \mathrm{d}t_k \wedge \mathrm{d}\bar{t}_k - (\sum_{j=1}^{n-1} \bar{t}_j \, \mathrm{d}t_j) \wedge (\sum_{k=1}^{n-1} t_k \, \mathrm{d}\bar{t}_k)}{(1 + |t|^2)^2}\,.$$

Up to a multiplicative constant this is exactly (a representative of) the generator of $H^2(\mathbb{P}^{n-1}, \mathbb{Z})$ known as Fubini–Study form $\omega_{FS}$. This can be checked immediately by writing down the standard representation of $\omega_{FS}$ which in homogeneous coordinates reads:

$$\omega_{FS} = \frac{\mathrm{i}}{2\pi}\,\frac{|z|^2\sum_{k=1}^{n}\mathrm{d}z_k\wedge\mathrm{d}\bar{z}_k - (\sum_{j=1}^{n}\bar{z}_j\,\mathrm{d}z_j)\wedge(\sum_{k=1}^{n}z_k\,\mathrm{d}\bar{z}_k)}{|z|^4} \tag{3.8}$$

and then passing to inhomogeneous coordinates. Following this recipe we find that:

$$\Omega_E \equiv 2\pi E\omega_{FS}\,. \tag{3.9}$$

The quantum bundles in which we are interested in can be selected using the well-known fact (see Griffiths and Harris [15])

$$c_1\left(T(\mathbb{P}^m)\right) = c_1(\mathbb{P}^m) = (m+1)\omega_{FS}\,, \qquad m\geq 1$$

and their definition (see Section 3.3). So, we have:

$$\frac{1}{2\pi}\Omega_E - \frac{n}{2}\omega_{FS} = N\omega_{FS} \qquad N = 0,1,2\ldots$$

and consequently

$$E_N = N + \frac{n}{2}\,, \qquad N = 0,1,2\ldots \tag{3.10}$$

is the quantum spectrum of the system. Next step in quantum mechanics is to describe the "wave" functions of the system. Following the scheme they are sections of the quantum line bundles $Q_N$ (associated with any energy level $E_N, N\geq 1$). Let us take and fix one such value $N$ which should be anticipated in the following considerations. As we already know from Section 3.3 the very first step in the description of the quantum line bundle $Q_N$ is the choice of the harmonic representative of $c_1(Q_N)$. In our case an appropriate choice turns out to be:

$$\eta_n = \mathrm{i}\frac{N}{2\pi}\frac{\sum_{j=1}^{n-1}\mathrm{d}t_j\wedge\mathrm{d}\bar{t}_j}{(1+|t|^2)^2} = \mathrm{i}\frac{N}{2\pi}\frac{\mathrm{d}t\wedge\mathrm{d}\bar{t}}{(1+|t|^2)^2}$$

where the index $n$ encodes the local chart. The symplectic form $\omega_\eta = 2\pi\eta$ possesses an adapted to the anti-holomorphic polarization $\left\{\frac{\partial}{\partial\bar{t}_1},\frac{\partial}{\partial\bar{t}_2},\ldots,\frac{\partial}{\partial\bar{t}_{n-1}}\right\}$ potential $\theta_\eta(\omega_\eta = \bar{\partial}\theta_\eta)$

$$\theta_n = -\mathrm{i}N\frac{\bar{t}\,\mathrm{d}t}{1+|t|^2}$$

and the transition functions for the bundle $Q_N$ are defined by the fundamental relation

$$\theta_j = \theta_k - \mathrm{i}\frac{\mathrm{d}c_{jk}}{c_{jk}}\,,$$

which is valid on $U_j \cap U_k \neq \emptyset$. The above results can be summarized as

**Lemma 3.1.** *The quantum line bundle $Q_N$ associated with the energy level $E_N = N + \frac{n}{2}, N \geq 0$ of the $n$-dimensional harmonic oscillator is the holomorphic line bundle of degree $N(\deg Q_N = \int\limits_{\mathbb{P}^{n-1}} c_1(Q_N) \wedge \omega_{FS}^{n-2})$, with transition functions $c_{jk} = (z_k/z_j)^N$.*

The Hermitian metric $h$ which is compatible with the connection $\theta$ is determined by the equation:

$$\theta = -\mathrm{i}h^{-1}\partial h\,. \tag{3.11}$$

Solving (3.11) we find that

$$h_n = (1 + |t|^2)^{-N}\,.$$

The local sections of $Q_N$ being a holomorphic functions without poles are polynomials of some degree which is dictated by the transition functions and obviously do not exceed $N$. From here we find the number of linearly independent sections of the line bundle $Q_N \to \mathbb{P}^{n-1}$ or what is the same, the degeneracy $m(E_N)$ of the states whose energy is $E_N$

$$\dim H^0(\mathbb{P}^{n-1}, \mathcal{O}(Q_N)) = m(E_N) = \binom{n+N-1}{N}. \tag{3.12}$$

The scalar product (see formula (3.4)) of the global sections $s_1$ and $s_2$ whose local representative on $U_n$ are the polynomials $p_1(t)$ and $p_2(t)$ is given (up to a scalar factor) by

$$\langle s_1, s_2\rangle = N^{n-1} \int\limits_{\mathbb{P}^{n-1}} \frac{p_1(t)\bar{p}_2(t)\,\mathrm{d}t_1 \wedge \ldots \mathrm{d}t_{n-1} \wedge \mathrm{d}\bar{t}_1 \wedge \ldots \mathrm{d}\bar{t}_{n-1}}{(1+|t|^2)^{N+2(n-1)}}$$

A natural question arises which functions have representation in $\mathcal{H}_N$ as operators? The answer is the content of the lemma that follows.

**Lemma 3.2.** *The subalgebra of the Lie algebra of the smooth functions over $\mathbb{P}^{n-1}$ which can be quantized is generated by the functions having the general form:*

$$f([z_1 : z_2 : \cdots : z_n]) = \frac{\sum_{j,k=1}^{n} a_{jk} z_j \bar{z}_k}{\sum_{j=1}^{n} |z_j|^2}\,,$$

*where the matrix $[a_{jk}]_{j,k=1}^{n}$ is hermitian and consequently its real dimensions is $n^2$.*

**Proof:** In any local chart (for example in $U_n$) the function $f$ can be written as $\dfrac{\tilde{f}}{(1+|t|^2)}$. The transition to this form is dictated just by the advantages it has in calculations in view of the explicit expression for the symplectic form $\omega_\eta = \bar{\partial}\theta_\eta$. Now the condition that the flow generated by $f$ preserves the chosen polarization is equivalent with the statement that $\tilde{f}$ is determined completely by the hermitian matrix $[a_{jk}]_{j,k=1}^n$. □

### 3.5. Multidimensional Kepler Problem

**Definition 3.3.** *The Hamiltonian system* $(M, \omega, H)$ *where* $M = \mathbb{R}^{2n}$ *with the global coordinates* $q_1, \ldots, q_n$, $p_1, \ldots, p_n$, $(q, p) \in \mathbb{R}^n \backslash \{0\} \times \mathbb{R}^n$ *and*

$$\omega = \mathrm{d}p \wedge \mathrm{d}q\,, \quad H = p^2/2 - 1/|q|\,, \quad |q|^2 = q_1^2 + q_2^2 + \cdots + q_n^2 \tag{3.13}$$

*is known as the* $n$*-dimensional* ***Kepler problem****.*

It turns out that this problem is closely related with another dynamical system which we will introduce immediately. For that, let as before $\xi = (\xi_0, \xi_1, \ldots, \xi_n)$ be the Euclidean coordinates in $\mathbb{R}^{n+1}$ and $\eta = (\eta_0, \eta_1, \ldots, \eta_n)$ — the coordinates in the dual space $(\mathbb{R}^{n+1})^*$. So defined $(\xi, \eta)$ are the global coordinates on $T^*\mathbb{R}^{n+1}$ in which the canonical symplectic form is $\sigma = \mathrm{d}\eta \wedge \mathrm{d}\xi$. Let us take as a hamiltonian function on this phase space the function $\Phi = |\xi|^2|\eta|^2/2$ and write down the hamiltonian equations of motion. They are:

$$\frac{\mathrm{d}\xi}{\mathrm{d}s} = \frac{\partial\Phi}{\partial\eta} = |\xi|^2\eta\,, \qquad \frac{\mathrm{d}\eta}{\mathrm{d}s} = -\frac{\partial\Phi}{\partial\xi} = -|\eta|^2\xi\,. \tag{3.14}$$

