

THE WITTEN CONJECTURE

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Abstract. Low-dimensional topology has experienced a number of revolutionary upheavals in the past twenty years. For many of these the seeds of the revolution were sown in theoretical physics and, more particularly, in the work of Edward Witten. The most recent such event occurred in 1994 when Witten suggested that the topological information about smooth 4-manifolds contained in the Donaldson invariants should also be contained in the much simpler and now famous Seiberg–Witten invariants. This lecture will provide an informal survey of some of the background behind the conjecture and how it came to be made.

1. Donaldson Theory

The first application of gauge-theoretic techniques to the study of smooth 4-manifolds was made by Donaldson [3] who proved that if M is a compact, simply connected, oriented, smooth 4-manifold and the intersection form $q_M: H_2(M, \mathbb{Z}) \times H_2(M, \mathbb{Z}) \rightarrow \mathbb{Z}$ is definite, then, in fact, q_M is diagonalizable over \mathbb{Z} . It follows that there exists a basis for $H_2(M, \mathbb{Z})$ over \mathbb{Z} relative to which the matrix of q_M is $\pm \text{Id}_{b_2(M)}$. Roughly, the proof goes something like this (in the negative definite case):

Consider the principal $SU(2)$ -bundle $SU(2) \hookrightarrow P_1 \rightarrow M$ over M with second Chern class $c_2(P_1) = 1$. Choose a Riemannian metric g on M . Taubes [11] has shown that the bundle admits connections that are **anti-self-dual** (ASD) relative to the Hodge star operator determined by g and the given orientation of M (such connections are called instantons). Two such connections are said to be gauge equivalent if they differ by an automorphism of P_1 and the collection $\mathcal{M}_1(g)$ of gauge equivalence classes of such connections is called the moduli

space of ASD connections on P_1 . For a generic choice of g , $\mathcal{M}_1(g)$ has the following properties:

1. If m denotes half the number of homology classes $x \in H_2(M, \mathbb{Z})$ for which $q_M(x, x) = -1$, then there exist m points p_1, \dots, p_m in $\mathcal{M}_1(g)$ such that $\mathcal{M}_1(g) - \{p_1, \dots, p_m\}$ is a smooth, orientable 5-manifold.
2. Each $p_i, i = 1, \dots, m$, has a neighborhood in $\mathcal{M}_1(g)$ that is homeomorphic to a cone over $\mathbb{C}\mathbb{P}^2$ with p_i at the vertex.
3. There is a compact set $K \subseteq \mathcal{M}_1(g)$ such that $\mathcal{M}_1(g) - K$ is a submanifold of $\mathcal{M}_1(g) - \{p_1, \dots, p_m\}$ diffeomorphic to $M \times (0, 1)$.

Now build a new space \mathcal{M} from $\mathcal{M}_1(g)$ by cutting off the (open) top half of each cone and the bottom half of the cylinder $M \times (0, 1)$. Then \mathcal{M} is compact (because K is compact). It is also a manifold with boundary whose boundary consists of the disjoint union of a copy $M \times \{\frac{1}{2}\}$ of M and m copies of $\mathbb{C}\mathbb{P}^2$. Thus, \mathcal{M} gives a cobordism between M and $\sqcup \mathbb{C}\mathbb{P}^2$. Now, the signature of the intersection form of a smooth 4-manifold is a cobordism invariant. This fact, together with the negative definiteness of q_M , the known intersection form of $\sqcup \mathbb{C}\mathbb{P}^2$ and a bit of integer linear algebra then suffice to prove Donaldson's Theorem.

Between 1983 and 1994 the use of moduli spaces of ASD connections to study the topology of smooth 4-manifolds expanded into a vast industry (called Donaldson theory). The product manufactured by this industry was the "Donaldson polynomial invariant" and we will now briefly describe the construction. For this we will assume that $b_2^+(M)$ is odd and greater than 1. For each $k = 1, 2, \dots$ we consider the principal $SU(2)$ -bundle

$$SU(2) \hookrightarrow P_k \longrightarrow M$$

with Chern class k . For a generic Riemannian metric g on M the moduli space $\hat{\mathcal{M}}_k(g)$ of irreducible ASD connections on P_k is a smooth orientable manifold of (formal) dimension

