Second International Conference on
Geometry, Integrability and Quantization
June 7–15, 2000, Varna, Bulgaria
Ivaïlo M. Mladenov and Gregory L. Naber, Editors
Coral Press, Sofia 2001

MAXWELL–BLOCH EQUATIONS WITH A QUADRATIC CONTROL ABOUT Ox_1 AXIS

MIRCEA PUTA[†], MARIUS BUTUR[‡], GHEORGHE GOLDENTHAL[#], IONEL MOŞ[⋆] and CIPRIAN RUJESCU[*]

[†] *European University Drâgan, 1800 Lugoj, Romania, and West University of Timişoara, 1900 Timişoara, Romania*

[‡] *U.M.F. Timişoara, Facultatea de Farmacie P-ţa E. Murgu Nr. 2, 1900 Timişoara, Romania*

[#] *Rudolf–Koch–Schule, Gymnasium der Stadt, Offenbach, M. Schloß Str. 50, Germany*

[⋆] *West University of Timişoara, 1900 Timişoara, Romania*

[*] *Universitatea de Ştiinţe Agricole şi Medicinâ Veterinarâ a Banatului Timişoara, Facultatea de Manegement Agricol, 1900 Timişoara, Romania*

Abstract. The Maxwell–Bloch equations with one quadratic control about Ox_1 axis are introduced and some of their dynamical and geometrical properties are pointed out.

1. Introduction

The Maxwell–Bloch equations with one control about Ox_1 axis can be written in the following form:

$$\begin{cases} \dot{x}_1 = x_2 + u_1 \\ \dot{x}_2 = x_1 x_3 \\ \dot{x}_3 = -x_1 x_2 \, . \end{cases} \tag{1.1}$$

In all that follows we shall employ the quadratic feedback:

$$u_1 = -k x_2 x_3 \tag{1.2}$$

where $k \in \mathbb{R}$ is the feedback gain parameter. We shall refer to the system (1.1), (1.2) as the controlled system.

The goal of our paper is to point out some geometrical and dynamical properties of this system.

2. Controlled System and Poisson Geometry

In this section we shall point out some properties of the controlled system (1.1), (1.2) from the Poisson geometry point of view.

Theorem 2.1. *The controlled system (1.1), (1.2) has Hamilton–Poisson realization.*

Proof: Indeed, let us consider on $\mathbb{R}^3$ the Poisson structure given by the matrix:

$$\Pi_- = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & 0 \\ -x_2 & 0 & 0 \end{bmatrix} \tag{2.1}$$

and the Hamiltonian H given by:

$$H(x_1,x_2,x_3) = \frac{1}{2}x_1^2 + \frac{k}{2}x_2^2 + x_3\,. \tag{2.2}$$

Then we have succesively:

$$\begin{aligned} \Pi_- \cdot \nabla H &= \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & 0 \\ -x_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} -kx_2x_3 + x_2 \\ x_1x_3 \\ -x_1x_2 \end{bmatrix} \\ &= \begin{bmatrix} x_2 - kx_2x_3 \\ x_1x_3 \\ -x_1x_2 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}, \end{aligned}$$

as desired. □

Remark 2.1. *The Poisson structure (2.1) is in fact a minus-Lie–Poisson structure. Indeed, let*

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \qquad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \qquad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

be the canonical basis of $\mathbb{R}^3$. *If we define now the bracket* $[\cdot,\cdot]$ *on* $\mathbb{R}^3$ *by:*

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	*0*	e_3	$-e_2$
e_2	$-e_3$	*0*	*0*
e_3	e_2	*0*	*0*

then $(\mathbb{R}^3,[\cdot,\cdot])$ *is a Lie algebra denoted by* $\mathbb{R}^3_{[\cdot,\cdot]}$. *It is in fact a Lie algebra of* *type VII in Bianchi classification and it is isomorphic to the Lie algebra of the Lie group* $E(2,\mathbb{R})$. *Moreover, an easy computation shows us that* Π_- *is in fact the minus-Lie– Poisson structure on* $(e(2,\mathbb{R}))^* \simeq (\mathbb{R}^3_{[\cdot,\cdot]})^*$.

Remark 2.2. *It is not hard to see that the function* C *given by:*

$$C(x_1,x_2,x_3) = \frac{1}{2}(x_2^2 + x_3^2) \tag{2.3}$$

is a Casimir of our configuration $(\mathbb{R}^3, \Pi_-)$.