From these equations follows:

$$\frac{\mathrm{d}}{\mathrm{d}s}\langle\xi, \eta\rangle = \left\langle \frac{\mathrm{d}\xi}{\mathrm{d}s}, \eta \right\rangle + \left\langle \xi, \frac{\mathrm{d}\eta}{\mathrm{d}s} \right\rangle = 0\,,$$

$$\frac{\mathrm{d}|\xi|^2}{\mathrm{d}s} = 2\langle\xi, \eta\rangle|\xi|^2 \qquad \text{and} \qquad \frac{\mathrm{d}|\eta|^2}{\mathrm{d}s} = 2\langle\xi, \eta\rangle|\eta|^2$$

which means that the cotangent bundle of the unit sphere $S^n$ with a removed zero section

$$P = T^+S^n = \{(\xi, \eta) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}\,;\ |\xi| = 1\,,\ \langle\xi, \eta\rangle = 0\,,\ |\eta| \neq 0\}$$

is preserved by the flow of $\Phi$. On $T^+S^n$ the trajectories of $\Phi$ as defined by the equations (3.14) take the form

$$\frac{\mathrm{d}\xi}{\mathrm{d}s} = \eta\,, \qquad \frac{\mathrm{d}\eta}{\mathrm{d}s} = -|\eta|^2\xi$$

and can be rewritten finally as

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}s^2} + |\eta|^2\xi = 0\,, \tag{3.15}$$

which are just the equations of the great circles on the sphere $S^n$. The above considerations can be shortly described by introducing the following notion:

**Definition 3.4.** *The Hamiltonian system $(P, \sigma, \Phi)$, where $\sigma$ and $\Phi$ should be considered as restrictions of the objects introduced above on $P$, is called a **geodesic flow** on $S^n$.*

Now, after these preliminary remarks we are ready to describe in details the aforementioned equivalence between the Kepler problem and the geodesic flow on the sphere.

For that purpose, let us consider in some details the stereographic projection of the sphere $S^n$ on the plane $(0, x_1, x_2, \dots x_n)$. The sphere $S^n$ in $\mathbb{R}^{n+1}$ will be represented as usually by:

$$S^n = \{(\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}\,;\ |\xi| = 1\}\,. \tag{3.16}$$

The stereographic projections from the north $N$, respectively the south $S$ pole of the sphere $S^n$ provide an atlas consisting of two charts $(U_N, \varphi_N)$, $(U_S, \varphi_S)$ where

$$\begin{aligned} U_N &= \{(\xi_0, \xi_1, \dots, \xi_n) \in S^n\,;\ \xi_0 \neq 1\} = S^n \backslash \{N\}\,,\\ U_S &= \{(\xi_0, \xi_1, \dots, \xi_n) \in S^n\,;\ \xi_0 \neq -1\} = S^n \backslash \{S\}\,, \end{aligned}$$

and the stereographic projections from the north, respectively the south pole of the sphere

$$\varphi_N \colon U_N \ni \xi \to x \in \mathbb{R}^n\,, \qquad \varphi_S \colon U_S \ni \xi \to \tilde{x} \in \mathbb{R}^n\,,$$

are

$$\varphi_N(\xi) = x_k = \frac{\xi_k}{1 - \xi_0}\,, \quad \varphi_S(\xi) = \tilde{x}_k = \frac{\xi_k}{1 + \xi_0}\,, \quad k = 1, 2, \dots, n\,. \tag{3.17}$$

The transition function $\varphi_{SN}$ between these charts is

$$\tilde{x} = \varphi_{SN}(x) = \varphi_S \circ \varphi_N^{-1}(x) = \left(\frac{x_1}{|x|^2}, \frac{x_2}{|x|^2}, \dots, \frac{x_n}{|x|^2}\right)\,. \tag{3.18}$$

Further on we shall work in the chart $(U_N, \varphi_N)$, i. e. assuming $\xi_0 \neq 1$ as the considerations in the other chart $(U_S, \varphi_S)$ are identical.

We will need also of the mapping $\chi_N\colon \mathbb{R}^n \to S^n\backslash\{N\}$ which is the inverse map of $\varphi_N$:

$$\xi_0 = \frac{|x|^2 - 1}{|x|^2 + 1}\,, \quad \xi_k = \frac{2x_k}{|x|^2 + 1}\,, \qquad k = 1, 2, \ldots, n\,. \tag{3.19}$$

The mapping $\varphi_N\colon S^n\backslash\{N\} \to \mathbb{R}^n$ and its inverse $\chi_N$ as given by the formulae (3.17) and (3.19) define a diffeomorphism between $S^n\backslash\{N\}$ and $\mathbb{R}^n$, which can be "lifted" canonically to a diffeomorphism of their cotangent bundles $T^*(S^n\backslash\{N\})$ and $T^*\mathbb{R}^n$. If $\eta\,\mathrm{d}\xi$ and $y\,\mathrm{d}x$ are the corresponding canonical one-forms the induced diffeomorphism transforms the first one into the other, i. e.

$$(\chi_N)^*\,(\eta\,\mathrm{d}\xi) = y\,\mathrm{d}x$$

which gives directly the formula

$$y_k = (1 - \xi_0)\eta_k + \eta_0\xi_k$$

and after some algebraic manipulations

$$\eta_0 = \langle x, y\rangle\,, \quad \eta_k = \frac{|x|^2 + 1}{2} y_k - \langle x, y\rangle x_k\,, \quad k = 1, 2, \ldots, n\,.$$

In order to transfer the hamiltonian equations of motion from $T^+(S^n\backslash\{N\})$ on $T^+\mathbb{R}^n$ it remains to compute the hamiltonian $\Phi$ as function of the coordinates $(x, y)$. Using previously derived relations we find that

$$F(x, y) = \Phi(\xi, \eta) = \frac{1}{2}|\xi|^2|\eta|^2 = \frac{(|x|^2 + 1)^2}{2}|y|^2\,,$$

and in parallel that the geodesics of "velocity" one ($|\eta|^2 = 1$) correspond to solutions with "energy" $F = 1/2$. On this hypersurface $F$ and $u(F)$, where $u'(1/2) = 1$ define identical trajectories. In particular, one can choose

$$G = u(F) = \sqrt{2F} - 1 = \frac{|x|^2 + 1}{2}|y| - 1\,,$$

which means that the evolution of the system is going on the hypersurface $G \equiv 0$. Now, we change the time parameter $s$ with $t = \int |y|\,\mathrm{d}s\,(|y| \neq 0)$, and the dynamical equations transforms accordindgly into the following system:

$$\begin{aligned} \dot{x} &= |y|^{-1}x' = |y|^{-1}G_y \\ \dot{y} &= |y|^{-1}y' = -|y|^{-1}G_x \end{aligned}$$

which is not of the canonical type. The Hamiltonian structure of these equations

$$\dot{x} = H_y\,, \qquad \dot{y} = -H_x$$

can be restored after an integration by which from

$$|y|^{-1}G_x = H_x \, ,$$

one obtains

$$|y|^{-1}G + C(y) = H \, .$$

If we differentiate the last equality with respect to $y$ and take into the account the constraint $G \equiv 0$, we find that $C(y) = \mathrm{const}$. Choosing this constant to be $-1/2$, we have

$$H = |y|^{-1}G - \frac{1}{2} = \frac{|x|^2}{2} - \frac{1}{|y|}$$

and correspondingly $G \equiv 0$ means $H = -1/2$.
The canonical transformation $x = -p$, $y = q$ sends finally $H$ into the Hamiltonian $H = \dfrac{|p|^2}{2} - \dfrac{1}{|q|}$ of the Kepler problem and the symplectic form $\mathrm{d}y \wedge \mathrm{d}x$ into $\mathrm{d}p \wedge \mathrm{d}q$ so that we can state:

**Theorem 3.2.** (Moser [43]) *The bundle of cotangent vectors of unit length to the punctured at the north pole sphere is mapped onto the hypersurface $E = -1/2$ of the Kepler problem in one-to-one way. The flow of the Kepler problem after a change of the time parameter is embedded into the geodesic flow on the sphere.*

**Remark 3.1.** *The hypersurface $P_\epsilon$ in the phase space $P$,*

$$P_\epsilon = \{(\xi, \eta) \in P \, ; \, \Phi(\xi, \eta) = \epsilon, \, \epsilon > 0\}$$

*is mapped diffeomorphically onto the hypersurface $\tilde{M}_E \subset M$,*

$$\tilde{M}_E = \{(q, p) \in M \, ; \, H(q, p) = E, \, E < 0\}$$

*if $E$ and $\epsilon$ are related as follows:*

$$E = -\frac{1}{4\epsilon} \, . \tag{3.20}$$

**Remark 3.2.** *Until now the north pole $N = (1, 0, \ldots, 0)$ of the sphere $S^n$ has been excluded from our considerations. By adding this point we compactify the energy hypersurfaces and regularize the Kepler flow as the geodesics through $N$ corresponds to the collisions with the central body in the Kepler problem.*

The so established equivalence explains the presence of the "hidden" $\mathbb{SO}(n+1)$ symmetry group of the Kepler problem as well of its "dynamical" extensions