$$8k - 3(1 + b_2^+(M)) .$$

$\hat{\mathcal{M}}_k(g)$ is not compact, but it has an "Uhlenbeck compactification" $\bar{\mathcal{M}}_k(g)$ which, for $k > \frac{3}{4}(1 + b_2^+(M))$ (the "stable range") carries a fundamental homology class $[\bar{\mathcal{M}}_k(g)]$. Donaldson constructs, for each k , a map

$$\bar{\mu}: H_2(M, \mathbb{Z}) \longrightarrow H^2(\bar{\mathcal{M}}_k(g), \mathbb{Z})$$

from the homology of M to the cohomology of $\bar{\mathcal{M}}_k(g)$. Now let

$$d_k = 4k - \frac{3}{2}(1 + b_2^+(M))$$

be half the dimension of the moduli space (an integer because $b_2^+(M)$ is assumed odd) and define

$$\gamma_k(M): \overbrace{H_2(M, \mathbb{Z}) \times \cdots \times H_2(M, \mathbb{Z})}^{d_k} \longrightarrow \mathbb{Z}$$

by

$$\begin{aligned} \gamma_k(M)(x_1, \dots, x_{d_k}) &= \langle \bar{\mu}(x_1) \cup \cdots \cup \bar{\mu}(x_{d_k}), [\bar{\mathcal{M}}_k(g)] \rangle \\ &= \int_{\bar{\mathcal{M}}_k(g)} \bar{\mu}(x_1) \wedge \cdots \wedge \bar{\mu}(x_{d_k}). \end{aligned}$$

This is a symmetric multilinear map which, via polarization, can be identified with a homogeneous polynomial. It is called the k -th (stable) **Donaldson polynomial** of M . One can show that fixing an orientation of $H_+^2(M, \mathbb{R})$ provides an orientation for each $\bar{\mathcal{M}}_k(g)$ and then $\gamma_k(M)$ is an orientation preserving diffeomorphism invariant of M . The proof that $\gamma_k(M)$ does not depend on the choice of the (generic) metric g uses the assumption that $b_2^+(M) > 1$, which ensures that a generic variation of the metric does not introduce any reducible connections.

Remark: There are various devices (e. g., “blow-up formulas”) for extending the definition of the Donaldson invariants outside of the stable range. When this is done the invariants can be collected together into a formal power series on $H_2(M, \mathbb{Z})$ called the “Donaldson series”

$$\mathbf{D}_M(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \gamma_k(M)(x).$$

In 1994, Kronheimer and Mrowka [6] isolated a (large) class of 4-manifolds (said to be of “simple type”) for which the Donaldson series can be described much more simply. They showed that, for manifolds of simple type, there exist finitely many cohomology classes $K_1, \dots, K_S \in H^2(M, \mathbb{Z})$ (“basic classes”) and rational numbers a_1, \dots, a_S (“coefficients”) such that

$$\mathbf{D}_M(x) = e^{1/2q_M(x,x)} \sum_{i=1}^S a_i e^{K_i(x)}.$$

Witten’s conjecture provides an alternative formula (“Witten’s Magical Formula”) for $\mathbf{D}_M(x)$.

2. Topological Quantum Field Theories

The construction of the Donaldson invariants outlined in the previous section is formally analogous to the Feynman path integral approach to the quantization of classical field theories. In fact, Witten [14] has shown that it is a great deal more than simply “analogous”.

In field theory one begins with a “configuration space” of classical fields $\{\xi\}$. For example, in Yang–Mills theory there is just one field (a connection on a principal bundle), but in Yang–Mills–Higgs theory there are two (a connection and a matter field coupled to it that is represented by a section of the adjoint bundle). One also specifies an “action” $S(\xi)$ (just a real-valued function on the fields), e. g., the Yang–Mills action is

$$S(\omega) = \frac{1}{8\pi^2} \int_M \text{Tr}(F_\omega \wedge *F_\omega),$$

where F_ω is the curvature of ω and the $*$ indicates the Hodge dual. The action generally has various symmetries (e. g., gauge invariance) so that the appropriate object of study is the quotient of the configuration space modulo these symmetries. This is called the “moduli space” \mathcal{B} and its elements represent the physical “states”. Real-valued functions