Remark 2.3. *Since* H *and* C *given respectively by (2.2) and (2.3) are constants of motion the phase curves of our controlled system are intersections of the surfaces:*

$$H = constant$$

and

$$C = constant.$$

Theorem 2.2. *The controlled system (1.1), (1.2) may be realized as an Hamilton–Poisson system in an infinite number of different ways.*

Proof: It is not hard to see that the triples $(\mathbb{R}^3, \{\cdot,\cdot\}_{ab}, H_{cd})$, where: $\{f,g\}_{ab} = -\nabla C_{ab}\cdot(\nabla f \times \nabla g)$, $\forall f,g \in C^\infty(\mathbb{R}^3,\mathbb{R})$; $C_{ab} = aC + bH$; $H_{cd} = cC + dH$; $a,b,c,d \in \mathbb{R}$, $ad - bc = 1$, are Hamilton– Poisson realizations of the controlled system (1.1), (1.2). □

Remark 2.4. *The above theorem telles us in fact that the traijectories of motion are unchanged if we replace* H *and* C *by* $SL(2,\mathbb{R})$ *combinations of* H *and* C.

Theorem 2.3. *The dynamics (1.1), (1.2) is equivalent to the dynamics of a perturbed pendulum.*

Proof: It is clear that

$$2C = x_2^2 + x_3^2$$

is a constant of motion. If we now take:

$$\begin{cases} x_2 = \sqrt{2C}\cos\theta \\ x_3 = \sqrt{2C}\sin\theta, \end{cases}$$

then we have:

$$\dot{x}_2 = -\dot{\theta}\sqrt{2C}\sin\theta = -\dot{\theta}x_3\,.$$

Hence:

$$\dot{\theta} = -\frac{x_1x_3}{x_3} = -x_1\,.$$

Differentiating again we obtain:

$$\ddot{\theta} = -\sqrt{2C}\cos\theta + Ck\sin 2\theta\,,$$

as required. □

Remark 2.5. *In the particular case $k = 0$, we refined some results from [1].*

3. Stability Problem

It is easy to see that equilibrium states of our system are:

$$\begin{aligned}
&(M, 0, \ 0\ ), \quad M \in \mathbb{R};\\
&\left(0, M, \frac{1}{k}\right), \quad M \in \mathbb{R},\ k \in \mathbb{R},\ k \neq 0;\\
&(\ 0,\ 0, M), \quad M \in \mathbb{R}.
\end{aligned}$$

We can now prove:

Theorem 3.1. *The equilibrium states $(M, 0, 0)$, $M \in \mathbb{R}$ are nonlinear stable.*

Proof: We shall make the proof using the energy-Casimir method, see for details [2] or [4]. Let us consider the energy-Casimir function:

$$H_\varphi(x_1, x_2, x_3) = \frac{1}{2}(x_2^2 + x_3^2) + \varphi\left(\frac{1}{2}x_1^2 + \frac{k}{2}x_2^2 + x_3\right)\,,$$

where $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$. Then we have:

i) $DH_\varphi = x_2\delta x_2 + x_3\delta x_3 + \dot{\varphi}(x_1\delta x_1 + kx_2\delta x_2 + \delta x_3)$.
Hence

$$DH_\varphi(M, 0, 0) = 0\,,$$

if and only if:

$$\dot{\varphi}\left(\frac{1}{2}M^2\right) = 0\,.$$

ii) $D^2H_\varphi = (\delta x_2)^2 + (\delta x_3)^2 + \ddot{\varphi}(x_1\delta x_1 + kx_2\delta x_2 + \delta x_3)^2 + \dot{\varphi}((\delta x_1)^2 + k(\delta x_2)^2)$.
Hence:

$$D^2H_\varphi(M, 0, 0) = (\delta x_2)^2 + (\delta x_3)^2 + \ddot{\varphi}\left(\frac{1}{2}M^2\right)(M\delta x_1 + \delta x_3)^2\,.$$

If we choose now the function φ such that:

$$\dot{\varphi}\left(\frac{1}{2}M^2\right) = 0$$

and

$$\ddot{\varphi}\left(\frac{1}{2}M^2\right) > 0\,,$$

then the second variation at the equilibrium of interest is positive definite and so the equilibrium states:

$$(M, 0, 0), \quad M \in \mathbb{R}$$

are nonlinear stable as required. Such a function φ is given for instance by:

$$\varphi(x) = \frac{1}{2}\left(x - \frac{1}{2}M^2\right)^2.$$

□

Similar arguments lead us to:

Theorem 3.2. *The equilibrium states* $\left(0, M, \frac{1}{k}\right)$, $M \in \mathbb{R}$, $k \in \mathbb{R}$, $k \neq 0$ *are:*

i) *unstable if* $k > 0$*;*
ii) *nonlinear stable if* $k < 0$.

Theorem 3.3. *The equilibrium states* $(0, 0, M)$, $M \in \mathbb{R}$ *are:*

i) *unstable if* $M(1 - kM) > 0$*;*
ii) *nonlinear stable if* $M < 0$ *and* $1 - kM > 0$.