$\mathbb{SO}(n+1,1)$ and $\mathbb{SO}(n+1,2)$. In order to clarify this point, let us consider the hypersurface

$$P_{1/2} = T^+S^n = \{(\xi,\eta) \in \mathbb{R}^{n+1}\times\mathbb{R}^{n+1};\ |\xi| = |\eta| = 1,\ \langle\xi,\eta\rangle = 0\}\,,$$

on which the group $\mathbb{SO}(n+1)$ act transitively. The stationary subgroup of any point in this submanifold is isomorphic to the group $\mathbb{SO}(n-1)$. The factor-space $\mathbb{SO}(n+1)/\mathbb{SO}(n-1) \cong P_{1/2}$ is known as Stiefel manifold of oriented orthogonal two-frames $V(2,n+1)$ in $\mathbb{R}^{n+1}$. The orbits of the geodesic flow on $P_{1/2}$ coincide with the orbits of the $\mathbb{SO}(2)$-action:

$$(\xi,\eta) \to (\xi\cos t + \eta\sin t, -\xi\sin t + \eta\cos t) \quad \text{for} \quad \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \in \mathbb{SO}(2)\,.$$

The orbit manifold (the factor-space of $V(2,n+1)$ with respect the above action) is the compact hermitian symmetric space of oriented two-planes in $\mathbb{R}^{n+1}$

$$\mathbb{SO}(n+1)/(\mathbb{SO}(n-1)\times\mathbb{SO}(2)) = Gr(2,n+1)\,.$$

The Grassmannian $Gr(2,n+1)$ is isometric to the non-degenerated $(n-1)$-dimensional complex quadric $Q_{n-1}$

$$Q_{n-1} = \{[z_1 : z_2 : \cdots : z_{n+1}] \in \mathbb{P}^n;\ \sum_{j=1}^{n+1} z_j^2 = 0\} \tag{3.21}$$

equipped with the canonical Kähler structure induced by the Fubini-Study metric on $\mathbb{P}^n$.

Let $\pi_{1/2}$ denotes the natural projection $\pi_{1/2}\colon P_{1/2} \to Q_{n-1}$. The projection $\pi_\epsilon\colon P_\epsilon \to Q_{n-1}$ coincides up to a scalar multiplier with $\pi_{1/2}$ as $P_\epsilon$ comes out of $P_{1/2}$ and the transformation $\xi \to \xi$, $\eta \to \sqrt{2\epsilon}\eta$.

If $\Omega$ is an invariant representative of the first Chern class of the hyperplane line bundle over $Q_{n-1}$ we can choose the scaling multiplier in such a way that,

$$\pi^*_{1/2}\Omega = \sigma_{1/2}\,.$$

In the context of the reduction theorem under above normalization for $\sigma_\epsilon$ we have:

$$\sigma_\epsilon = \pi^*_\epsilon(\Omega_\epsilon) = \pi^*_\epsilon(\sqrt{2\epsilon}\Omega) = \pi^*_\epsilon(2\pi\sqrt{2\epsilon}\omega_{FS})\,.$$

Summing up the above we state:

**Lemma 3.3.** *The quantization of the $n$-dimensional Kepler problem reduces to geometric quantization of the $(n-1)$-dimensional quadric $Q_{n-1}$ equipped with the Kähler form $\sqrt{2\epsilon}\Omega$.*

As we will see soon, the existence of quantum line bundles on the quadric is a condition on $\epsilon$, which defines the energy spectrum while dimensionalities of the respective spaces of sections are the multiplicities.

### 3.6. Wave Functions

**Definition 3.5.** *The Fubini–Study metric on the $n$-dimensional complex projective space is the Kählerian metric $g$ corresponding to the Fubini–Study form*

$$\omega_{FS} = \frac{\mathrm{i}}{2\pi}\partial\bar{\partial}\log|z|^2\,, \qquad |z|^2 = \sum_{j=1}^{n+1} z_j\bar{z}_j\,.$$

As we already know $\omega_{FS}$ belongs to the first Chern class of the hyperplane line bundle over $\mathbb{P}^n$ and generates $H^2(\mathbb{P}^n,\mathbb{Z})$ (see also Griffiths and Harris [15]). The induced Kählerian structure over $Q_{n-1}$ which is embedded standardly into $\mathbb{P}^n$ (cf. (3.21)) will be denoted also by $\omega_{FS}$. This form coincides with the invariant Kählerian form over the symmetric space $\mathbb{SO}(n+1)/(\mathbb{SO}(n-1)\times\mathbb{SO}(2))$ (cf. Kobayashi and Nomizu [24]). By functoriality $[\omega_{FS}]$ is also the first Chern class of the hyperplane bundle on $Q_{n-1}$. From now on we will write $Q$ in place of $Q_{n-1}$ and if not stated definitely something different we will assume that $n>3$. For the smooth hypersurface $Q$ in the $n$-dimensional projective space $\mathbb{P}^n$ we have:

**Theorem 3.3.** (Lefschetz, see Griffiths and Harris [15]) *The map*

$$H^q(\mathbb{P}^n,\mathbb{Z}) \to H^q(Q,\mathbb{Z})$$

*induced by the embedding $i\colon Q\to\mathbb{P}^n$ is an isomorphism for $q<n-2$, and when $q=n-1$ is an injection.*

According the above cited theorem

$$H^2(Q,\mathbb{Z}) = H^2(\mathbb{P}^n,\mathbb{Z}) = \mathbb{Z}$$

and therefore $H^2(Q,\mathbb{Z})$ is generated by $[\omega_{FS}]$. Besides,

$$H^1(Q,\mathcal{O}) = H^2(Q,\mathcal{O}) = 0\,,$$

where $\mathcal{O}$ denotes the structure sheaf over $Q$. If $\mathcal{O}^*$ denotes the sheaf of nowhere vanishing holomorphic function we have the short exact sequence:

$$0\to\mathbb{Z}\to\mathcal{O}\xrightarrow{\exp}\mathcal{O}^*\to 0\,.$$

The interesting part of the corresponding long exact cohomology sequence is

$$H^1(Q,\mathcal{O})\to H^1(Q,\mathcal{O}^*)\to H^2(Q,\mathbb{Z})\to H^2(Q,\mathcal{O})\,.$$

As both extreme elements are zero we have:

$$H^1(Q, \mathcal{O}^*) = H^2(Q, \mathbb{Z}) = \mathbb{Z}$$

i. e., there exists an isomorphism between the group $H^1(Q, \mathcal{O}^*)$ of equivalent classes of holomorphic line bundles over $Q$ and $\mathbb{Z}$. Even more, every holomorphic line bundle over $Q$ is a tensor power of the hyperplane line bundle $L$. If $L_k$ denotes its $k$-th power we have:

**Lemma 3.4.** (Mladenov and Tsanov [38]) *The Picard group* $\mathrm{Pic}(Q_{n-1})$ *consisting of all holomorphic line bundles on the quadric* $Q_{n-1}$ *is isomorphic to* $H^2(Q_{n-1}, \mathbb{Z}) \cong \mathbb{Z}$.

The Chern class of the line bundle $L_k$ is $c_1(L_k) = k\omega_{FS}$. Quantum line bundles are those for which $k \in \mathbb{Z}^+$. As we are going to describe them in more details let us introduce some notation. $G$ will denote the group $\mathbb{SO}(n+1, \mathbb{R})$, $H$ and $K$ will denote respectively the groups $\mathbb{SO}(n-1, \mathbb{R})$ and $\mathbb{SO}(2, \mathbb{R}) \times \mathbb{SO}(n-1, \mathbb{R})$ realized as follows:

$$H = \left\{ \begin{bmatrix} 1 & 0 & \\ 0 & 1 & \\ & & h \end{bmatrix} ;\ h \in \mathbb{SO}(n-1, \mathbb{R}) \right\}$$

$$K = \left\{ \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} ;\ k_1 \in \mathbb{SO}(2, \mathbb{R}), k_2 \in \mathbb{SO}(n-1, \mathbb{R}) \right\} .$$

The group $\mathbb{SO}(2, \mathbb{R}) = k_\theta = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$ acts on Stiefel manifold $V(2, n+1) = G/H$ with right translations

$$(gH)u_\theta = g \begin{bmatrix} u_\theta & \\ & I_{n-1} \end{bmatrix} H .$$

The space $G/H$ is a principal fibre bundle with fibre $\mathbb{SO}(2, \mathbb{R})$ over $G/K \cong Gr(2, n+1) \cong Q_{n-1}$. For any integer $m$ we define a character $\tilde{\chi}_m$ of the group $\mathbb{SO}(2, \mathbb{R})$ by the formula:

$$\tilde{\chi}_m(u_\theta) = \mathrm{e}^{\mathrm{i}m\theta} , \qquad u_\theta \in \mathbb{SO}(2, \mathbb{R}) .$$