$$\mathcal{O}: \mathcal{B} \longrightarrow \mathbb{R}$$

on the moduli space are called “observables”. In Yang–Mills theory, for example, the symmetry is gauge invariance and the instanton number (Chern class) is an observable. Observables are assigned “expectation values” $\langle \mathcal{O} \rangle$ via Feynman integrals

$$\langle \mathcal{O} \rangle = \int_{\mathcal{B}} e^{-S(\xi)/e^2} \mathcal{O}(\xi) \mathcal{D}\xi,$$

where e is a “coupling constant” and $\mathcal{D}\xi$ signifies a (nonexistent) measure on \mathcal{B} . Such integrals generally have no precise mathematical definition, but physicists formally manipulate them to great effect. The process of assigning these expectation values is known as “quantization” and the result is a **quantum field theory** (QFT). A QFT is called a **topological quantum field theory** (TQFT) if, for some distinguished set of observables, these expectation values are independent of the Riemannian metric g on M .

Witten [14] constructed the first example of such a TQFT with the specific objective of exhibiting the Donaldson polynomial invariants as expectation values of certain observables. We will briefly describe Witten’s TQFT, but in a somewhat more restricted context than one finds in [14]. Thus, we will assume that M is a compact, simply connected, oriented, smooth 4-manifold with

$b_2^+(M) > 1$ and odd and will restrict our attention to the group $G = SU(2)$. We let P denote some principal $SU(2)$ -bundle over M , $\text{ad } P$ the vector bundle associated to P by the adjoint representation of $SU(2)$ on its Lie algebra $\mathfrak{su}(2)$ and $\mathcal{A}(P)$ the space of connections on P . For any $\omega \in \mathcal{A}(P)$ we denote by $F = F_\omega$ the curvature of ω and by $D = D_\omega$ the corresponding covariant derivatives. Now choose a generic Riemannian metric g on M . The fields in the configuration space are as follows:

$$\begin{aligned}\omega &\in \mathcal{A} \\ \chi &\in \Omega_+^2(M, \text{ad } P) \\ \psi &\in \Omega^1(M, \text{ad } P) \\ \eta, \phi &\in \Omega^0(M, \text{ad } P).\end{aligned}$$

Here $\Omega^k(M, \text{ad } P)$ is the space of k -forms on M with values in $\text{ad } P$ and $\Omega_+^2(M, \text{ad } P)$ is the space of self-dual 2-forms with values in $\text{ad } P$. The action $S = S(\omega, \chi, \psi, \eta, \phi)$ is given by

$$\begin{aligned}S = \int_M \sqrt{g} d^4x \text{Tr} &\left(\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - 2\chi_{\alpha\beta} D^\alpha \psi^\beta + \eta D_\alpha \psi^\alpha \right. \\ &\left. + \bar{\phi} [\psi_\alpha, \psi^\alpha] - \bar{\phi} D_\alpha D^\alpha \phi - \frac{1}{2} \phi [\chi_{\alpha\beta}, \chi^{\alpha\beta}] \right).\end{aligned}$$

Witten introduced these fields and this Lagrangian in order to achieve gauge invariance and what he referred to as a ‘‘BRST-like’’ symmetry. An interesting alternative, and purely geometrical derivation of the same action has been obtained by Atiyah and Jeffery [1]. We will denote by \mathcal{B} the corresponding moduli space of equivalence classes of configurations modulo these symmetries. For each $k = 0, 1, 2, 3, 4$, Witten introduces a k -form on M for each ω, ψ, ϕ as follows:

$$\begin{aligned}W_0 &= \frac{1}{2} \text{Tr}(\phi^2) & W_1 &= \text{Tr}(\phi\psi) \\ W_2 &= \text{Tr}\left(\frac{1}{4}\psi \wedge \psi + \phi F_\omega\right) & W_3 &= \text{Tr}(F_\omega \wedge \psi) \\ W_4 &= \frac{1}{2} \text{Tr}(F_\omega \wedge F_\omega)\end{aligned}$$

Now, for each homology k -cycle $x_k \in H_k(M, \mathbb{Z})$, $k = 0, 1, 2, 3, 4$, we define

$$\mathcal{O}^{(k)}(x_k) = \int_{x_k} W_k.$$

These turn out to be metric independent, gauge invariant and ‘‘BRST’’ invariant so, for each x_k , $\mathcal{O}^{(k)}(x_k)$ is defined on the moduli space:

$$\mathcal{O}^{(k)}(x_k): \mathcal{B} \longrightarrow \mathbb{R}.$$

Thus, $\mathcal{O}^{(k)}$ maps $H_k(M, \mathbb{Z})$ to observables.