4. Integrability Via Elliptic Functions and Numerical Integration

Let us start with our controlled system (1.1), (1.2). Then we have:

Theorem 4.1. *The equations* (1.1), (1.2) *may be integrated via elliptic functions.*

Proof: We have:

$$2H - 2x_3 = x_1^2 + kx_2^2$$

and

$$2C - x_3^2 = x_2^2\,.$$

Therefore:

$$x_1^2 = 2H - 2Ck + kx_3^2 - 2x_3\,,$$

which leads us finally to:

$$(\dot{x}_3^2) = (2H - 2Ck + kx_3^2 - 2x_3)(2C - x_3^2)\,,$$

and this may be integrated via elliptic functions, see for details [3]. □

Let us observe now that the Hamiltonian vector field X_H splits as follows:

$$X_H = X_{H_1} + X_{H_2} + X_{H_3}\,,$$

where

$$\begin{aligned}
H_1(x_1,x_2,x_3) &= \frac{1}{2}x_1^2\,;\\
H_2(x_1,x_2,x_3) &= \frac{k}{2}x_2^3\,;\\
H_3(x_1,x_2,x_3) &= x_3\,.
\end{aligned}$$

Their flows are respectively given by:

$$\begin{bmatrix} x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0\\ 0 & \cos x_1(0)t & \sin x_1(0)t\\ 0 & -\sin x_1(0)t & \cos x_1(0)t\end{bmatrix}\begin{bmatrix} x_1(0)\\ x_2(0)\\ x_3(0)\end{bmatrix}\;;$$

$$\begin{bmatrix} x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} = \begin{bmatrix} 1 & -kx_3(0)t & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} x_1(0)\\ x_2(0)\\ x_3(0)\end{bmatrix}\;;$$

$$\begin{bmatrix} x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} = \begin{bmatrix} 1 & t & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} x_1(0)\\ x_2(0)\\ x_3(0)\end{bmatrix}\;.$$

Then the Lie–Trotter integrator, [6] can be written in the following form:

$$\begin{cases} x_1^{n+1} = x_1^n + t[1-kx_3(0)]x_2^n\\ x_2^{n+1} = x_2^n\cos x_1(0)t + x_3^n\sin x_1(0)t\\ x_3^{n+1} = -x_2^n\sin x_1(0)t + x_3^n\cos x_1(0)t\,. \end{cases} \tag{4.1}$$

We can now prove:

Theorem 4.2. *The first order integrator (4.1) has the following properties:*

i) *It is a Poisson integrator;*

ii) *Its restriction to the coadjoint orbits $(\mathcal{O}_k,\omega_k)$, where:*

$$\mathcal{O}_k = \{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_2^2+x_3^2 = k\}\,,$$

and $$\omega_k = \frac{1}{k}(x_3\,\mathrm{d}x_1\wedge\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1\wedge\,\mathrm{d}x_3)\,,$$

gives rise to a symplectic integrator;

iii) *It doesn't preserve the Hamiltonian H given by (2.2).*

Proof: The assertions (i) and (ii) are consequence of the fact that X_{H_1}, X_{H_2}, X_{H_3} are Hamiltonian vector fields and so their flows are Poisson ones and their restrictions to the coadjoint orbits are symplectic. The same results can be obtained directly by a long computation or using eventualy MAPLE V.

The last assertion is a consequence of the fact that:

$$\{H_1, H_2\}_- \neq 0$$

and

$$\{H_1, H_3\}_- \neq 0\,.$$

□

Remark 4.1. *In the particular case $k = 0$, we refined the results from [5].*

References

[1] David D. and Holm D., *Multiple Lie–Poisson Structures Reduction and Geometric Phases for the Maxwell–Bloch Traveling Wave Equations* , J. Nonlinear Sciences **2** (1992) 241–262.

[2] Holm D., Marsden J., Raţiu T. and Weinstein A., *Nonlinear Stability of Fluid and Plasma Equilibria*, Phys. Reports **123** (1985) 1–116.

[3] Lawden D., *Elliptic Functions and Applications, Applied Mathematical Sciences*, Vol. 80, Springer Verlag 1989.

[4] Puta M., *Hamiltonian Mechanics and Geometric Quantization*, Math. and Appl., Vol. 260, Kluwer Academic Publishers 1993.

[5] Puta M., *Lie–Trotter Formula and Poisson Dynamics*, Int. J. of Biffurcation and Chaos, **9**(3) (1999) 555–559.

[6] Trotter H., *On the Product of Semigroups of Operators*, Proc. Amer. Math. Soc. **10** (1959) 545–551.