The line bundle over $G/H$ associated with the character $\tilde{\chi}_m$ will be denoted by $\tilde{L}_m$. The space of smooth sections $\Gamma(\tilde{L}_m)$ of $\tilde{L}_m$ is a $G$-module with respect to the left translations and is isomorphic to the $G$-module

$$\{f \in G^\infty(G/H);\ f(xu_\theta) = \tilde{\chi}_m^{-1}(u_\theta)f(x),\ x \in G/H\}$$

Until now we have not used the existing $G$-invariant complex structure over $G/K$, which by Borel theorem is holomorphically isomorphic with that of

$G^C/K^C P_+$ (cf. Helgason [18]). Here $G^C$ and $K^C$ are complexifications of $G$ and $K$, and $P_+$ is a subgroup of $G^C$ whose elements are of the form [26]:

$$\begin{bmatrix} 1-(z_1^2+\cdots+z_{n+1}^2)/2 & \mathrm{i}(z_1^2+\cdots+z_{n+1}^2)/2 & -z_3 & \dots & -z_{n+1} \\ \mathrm{i}(z_1^2+\cdots+z_{n+1}^2)/2 & 1+(z_1^2+\cdots+z_{n+1}^2)/2 & \mathrm{i}z_3 & \dots & \mathrm{i}z_{n+1} \\ z_3 & -\mathrm{i}z_3 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ z_{n+1} & -\mathrm{i}z_{n+1} & 0 & \dots & 1 \end{bmatrix}$$

where $z_1, z_2, \dots, z_{n+1} \in \mathbb{C}$.

For any non-negative integer $m$ we can define a holomorphic character $\chi_m$ of $K^C P_+$ via the formula

$$\chi_m(kz) = \mathrm{e}^{im\theta} \quad \text{for every} \quad k = \begin{bmatrix} u_\zeta & \\ & k' \end{bmatrix} \in K^C \quad \text{and} \quad z \in P_+\,,$$

where

$$u_\zeta = \begin{bmatrix} \cos\zeta & \sin\zeta \\ -\sin\zeta & \cos\zeta \end{bmatrix} \in \mathbb{SO}(2,\mathbb{C}) \quad \text{and} \quad k' \in \mathbb{SO}(n-1,\mathbb{C})\,.$$

The associated with $\chi_m$ $G^C$-homogeneous holomorphic line bundle $L_m$ on $G^C/K^C P_+$ is $C^\infty$-isomorphic with the line bundle $\tilde{L}_m$. The space of all holomorphic sections $\Gamma(L_m)$ can be identified with the space

$$\left\{ f \in \mathrm{Hol}\left(\mathbb{SO}(n+1,\mathbb{C})\right) \,\middle|\, \begin{array}{l} f(g\gamma) = \chi_m^{-1}(\gamma) f(g)\,, \\ g \in \mathbb{SO}(n+1,\mathbb{C}),\ \gamma \in K^C P_+ \end{array} \right\},$$

on which $G$ act by left translations. The Borel–Weil theorem guarantee that the representation $\pi_m$ of $G$ on $\Gamma(L_m)$ is irreducible. For each multi-index $(i_1, \dots, i_{n+1})$ of non-negative integers with "length" $m = \sum_{k=1}^{n+1} i_k$, we define a function $\varphi_{i_1,\dots,i_{n+1}}$ on $\mathbb{SO}(n+1,\mathbb{C})$ by the formula:

$$\varphi_{i_1,\dots,i_{n+1}}(g) = (x_1 - iy_1)^{i_1} \cdots (x_{n+1} - iy_{n+1})^{i_{n+1}}\,,$$

where $g = \begin{bmatrix} x_1 & y_1 & \\ \vdots & \vdots & * \\ x_{n+1} & y_{n+1} & \end{bmatrix}$ is any element of $\mathbb{SO}(n+1,\mathbb{C})$.

An easy check shows that $\varphi_{i_1,\dots,i_{n+1}}$ satisfies

$$\varphi_{i_1,\dots,i_{n+1}}(g\gamma) = \chi_m^{-1}(\gamma)\varphi_{i_1,\dots,i_{n+1}}(g),$$

for any $g \in \mathbb{SO}(n+1,\mathbb{C})$ and $\gamma \in K^C.P_+$ and therefore $\varphi_{i_1,\dots,i_{n+1}} \in \Gamma(L_m)$. Moreover, $\left\{\varphi_{i_1,\dots,i_{n+1}}\,;\ \sum_{k=1}^{n+1} i_k = m\right\}$ span a bases in $\Gamma(L_m)$, as the last space can be identified with a subspace in $\mathbb{C}[z_1,\dots,z_{n+1}]/(z_1^2+\cdots+z_{n+1}^2)$, where

$\mathbb{C}[z_1, \dots, z_{n+1}]$ denotes the ring of complex polynomials and $(z_1^2 + \cdots + z_{n+1}^2)$ is the ideal in $\mathbb{C}[z_1, \dots, z_{n+1}]$ generated by $z_1^2 + \cdots + z_{n+1}^2$. The identification is realized following the rule: to $\varphi_{i_1,\dots,i_{n+1}}$ one juxtaposes the monomial $\Pi z_k^{i_k} = z_1^{i_1} \cdots z_{n+1}^{i_{n+1}}$.

### 3.7. Spectra and Multiplicities

**Definition 3.6.** *Let $M$ be a $n$-dimensional complex manifold and $T^*M$ is its cotangent bundle. A canonical bundle of $M$ will be called the $n$-th exterior power $\Lambda^n(T^*M)$ of the bundle $T^*M$ which will be denoted further by $K_M$.*

If $N$ is a smooth analytical hypersurface in $M$, we will denote by $[N]$ the line bundle associated with the divisor $N$. There exists a close relation (known as adjunction formula) between the canonical bundles of $M$ and $N$, namely

$$K_N = (K_M \otimes [N])|_N \,. \tag{3.22}$$

In particular if $V$ is a smooth hypersurface in $\mathbb{P}^n (n \geq 4)$ of degree $d \neq n+1$ this formula reads:

$$K_V = (K_{\mathbb{P}^n} \otimes [V])|_V = [(d - n - 1)\mathbb{P}^{n-1}] \,.$$

Concretely for $Q \subset \mathbb{P}^n$ which is the case we are interested we have

$$K_Q = [(-n+1)\mathbb{P}^{n-1}] \,.$$

The canonical bundle $K_Q$ was introduced having in mind the equality:

$$c_1(Q_{n-1}) = -c_1(K_Q) \,,$$

from which we obtain

$$c_1(Q_{n-1}) = (n-1)\omega_{FS} \,.$$

Let $L_{N-1} (N = 1, 2, \dots)$ is an arbitrary quantum bundle over $Q_{n-1}$. Following the definition of the quantum bundle given in Section 3.3 we can write for $L_{N-1}$

$$c_1(L_{N-1}) = (N-1)\omega_{FS} = \sqrt{2\epsilon}\,\omega_{FS} - \frac{n-1}{2}\omega_{FS} \,,$$

i. e.

$$\sqrt{2\epsilon} = N - 1 + \frac{n-1}{2} \,,$$

and this is enough to obtain the spectrum of the geodesic flow on $S^n$:

$$\epsilon_N = \frac{1}{2}\left(N + \frac{n-3}{2}\right)^2, \qquad N = 1, 2, 3, \dots \tag{3.23}$$

In order to find the corresponding multiplicities, which coincide with dimensionalities of the spaces $H^0(Q, L_k)$ use will be made of the short exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^n}(L_k \otimes L_{-2}) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}(L_k) \xrightarrow{r} \mathcal{O}_Q(L_k) \to 0\,,$$

in which $\alpha$ is a multiplication of the sections of $L_{k-2}$ with the polynomial $\sum\limits_{j=1}^{n+1} z_j^2$ defining the quadric $Q$ in $\mathbb{P}^n$ and $r$ means the restriction on $Q$.
The corresponding long exact cohomological sequence starts up with

$$0 \to H^0(\mathbb{P}^n, L_{k-2}) \to H^0(\mathbb{P}^n, L_k) \to H^0(Q, L_k) \to H^1(\mathbb{P}^n, L_{k-2})\,.$$

The extreme element at the right is zero by the Kodaira vanishing theorem (see Griffiths and Harris [15]). So, we can conclude that

$$\dim H^0(Q, L_k) = \binom{n+k}{k} - \binom{n+k-2}{k-2}. \tag{3.24}$$

Summing up all that above, we can state:

**Theorem 3.4.** (Mladenov and Tsanov [38]) *The energy spectrum of the geodesic flow on the $n$-dimensional sphere is*

$$\epsilon_N = \frac{1}{2}\left(N + \frac{n-3}{2}\right)^2 \tag{3.25}$$

*with multiplicities*

$$m(\epsilon_N) = \frac{2N+n-3}{N+n-2}\binom{N+n-2}{N-1} \tag{3.26}$$

*where $N = 1, 2, 3, \ldots$*

By this theorem and Remark 3.20 we can formulate also the corresponding results for the energy spectrum spectrum of the $n$-dimensional Kepler problem (details can be found in [38]).