Remark: For a simply connected 4-manifold, $H_1(M, \mathbb{Z}) = H_3(M, \mathbb{Z}) = 0$ so $\mathcal{O}^{(1)} = \mathcal{O}^{(3)} = 0$. For $k = 0$, $\mathcal{O}^{(0)}$ essentially evaluates W_0 at the point x_0 . For $k = 4$, x_4 is essentially the fundamental class $[M]$ so $\mathcal{O}^{(4)}$ is essentially the Chern class. New information arises only for $k = 2$.

Products $\mathcal{O}^{(k_1)} \cdots \mathcal{O}^{(k_p)}$ map $H_{k_1}(M, \mathbb{Z}) \times \cdots \times H_{k_p}(M, \mathbb{Z})$ to observables and these have expectation values

$$\langle \mathcal{O}^{(k_1)}(x_{k_1}) \cdots \mathcal{O}^{(k_p)}(x_{k_p}) \rangle = \int_{\mathcal{B}} e^{-S/e^2} \mathcal{O}^{(k_1)}(x_{k_1}) \cdots \mathcal{O}^{(k_p)}(x_{k_p}) \mathcal{D}\xi.$$

Witten then associates with each $\mathcal{O}^{(k)}(x_k)$ a closed k -form $\alpha^{(k)}(x_k)$ on the moduli space $\mathcal{M}(g)$ of ASD connections on P and shows that

$$\langle \mathcal{O}^{(k_1)}(x_{k_1}) \cdots \mathcal{O}^{(k_p)}(x_{k_p}) \rangle = \int_{\bar{\mathcal{M}}(g)} \alpha^{(k_1)}(x_{k_1}) \wedge \cdots \wedge \alpha^{(k_p)}(x_{k_p}).$$

Finally, Witten shows that, for $k = 2$, the cohomology class of $\alpha^{(2)}(x)$ can be identified with $\bar{\mu}(x)$ so that these expectation values coincide with the Donaldson invariants.

The essential feature of Witten’s field theory that accounts for the reduction of the functional integral to an integral over the moduli space of instantons is the BRST-like symmetry which implies that the expectation values are independent not only of the Riemannian metric g , but also of the value of the coupling constant e . The calculations leading from the functional integral to the Donaldson invariants are perturbation calculations done in the “weak coupling limit” (small e). This suggests that a calculation in the strong coupling limit should give a alternative (dual) view of these same invariants. Until 1994, however, such calculations were quite intractable and one did not know what this dual version of Donaldson theory might look like. We summarize the pre-1994 situation in the following table.

Duality in Witten’s TQFT

$e \rightarrow 0$	$e \rightarrow \infty$
“Weak coupling”	“Strong coupling”
“Ultraviolet”	“Infrared”
“Magnetic”	“Electric”
Perturbative	Nonperturbative
Computable	Intractable
Donaldson invariants	?

So matters stood until 1994 when Seiberg and Witten [10] learned how to do exact calculations in the infrared regime of Witten's TQFT and filled in the empty box:

Seiberg–Witten invariants

We will now have a brief look at the construction of these new invariants.

3. Seiberg–Witten Invariants

We assume that M is a compact, simply connected, oriented, smooth 4-manifold with $b_2^+(M) > 1$ and odd and will restrict attention to the structure group $G = SU(2)$. The Seiberg–Witten invariants arise from a moduli space of solutions to a certain (mildly) nonlinear system of partial differential equations. To write down these equations we make two choices. Begin by choosing a Riemannian metric g on M . Next we must select what is called a “spin^c-structure” \mathcal{L} for M .