**Remark 3.3.** *For $n = 1, 2$ and $3$ the formulae for the spectra and multiplicities of the hydrogen atom (Kepler problem) reproduce classical quantum-mechanical results but they can not be considered as proven because at many places in our treatment these cases have been excluded in advance (cf. also the Cartan's list in Section 3.3).*

Because they are specific and of real importance for physics, we will study them case by case. We will start with the most interesting case $n = 3$.

### 3.8. Hydrogen Atom

In our notation the relevant orbit manifold in this case is the quadric $Q_2$ in $\mathbb{P}^3$ which is an interesting object even by itself. Algebraically, $Q_2$ is doubly ruled complex surface and any of its points is the unique intersection of two complex lines lying on $Q_2$. Let us consider e. g. the point $\tilde{z} = [\mathrm{i} : 1 : 0 : 0]$ on the quadric

$$Q_2 : z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 . \tag{3.27}$$

The hyperplane $H_2$: $z_1 - \mathrm{i}z_2 = 0$ contains $\tilde{z}$ and is tangential to $Q_2$ at that point. Every point in $Q_2 \cap H_2$ satisfies $z_3^2 + z_4^2 = 0$ and consequently lies on one of the planes:

$$H_2' \colon z_3 + \mathrm{i}z_4 = 0 , \qquad H_2'' \colon z_3 - \mathrm{i}z_4 = 0 ,$$

i. e.

$$Q_2 \cap H_2 = (H_2 \cap H_2') \cup (H_2 \cap H_2'') . \tag{3.28}$$

The right part represents an union of a pair of complex lines $l$ and $l'$ lying on $Q_2$ which intersects at the point $\tilde{z}$, i. e.

$$Q_2 \cap H_2 = l' \cup l'' , \qquad l' \cap l'' = [\mathrm{i} : 1 : 0 : 0] .$$

If a point $(t_1, t_2) \in \mathbb{C} \times \mathbb{C}$ is given, then the point

$$[1 + t_1 t_2 : \mathrm{i}(1 - t_1 t_2) : t_1 - t_2 : \mathrm{i}(t_1 + t_2)]$$

lies on $Q_2$, and this means that we have a holomorphic map $f \colon \mathbb{C} \times \mathbb{C} \to Q_2$ which is invertible on $Q_2^* = Q_2 \backslash H_2$ by the formula:

$$(t_1, t_2) = \left( \frac{z_3 + \mathrm{i}z_4}{z_1 - \mathrm{i}z_2} \quad - \frac{z_3 - \mathrm{i}z_4}{z_1 - \mathrm{i}z_2} \right) . \tag{3.29}$$

The image of $f$ is $Q_2 \backslash \{l' \cup l''\}$. If $t_1, t_2$ have to be considered as non-homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$ it is clear that the map $f$ defined above can be extended to continuous bijection between $\mathbb{P}^1 \times \mathbb{P}^1$ and $Q_2$. Now, let $L_1$ be the hyperplane bundle over $Q_2$ and $\Omega$ be the Kählerian form of the invariant Kählerian metric. We can normalizie $\Omega$ in order to have $c_1(L) = [\Omega]$. In the space $H^2(Q_2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$, $[\Omega]$ is just the sum of both generators. Under reduction, the symplectic form $\sigma$ in the phase space $P$ "falls" onto $\Omega$ up to a real factor and obviously it belongs to integer cohomological class if and only if this factor is integer. Therefore, all quantum bundles over $Q_2$ are again positive degrees of $L_1$ and taking into account that $c_1(Q_2) = 2[\Omega]$ the proof of the Theorem 3.4 is completed also in the case $n = 3$.

### 3.9. Space Rotator

In this case, let us consider the quadric $Q_1$ in $\mathbb{P}^2$

$$Q_1 = z_1^2 + z_2^2 + z_3^2 = 0\,.$$

Let

$$Q_1^* = \{z \in Q_1\,;\; z_3 \neq 0\}$$

and introduce

$$\zeta = \frac{z_1}{z_3} + \mathrm{i}\frac{z_2}{z_3}\,.$$

From the equation of the quadric (when $z_3 \neq 0$) we have:

$$\left(\frac{z_1}{z_3} + \mathrm{i}\frac{z_2}{z_3}\right)\left(\frac{z_1}{z_3} - \mathrm{i}\frac{z_2}{z_3}\right) = -1\,,$$

which means that $\zeta \neq 0$ if $z \in Q_1^*$ and by the same reason as well:

$$\frac{z_1}{z_3} - \mathrm{i}\frac{z_2}{z_3} = -\frac{1}{\zeta}\,.$$

In this way we obtain

$$\frac{z_1}{z_3} = \frac{1}{2}\left(\zeta - \frac{1}{\zeta}\right)\,, \qquad \frac{z_2}{z_3} = \frac{1}{2\mathrm{i}}\left(\zeta + \frac{1}{\zeta}\right)\,, \quad \zeta \in \mathbb{C}^*$$

which tell us that we have a holomorphic map $f\colon \mathbb{C}^* \to Q_1^*$ that can be obviously extended to a continuous bijection between $\mathbb{P}^1$ and $Q_1$. Straightforward application of the modified qeometric quantization scheme to $\mathbb{P}^1$ leads to the following result:

*The quantum energy levels of the geodesic flow on $S^2$ are:*

$$\epsilon_N = \frac{1}{2}\left(\frac{N}{2}\right)$$

*with multiplicities* $m(\epsilon_N) = N\,,$ *where* $N = 1, 2, 3, \ldots$

The reason for the discrepancy with the statement of Theorem 3.4 can be pointed out immediately. Quasiclassical and quantum-mechanical results are in complete agreement for the systems which we have studied. On the other hand in Simms [53] one can find a proof that geometric and quasiclassical quantization coincide in the case of Hamiltonian sytems with simply-connected energy hypersurfaces. Just this condition is not fullfield in the case under consideration. The Stiefel manifold $V(2,3)$ of the orthonormal two-frames in $\mathbb{R}^3$ is isomorphic to the group $\mathbb{SO}(3)$ for which we know that

$$\pi_1\left(\mathbb{SO}(3)\right) = H_1\left(\mathbb{SO}(3), \mathbb{Z}\right) = \mathbb{Z}_2 \neq 0$$

and therefore, not simply-connected. The correct energy levels and multiplicities can be obtained if we take into account only those quantum line bundles over $Q_1$ which are restrictions of such on $\mathbb{P}^2$. In our notation these are exactly the bundles $L_{2k}$, for $k = 0, 1, 2, \ldots$ When $n > 2$ such problem does not appear as all line bundles over $Q_{n-1}$ are restrictions of line bundles on $\mathbb{P}^n$. Processing in the way proposed above we can write

$$\sqrt{2\epsilon}[\omega_{FS}] - [\omega_{FS}] = (2N - 2)[\omega_{FS}], \quad N = 1, 2, 3, \ldots$$

where use has been made of the fact that $c_1(\mathbb{P}^1) = 2[\omega_{FS}]$. Now we can state:

**Theorem 3.5.** (Mladenov and Tsanov [39]) *The quantum energy levels of the geodesic flow on the two-dimensional sphere are given by the formula:*

$$\epsilon_N = \frac{1}{2}\left(N - \frac{1}{2}\right)^2, \tag{3.30}$$

*and their multiplicities are:*

$$m(\epsilon_N) = 2N - 1, \qquad N = 1, 2, 3, \ldots \tag{3.31}$$

Using the relation $E = -\dfrac{1}{4\epsilon}$ we can formulate immediately the corresponding results for the Kepler problem in the flat two-dimensional space which coincide with quantum-mechanical findings.

## 3.10. Plane Rotator

Despite its simplicity this model could be quite profitable as it has much in common with the general problem of quantizing completely integrable systems. The first remark that should be made is based on the quite simple observation concerning the scheme that we have followed. It does not work! Why? Because the orbit manifold consists either of a point or two ponits and our computations involving the characteristic classes can not go through in this case. Fortunately there exists another cohomological theory which is meaningful in the cases when the manifold is just a pont. It is called an equivariant cohomology and we will make a short digression in this theory just to state the relevant definitions and results by using the notation of Atiyah and Bott [3]. Naïvely speaking the equivariant cohomology of the $G$-manifold $M$ should be the cohomology of the quotient space $M/G$. Unfortunately this gives a useful notion only when $G$ acts freely on $M$. The way to get a good theory in the general case is to find the right notion of quotient, as given e. g. by the Borel construction described below. Let $EG, BG = EG/G$ be respectively the universal principal bundle and the classifying space for the group $G$. We denote by $M_G$ the associated $M$-bundle

$$M_G = M \times_G EG\,.$$

Then the equivariant cohomology ring with coefficients in the ring $F$ is defined by

$$H^*_G(M,F) \cong H^*(M_G,F)\,. \tag{3.32}$$

When $K \subset G$ is a Lie subgroup and $M$ is the homogeneous space $G/K$, we have

$$H^*_G(M,F) \cong H^*(EG/K,F) = H^*(BK,F)\,.$$

In particular

$$H^*_G(pt,F) \cong H^*(BG,F) \tag{3.33}$$

which explains why the equivariant cohomology of a point is so rich.
It is well-known that if $K$ is a torus of dimension $k$ the above cohomology ring is just the (cut) polynomial ring of $k$ generators of degree two with coefficients in $F$, i. e.

$$H^*(BK,F) \cong F(u_1,\ldots,u_k).$$

If $G$ is a compact Lie group with maximal torus $K$ and Weyl group $W$ then we have

$$H^*(BG,F) \cong H^*(BK,F)^W \cong F(u_1,\ldots,u_k)^W \tag{3.34}$$

i. e. the cohomology ring of the classifying space $BK$ of the group $K$ consists of the $W$-symmetric polynomials and is again generated by $k$ elements of even degree (the "elementary symmetric functions"). In any case the equivariant cohomology ring $H^*(BG,\mathbb{Z}) = H^*_G$ labels the irreducible representations of the group $G$.
We shall always interpret $H^*_G(M,\mathbb{R})$ as the equivariant de Rham cohomology ring of $M$ as described in Atiyah and Bott [3]. Let $(M,\sigma)$ be a symplectic manifold with a $G$-invariant symplectic form $\sigma$, and let

$$J\colon M \to \mathfrak{g}^* \tag{3.35}$$

be the moment map for the Hamiltonian action of $G$ on $M$. Then the map $J$ determines an unique "equivariant extension"

$$\sigma \to \sigma^\# \in H^*_G(M,\mathbb{R})$$

(see Atiyah and Bott [3], Prop. 6.18).

Let us return for the moment to the case when we have $\mathbb{SO}(2)$ action defined by the geodesic flow on the sphere of arbitrary dimension. The momentum map of this action on the symplectic manifold $(P, \sigma)$ is

$$\Phi \colon P \to \mathbb{R}$$

and it is obvious that it commutes with the natural symplectic action of $\mathbb{SO}(n+1)$ on the same manifold (we take the standard action of $\mathbb{SO}(n+1)$ on $S^n$ and lift it to the cotangent bundle). We identify $\mathfrak{so}(n+1)^*$ (via the Killing form) with the space of all antisymmetric matrices where one has a natural (co)adjoint action of $\mathbb{SO}(n+1)$. The moment map

$$J \colon P \to \mathfrak{so}(n+1)$$

of the lifted $\mathbb{SO}(n+1)$ action on $P = T^+S^n$ is given by

$$J_{ij}(\xi, \eta) = \eta_i \xi_j - \eta_j \xi_i \,, \qquad i, j = 1, \ldots, n+1 \,.$$

Obviously

$$\{\Phi, J_{ij}\} = 0$$

for all $i, j$, because the Hamiltonian $\Phi$ is invariant with respect to the action of $\mathbb{SO}(n+1)$.

Thus the equivariant extension $\sigma^{\#}$ of $\sigma$ is invariant under the $\mathbb{SO}(2)$ action defined by $(P, \sigma, \Phi)$ (the geodesic flow). This allows us to "reduce" $\sigma^{\#} \in H^*_{\mathbb{SO}(n+1)}(P, \mathbb{R})$ to an element

$$\sigma^{\#}_{\epsilon} \in H^*_{\mathbb{SO}(n+1)}(Q_{n-1}, \mathbb{R}) = H^*\Big(B\Big(\mathbb{SO}(n-1)\times\mathbb{SO}(2)\Big), \mathbb{R}\Big) \,.$$

The admissibility condition for the parameter $\epsilon$ now reads

$$\sigma^{\#}_{\epsilon} - \frac{1}{2} c_1^{\#}(Q_{n-1}) \in H^*_{\mathbb{SO}(n+1)}(Q_{n-1}, \mathbb{Z}) = H^*\Big(B\Big(\mathbb{SO}(n-1)\times\mathbb{SO}(2)\Big), \mathbb{Z}\Big)$$

and this gives the spectrum (3.25) and multiplicities (3.26). It should be pointed out that the multiplicitity formula is valid for all values of $n$ and $N$, except $n = N = 1$ (see bellow).

Now let $n = 1$. Then

$$\sigma^{\#} = \sigma - Ju \tag{3.36}$$

where $u$ is the generator of $H^*_{\mathbb{SO}(2)}(pt, \mathbb{R}) = H^*\Big(B\mathbb{SO}(2), \mathbb{R}\Big)$, and $J$ is the momentum map

$$J(\xi, \eta) = \xi_1 \eta_2 - \xi_2 \eta_1 \,.$$

One computes easily that

$$J^2 = 2\Phi\,.$$

Now if we reduce the element in (3.36) at $\Phi = \epsilon$, the quantization condition becomes

$$\pm\sqrt{2\epsilon}u = \sigma_\epsilon^{\#} \in H^*\Big(B\mathbb{SO}(2), \mathbb{Z}\Big)\,,$$

whence

$$\pm\sqrt{2\epsilon}u = lu$$

for some integer $l$. The admissibility condition for the energy thus reduces to

$$2\epsilon = l^2$$

and as usual we introduce tne standard index $N = |l| + 1$.
This gives the energy values for the case $n = 1$. The dimensions of the irreducible representations of $\mathbb{SO}(2)$ are of course known to be equal to one, and as there are two representations (values of $J$) corresponding to the eigenvalue

$$\epsilon_N = \frac{1}{2}(N-1)^2\,, \qquad N = 1, 2, 3, \ldots$$

we have

$$m_N = 2 \;\text{ for all }\; N > 1\,,$$

and

$$m_1 = 1\,.$$

Of course the double degeneracy of the spectrum corresponds topologically to the fact that for $\Phi > 0$ the orbit space $Q_0$ consists of two points.

**Remark 3.4.** *We have introduced the equivariant cohomologies in order to complete our study of the geodesic flow on the spheres in all dimensions [42]. However, the idea to use this theory in the context of Marsden-Weinstein reduction is much more general and should work in other important cases.*

## 4. Constrained Quantum Mechanics

The alternative to the Kostant–Souriau quantization of curved manifolds has been introduced in a few years by Jensen and Koppe [23] under the name "constrained" quantum mechanical systems. As a matter of fact, at that time they have considered these systems as pure mathematical ones, "since such systems do no exist" [23]. Fifteen years later the experimental search for new materials arrived at the exotic $C_{60}$ and $C_{70}$ molecules, and subsequently the quantization of these molecules was faced with an entirely new puzzle of curiosity, which was nothing but the above mentioned "unphysical" problem. Nevertheless, quantum-mechanical study of these molecules remained so far mainly in the framework of quantum chemistry and its simplest approximations (cf. e. g. Fowler [12], Haddon [17] and references therein). This situation is a little strange in view of the fact that the rigorous quantum mechanical description of the behaviour of a particle constrained on a curved manifold is relatively old and well-known problem (see Jensen and Koppe [23, 25]) and for later developments da Costa [7], Fujii *et al* [13], Ikegami *et al* [21], Matsutani [32, 33], Ogawa [45], Ohnuki and Kitagado [46], Tanimura [60] and Tolar [61]. Quantum channels, strips, tubes wavegides and wires attract great attention as well, see Exner and Seba [11], Goldstone and Jaffe [14], and Takagi and Tanzawa [59]. A trully remarkable result in this field is that the bends and bulges in infinite tubes of constant normal cross section produce always at least one bound state (more details can be found in the book by Londergan *et al* [30]). Besides, scattering states can be treated effectively via the method of transfer matrices and one can consider heterostructures of coupled bends. Another strong result in this field can be provided by investigating the validity of the Saxon–Hutner conjecture for periodic structures of the above kind. Presice statement and some criteria for validity of this conjecture can be found in [35] and references therein.