A **spin^c-structure** is a complex analogue of a spin structure, but, unlike spin structures, such things exist on every orientable, smooth 4-manifold (see [8]). Briefly, the definition is as follows: Since M is oriented and has a Riemannian metric there is an oriented, orthonormal frame bundle with structure group $SO(4)$.

$$SO(4) \hookrightarrow Fr(M) \xrightarrow{\pi_{Fr}} M$$

The group $\text{Spin}^c(4)$ is the double cover of $SO(4) \times U(1)$, i. e.,

$$\text{Spin}^c(4) = SU(2) \times SU(2) \times U(1) / \pm 1,$$

where $\mathbb{Z}_2 = \pm 1$ acts on all three factors simultaneously. By “forgetting” various of the factors in $\text{Spin}^c(4)$ we obtain homomorphisms δ , s_+ , s_- and π onto $U(1)$, $U(2)$ and $SO(4)$.

$$\begin{aligned} \delta: \text{Spin}^c(4) &\longrightarrow U(1) / \pm 1 \cong U(1) \\ s_{\pm}: \text{Spin}^c(4) &\longrightarrow SU(2) \times U(1) / \pm 1 \cong U(2) \\ \pi: \text{Spin}^c(4) &\longrightarrow SU(2) \times SU(2) / \pm 1 \cong SO(4) \end{aligned}$$

Now, a spin^c-structure on M is a lift of the frame bundle to a $\text{Spin}^c(4)$ -bundle. More precisely, a spin^c-structure \mathcal{L} on M consists of a principal $\text{Spin}^c(4)$ -bundle

$$\text{Spin}^c(4) \hookrightarrow S^c(M) \xrightarrow{\pi_{S^c}} M$$

over M and a smooth map

$$\prod: S^c(M) \longrightarrow Fr(M)$$

satisfying

$$\pi_{Fr} \circ \prod = \pi_{S^c}$$

and

$$\prod(p \cdot a) = \prod(p) \cdot \pi(a)$$

for all $p \in S^c(M)$ and $a \in \text{Spin}^c(4)$. The homomorphisms δ , s_+ and s_- give rise to natural representations of $\text{Spin}^c(4)$ on \mathbb{C} and \mathbb{C}^2 and thereby vector bundles associated to the $\text{Spin}^c(4)$ -bundle. The associated complex line bundle

$$L = S^c(M) \times_{\delta} \mathbb{C}$$

is called the determinant line bundle of \mathcal{L} , while the complex 2-plane bundles

$$S_{\pm}^c(M) = S^c(M) \times_{s_{\pm}} \mathbb{C}^2$$

are the positive and negative (complex) spinor bundles associated to \mathcal{L} . Finally, we denote by L^0 the principal $U(1)$ -bundle associated to L .

The Seiberg–Witten equations are defined for a pair (A, ψ) , where A is a connection on the $U(1)$ -bundle L^0 and ψ is a positive spinor field, i. e., a section of $S_+^c(M)$. One thinks of ψ as a massless spin $\frac{1}{2}$ particle coupled to the $U(1)$ -gauge field A . The equations read

$$\begin{aligned} (SW\ 1) \quad \mathcal{D}_A \psi &= 0 && \text{Dirac equation} \\ (SW\ 2) \quad \rho^+(F_A^+) &= (\psi \otimes \psi^*)_0 && \text{Curvature equation} \end{aligned}$$

where

$$\mathcal{D}_A : \Gamma(S_+^c(M)) \longrightarrow \Gamma(S_-^c(M))$$

is a (coupled) Dirac operator and ρ^+ is an isomorphism of the complex self-dual 2-forms on M onto the trace-free endomorphisms of $S_+^c(M)$ (ψ^* is the Hermitian conjugate of ψ and $(\psi \otimes \psi^*)_0$ is the trace-free part of the endomorphism $\psi \otimes \psi^*$). Locally, in orthonormal coordinates, if we write $\partial^\alpha = \frac{\partial}{\partial x^\alpha}$, $A = A_\alpha dx^\alpha$, $F_A = \sum_{\alpha < \beta} F_{\alpha\beta} dx^\alpha \wedge dx^\beta = \sum_{\alpha < \beta} (\partial_\alpha A_\beta - \partial_\beta A_\alpha) dx^\alpha \wedge dx^\beta$,