### 4.1. Surface Geometry

As the constrained quantum mechanics approach is based on some notions of the classical differential geometry the respective definitions are reviewed as far as they are needed for the discussions to follow. Modern exposition of the subject can be found, e. g. in the books by Berger and Gostiaux [4], McLeary [28], and Oprea [47]. There in full depth is explained that any such surface $S$

$$\mathbf{x} = \mathbf{x}[u, v] = \Big(x(u, v), y(u, v), z(u, v)\Big)$$

is specified by its first and second fundamental forms

$$\begin{aligned} \mathrm{I} &= E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2 \quad \text{and} \\ \mathrm{II} &= L\,\mathrm{d}u^2 + 2M\,\mathrm{d}u\,\mathrm{d}v + N\,\mathrm{d}v^2\,, \end{aligned} \tag{4.1}$$

and that their coefficients are given by

$$\begin{aligned} E &= E[u,v] = \mathbf{x}_u.\mathbf{x}_u\,, \quad F = F[u,v] = \mathbf{x}_u.\mathbf{x}_v\,, \quad G = G[u,v] = \mathbf{x}_v.\mathbf{x}_v\,, \\ L &= L[u,v] = \mathbf{x}_{uu}.\mathbf{n}\,, \quad M = M[u,v] = \mathbf{x}_{uv}.\mathbf{n}\,, \quad N = N[u,v] = \mathbf{x}_{vv}.\mathbf{n}\,, \end{aligned}$$

where $\mathbf{n}$ is the unit vector normal to $\mathcal{S}$

$$\mathbf{n} = \mathbf{n}[u,v] = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}\,. \tag{4.2}$$

By definition the **normal curvature** $\mathbf{k}_n$ in the direction $(\,\mathrm{d}u : \mathrm{d}v)$ is

$$\mathbf{k}_n = \frac{\mathrm{II}}{\mathrm{I}} = \frac{L\,\mathrm{d}u^2 + 2M\,\mathrm{d}u\,\mathrm{d}v + N\,\mathrm{d}v^2}{E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2}\,, \tag{4.3}$$

and the directions at which it attains extremal values (maximum and minimum) are called **principal directions**. If the coordinate curves coincide with the principal directions then

$$F = M \equiv 0 \tag{4.4}$$

and the corresponding curvatures of these directions can be found by the formulae

$$\mathbf{k}_1 = \frac{L}{E}\,, \qquad \mathbf{k}_2 = \frac{N}{G}\,. \tag{4.5}$$

Besides, it should be noted also that in this situation $\mathbf{k}_1$ and $\mathbf{k}_2$ are the principal curvatures along the meridians and parallels of latitude respectively. Classical differential geometry operates also with other important notions which are of immediate interest for us. These are the **Gaussian curvature** $K$ and the **mean curvature** $\bar{H}$

$$K = \mathbf{k}_1\mathbf{k}_2\,, \qquad \bar{H} = \frac{\mathbf{k}_1 + \mathbf{k}_2}{2}\,, \tag{4.6}$$

and the surface area element $\mathrm{d}A$

$$\mathrm{d}A = \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v = \sqrt{EG}\,\mathrm{d}u\,\mathrm{d}v\,. \tag{4.7}$$

### 4.2. Quantum Mechanics on Surfaces in $\mathbb{R}^3$

The systems in which we are interested after Jensen and Koppe [23] are of the following type: a particle of mass $m$ is constrained to move on some surface $\mathcal{S}$. In this setting the naïve approach to quantization of such systems refers to association of the kinetic energy with the Laplacian operator of the natural Riemannian metric induced on this surface. More consistent quantum-mechanical considerations show that a particle permanently attached to the surface of parametric equations $\mathbf{x}[u,v]$ violates Heisenberg's uncertainty principle so that we are obliged to consider the portion of the space in an immediate neighborhood of $\mathcal{S}$ which can be parametrized as

$$\mathbf{r}[u,v,\tau] = \mathbf{x}[u,v] + \tau\mathbf{n}[u,v] \tag{4.8}$$

where the absolute value of $\tau$ gives the distance between the surface and the point with coordinates $(u,v,\tau)$. As our idea is to bent the particle on the surface we need of an infinite squeezing force which simulate the constraints in classical mechanics. For that purpose we consider the potential $V_\lambda(\tau)$ where $\lambda$ is the squeezing parameter that measures the strength of the potential

$$\lim_{\lambda\to\infty} V_\lambda(\tau) = \begin{cases} 0\,, & \tau = 0\,, \\ \infty\,, & \tau \neq 0\,. \end{cases} \tag{4.9}$$

From (4.8) it follows that

$$\mathrm{d}\mathbf{r} = (\mathbf{x}_u + \tau\mathbf{n}_u)\,\mathrm{d}u + (\mathbf{x}_v + \tau\mathbf{n}_v)\,\mathrm{d}v + \mathbf{n}\,\mathrm{d}\tau \tag{4.10}$$

Since the derivatives of $\mathbf{x}$ and $\mathbf{n}$ with respect of $u$ and $v$ are orthogonal to $\mathbf{n}$ the matrix of the metric tensor $\tilde{g}$ in our three-dimensional neighborhood of $\mathcal{S}$ associated with the line element

$$\mathrm{d}\ell^2 = \mathrm{d}\mathbf{r}.\,\mathrm{d}\mathbf{r}$$

is of $2+1$ block-diagonal form. Now we can turn our attention to the Schrödinger equation. Writing the Laplacian $\Delta_{\tilde{g}}$ in the curvilinear coordinates $(u,v,\tau)$

$$\Delta_{\tilde{g}} = -\frac{1}{\sqrt{|\tilde{g}|}}\partial_i(\tilde{g}^{ij}\sqrt{|\tilde{g}|}\partial_j) \quad \text{for} \quad i,j = u,v,\tau \tag{4.11}$$

where $|\tilde{g}|$ is the determinant of the metric $\tilde{g}$, and $\tilde{g}^{ij} = (\tilde{g}^{-1})_{ij}$, we obtain

$$-\frac{1}{2m}\Delta_{\tilde{g}}\Phi + V_\lambda(\tau)\Phi = \mathrm{i}\frac{\partial\Phi}{\partial t}\,. \tag{4.12}$$

Due to the structure of the metric tensor $\tilde{g}$ and setting $\Phi(u, v, \tau, t) = \Psi(u, v, t).\chi(\tau, t)$ the Schrödinger equation breaks into two parts: the surface part

$$-\frac{1}{2m}\Delta_S\Psi + U_S\Psi = \mathrm{i}\frac{\partial\Psi}{\partial t}\,, \tag{4.13}$$

and the normal part

$$-\frac{1}{2m}\frac{\partial^2\chi}{\partial\tau^2} + V_\lambda(\tau)\chi = \mathrm{i}\frac{\partial\chi}{\partial t} \tag{4.14}$$

Equation (4.14) is just the one-dimensional Schrödinger equation for a particle bounded by the transversal potential $V_\lambda(\tau)$, and can be ignored in all future calculations.

Equation (4.13) is however much more interesting due to the presence of the surface potential $U_S$ which takes into account the actual embedding of $\mathcal{S}$. Using the mean $\bar{H}$ and Gaussian curvature $K$ or the principal curvatures $\mathbf{k}_1, \mathbf{k}_2$ of the surface $\mathcal{S}$, this additional term can be expressed as follows:

$$U_S = -\frac{1}{2m}(\bar{H}^2 - K) = -\frac{1}{8m}(\mathbf{k}_1 - \mathbf{k}_2)^2\,. \tag{4.15}$$

It should be noted also that the only two-dimensional surface for which this potential vanishes is the sphere, since in this case the two principal curvatures are equal. For other surfaces however the above scheme results in a heavy mathematical problem. In the next two sections we will combine the techniques presented in the previous ones to some surfaces which are related to fullerenes since from the mathematical point of view they can be considered to represent a family of closed curved two-dimensional manifolds.

## 4.3. Quantization of the Prolate Ellipsoid

The shape of $C_{60}$ as well of certain multiple-shell fullerenes is classified as rather spherical, whereas the configurational space of $C_{70}$ has the form of prolate rotational ellipsoid. In Cartesian coordinates this surface is described implicitly by

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1\,, \tag{4.16}$$

where $c = a\sqrt{1+\mu^2} > a > 0$ for some fixed $\mu \in \mathbb{R}^+$ and can be parameterized as follows:

$$\mathbf{x}[u, v] = \left(\frac{a\sin u\cos v}{\sqrt{1+\mu^2\cos^2 u}}, \frac{a\sin u\sin v}{\sqrt{1+\mu^2\cos^2 u}}, \frac{a(1+\mu^2)\cos u}{\sqrt{1+\mu^2\cos^2 u}}\right)\,,$$

$$u \in [0, \pi], \quad v \in [0, 2\pi]\,.$$

The corresponding induced Riemannian metric is

$$\mathrm{d}s^2 = \frac{a^2(1+\mu^2)^2}{(1+\mu^2\cos^2 u)^3}\,\mathrm{d}u^2 + \frac{a^2\sin^2 u}{1+\mu^2\cos^2 u}\,\mathrm{d}v^2 \tag{4.17}$$

and the surface area element is

$$\mathrm{d}A = \frac{a^2(1+\mu^2)\sin u}{(1+\mu^2\cos^2 u)^2}\,\mathrm{d}u \wedge \mathrm{d}v\,. \tag{4.18}$$

On any two-dimensional manifold the symplectic form $\omega$ coincides up to a multiplicative factor with the respective surface element $\mathrm{d}A$. In conjunction with (3.1) this means that the integration of $\frac{\mathrm{d}A}{2\pi}$ over $\mathcal{S}$ should produce integers. Accordingly, in our case we will have

$$\frac{\text{Area}}{2\pi} = \frac{A(\mathcal{S})}{2\pi} = \left[1 + \left(\mu + \frac{1}{\mu}\right)\arctan\mu\right] a^2 = N \in \mathbb{Z}^+ \tag{4.19}$$

which means that the axes of the ellipsoid are quantized!

The Laplacian and the Eulerian difference $\bar{H}^2 - K$ which enters into expressions for $V_S$ are easily found as well so that our quantization procedure leads to well posed analytical problem on the chosen coordinate patch. Unfortunately, it turns out that the resulting differential equation is of a formidable complexity for analytical treatment and this prevents us from the possibility to find in a closed form neither the wave functions nor the spectrum of the problem in question.

Fortunately we can go back to the sphere using an old observation by Neumann [44] that the geodesic flow on the ellipsoid is equivalent with the motion of a particle on the sphere under the influence of the quadratic potential specified by the axes of the ellipsoid.

In our case this equivalence amounts to work with the potential

$$V(x, y, z) = a^2(x^2 + y^2) + c^2 z^2\,, \tag{4.20}$$

and in this way we end with a symmetric harmonic oscillator system constrained on $S^2$. Introducing standard coordinatization of the sphere

$$x = \sin\theta\cos\phi\,, \qquad y = \sin\theta\sin\phi, \qquad z = \cos\theta \tag{4.21}$$

the Laplacian and the potential take the forms

$$\Delta_{S^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\,, \tag{4.22}$$

and

$$V(\phi,\theta) = a^2 \sin^2\theta + c^2\cos^2\theta\,, \tag{4.23}$$

respectively.

After separation of the variables in the quantum-mechanical time-independent Schrödinger equation

$$\hat{H}\Psi = \left[-\frac{1}{2m}\Delta_{S^2} + V\right]\Psi = E\Psi \tag{4.24}$$

by introducing

$$\Psi(\theta,\phi) = \tilde{Y}(\theta)\,\mathrm{e}^{-\mathrm{i}k\phi}, k \in \mathbb{Z} \tag{4.25}$$

and $\cos\theta = \zeta$ we end up with Sturm–Liouville type problem

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[(1-\zeta^2)\frac{\mathrm{d}Y(\zeta)}{\mathrm{d}\zeta}\right] + \left[\lambda - \varepsilon^2\zeta^2 - \frac{k^2}{1-\zeta^2}\right]Y(\zeta) = 0\,. \tag{4.26}$$

Here

$$\lambda = 2m(E-a^2)\,, \quad \text{and} \quad \varepsilon^2 = 2m(c^2-a^2) = 2m\mu^2a^2 > 0\,. \tag{4.27}$$

One can easily recognize in (4.26) the defining equation for the prolate angular spheroidal functions $S_{kl}(\varepsilon,\zeta)$, $l \geq k$, corresponding to the eigenvalues

$$\lambda_{kl} = l(l+1) + \sum_{\sigma=1}^{\infty} b_{2\sigma}\varepsilon^{2\sigma}\,, \tag{4.28}$$

which can be evaluated with any desired precision using various type of the existing formulae for the coefficients $b_{2\sigma}$, e. g.

$$b_2 = \frac{1}{2}\left[1 - \frac{(2k-1)(2k+1)}{(2l-1)(2l+1)}\right],$$

$$b_4 = \frac{(l-k-1)(l-k)(l+k-1)(l+k)}{2(2l-3)(2l-1)^3(2l+1)} - \frac{(l-k+1)(l-k+2)(l+k+1)(l+k+2)}{2(2l+1)(2l+3)^3(2l+5)}\,,$$

and so on. For more details see Abramowitz and Stegun [2].

What is more interesting here is that the above formula for $\lambda_{kl}$ combined with (4.27) produces the energy spectrum of the geodesic flow on the prolate symmetric ellipsoid as given below

$$E_{kl} = a^2 + \frac{l(l+1)}{2m} + \frac{1}{2m}\sum_{\sigma=1}^{\infty} b_{2\sigma}\varepsilon^{2\sigma}\,. \tag{4.29}$$

Let us remember however that the axes of our ellipsoid in accordance with (4.19) are discretized and the above formula should be written as

$$E_{Nkl} = a_N^2 + \frac{l(l+1)}{2m} + \frac{1}{2m}\sum_{\sigma=1}^{\infty} b_{2\sigma}\varepsilon^{2\sigma}\,. \tag{4.30}$$

Having the spectrum we have to comment the wave functions as well. Actually their properties and other spectral results follow directly from the general Sturm-Liouville theory. The wave functions $S_{kl}(\varepsilon,\zeta)$, with fixed $k$ form a complete orthogonal system in $\mathcal{L}^2(-1,1)$. Besides, any of these functions has $l-k$ zeros in the interval $(-1,1)$ and the energy levels $E_{Nkl}$ obviously increase when the indices $l$ and $N$ increase.

Finally, the "principal" quantum number $N$ enters implicitly via the definition of the first argument $\varepsilon$ given in (4.27).

### 4.4. Quantization of the Oblate Ellipsoid

For definiteness we will denote this surface by $\tilde{\mathcal{S}}$. As most of the considerations in this case are parallel to that ones in the previous section we will indicate only the differences. First, the parametrization where this time

$$a > c = a\sqrt{1-\nu^2} \quad \text{for some fixed} \quad \nu \in (0,1)$$

is given by

$$\tilde{\mathbf{x}}[u,v] = \left(\frac{a\sin u\cos v}{\sqrt{1-\nu^2\cos^2 u}}, \frac{a\sin u\sin v}{\sqrt{1-\nu^2\cos^2 u}}, \frac{a(1-\nu^2)\cos u}{\sqrt{1-\nu^2\cos^2 u}}\right)$$

and the induced Riemannian metric is respectively

$$\mathrm{d}\tilde{s}^2 = \frac{a^2(1-\nu^2)^2}{(1-\nu^2\cos^2 u)^3}\,\mathrm{d}u^2 + \frac{a^2\sin^2 u}{1-\nu^2\cos^2 u}\,\mathrm{d}v^2\,. \tag{4.31}$$

Correspondingly the surface area element is

$$\mathrm{d}\tilde{A} = \frac{a^2(1-\nu^2)\sin u}{(1-\nu^2\cos^2 u)^2}\,\mathrm{d}u\wedge\mathrm{d}v \tag{4.32}$$

and the quantization condition is

$$\frac{\text{Area}}{2\pi} = \frac{A(\tilde{\mathcal{S}})}{2\pi} = \left[1 - \left(\nu - \frac{1}{\nu}\right)\operatorname{arctanh}\nu\right]a^2 = \tilde{N} \in \mathbb{Z}^+\,. \tag{4.33}$$

The Sturm-Liouville equation is

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[(1-\zeta^2)\frac{\mathrm{d}Y(\zeta)}{\mathrm{d}\zeta}\right]+\left[\tilde{\lambda}+\varepsilon^2\zeta^2-\frac{k^2}{1-\zeta^2}\right]Y(\zeta)=0\,. \tag{4.34}$$

with eigenvalues

$$\tilde{\lambda}_{kl}=l(l+1)+\sum_{\sigma=1}^{\infty}(-1)^{\sigma}b_{2\sigma}\varepsilon^{2\sigma}\,. \tag{4.35}$$

Finally, the energy levels of the free particle motion on this surface is

$$E_{\tilde{N}kl}=a_{\tilde{N}}^2+\frac{l(l+1)}{2m}+\frac{1}{2m}\sum_{\sigma=1}^{\infty}(-1)^{\sigma}b_{2\sigma}\varepsilon^{2\sigma}\,. \tag{4.36}$$

## 5. Concluding Remarks

The geodesic flows on the axisymmetric prolate and oblate ellipsoids are quantized using a combination of methods from geometric quantization and constrained quantum mechanics. While geometric quantization scheme has found many concrete applications there were not such up to now of the constrained quantum mechanics. The reason is quite simple — the extra correction term resulting of surface embedding leads to a heavy analytical problem and this prevents the possibility of obtaining analytical results. One has to notice also that for two isometric surfaces (i. e. with the same induced metrics) these correction terms will depend on their second fundamental forms as well. This is in great contrast with the situation in the classical mechanics where the surface motion depends only on the metric properties of the surface. At the same time this hints also to make a search for surfaces for which the Eulerian difference is a simple one as much as possible. Potential candidates are at first place within the class of the so called Weingarten surfaces, i. e. those with a functional dependence among their principal curvatures. The sphere and the axisymmetric ellipsoids are just in this class — for the sphere one has $\mathbf{k}_1=\mathbf{k}_2$ and in the case of the rotational ellipsoids $\mathbf{k}_1\sim\mathbf{k}_2^3$. Quite recently a new surface, the so-called mylar balloon [37] with a remarkably simple relationship $\mathbf{k}_1=2\mathbf{k}_2$, has been found and its quantization will be discussed elsewhere. Other well studied classes in the classical differential geometry are that of surfaces with constant curvatures — both mean and Gaussian — present also a challenge and deserve profound study as well.

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