and $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, the equations become

$$\begin{aligned} &\begin{pmatrix} -(\partial_0 + A_0) + i(\partial_1 + A_1) & (\partial_2 + A_2) + i(\partial_3 + A_3) \\ -(\partial_2 + A_2) + i(\partial_3 + A_3) & -(\partial_0 + A_0) - i(\partial_1 + A_1) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\begin{cases} (\partial_0 A_1 - \partial_1 A_0) + (\partial_2 A_3 - \partial_3 A_2) = -\frac{1}{2}i(|\psi_1|^2 - |\psi_2|^2) \\ (\partial_0 A_2 - \partial_2 A_0) + (\partial_3 A_1 - \partial_1 A_3) = -i \text{Im}(\overline{\psi_1} \psi_2) \\ (\partial_0 A_3 - \partial_3 A_0) + (\partial_1 A_2 - \partial_2 A_1) = -i \text{Re}(\overline{\psi_1} \psi_2) . \end{cases} \end{aligned}$$

There is a symmetry group \mathcal{G} for these equations as there is for the ASD equations. One can describe \mathcal{G} either as the group of automorphisms of the

Spin^c(4)-bundle of \mathcal{L} that cover the identity in the frame bundle or, equivalently, as the group $C^\infty(M, S^1)$ of smooth maps from M to the circle S^1 . Thought of in the latter way the action of \mathcal{G} on (A, ψ) is given by

$$\begin{aligned} \mu \cdot (A, \psi) &= (\mu^{-1}A\mu + \mu^{-1}d\mu, \mu^{-1}\psi) \\ &= (A + \mu^{-1}d\mu, \mu^{-1}\psi) \end{aligned}$$

and one finds that (A, ψ) satisfies (SW) if and only if $\mu \cdot (A, \psi)$ satisfies (SW). The orbit space of this action on the space of solutions is the moduli space $\mathcal{M}_{\mathcal{L}}$ of solutions (A, ψ) to the Seiberg–Witten equations. Generically (this now means for a generic perturbation of the curvature equation) this moduli space $\mathcal{M}_{\mathcal{L}}$ is a smooth, oriented, compact manifold of (formal) dimension

$$d_{\mathcal{L}} = -\frac{1}{4} (2\chi(M) + 3\sigma(M)) + \frac{1}{4} c_1^2(L^0)$$

where $\chi(M)$ is the Euler characteristic of M , $\sigma(M)$ is the signature of (the intersection form of) M and $c_1^2(L^0) = \langle c_1(L^0) \cup c_1(L^0), [M] \rangle$.

The relative simplicity of Seiberg–Witten theory over Donaldson theory is due to the compactness of the moduli space $\mathcal{M}_{\mathcal{L}}$. One no longer requires a compactification, or conditions to ensure that the compactification admits a fundamental homology class with which to pair cohomology classes, or “devices” (e. g., blow-up formulas) for extending the definitions beyond such restrictions. Indeed, the Seiberg–Witten invariant $SW(M, \mathcal{L})$ associated with M and \mathcal{L} can now be defined as follows:

1. If $d_{\mathcal{L}} < 0$ one takes $SW(M, \mathcal{L}) = 0$ (the moduli space $\mathcal{M}_{\mathcal{L}}$ is generically empty in this case).
2. If $d_{\mathcal{L}} = 0$, then $\mathcal{M}_{\mathcal{L}}$ is a finite set of points, each with a sign coming from the orientation of $\mathcal{M}_{\mathcal{L}}$ and $SW(M, \mathcal{L})$ is the corresponding signed sum:

$$SW(M, \mathcal{L}) = \sum_{\mathcal{M}_{\mathcal{L}}} \pm 1.$$

3. If $d_{\mathcal{L}} > 0$, then

$$SW(M, \mathcal{L}) = \left\langle \overline{\mu \cup \cdots \cup \mu}^{d_{\mathcal{L}}}, [\mathcal{M}_{\mathcal{L}}] \right\rangle = \int_{\mathcal{M}_{\mathcal{L}}} \mu \wedge \cdots \wedge \mu,$$

where μ is the 1st Chern class of the principal $U(1)$ -bundle

$$U(1) \hookrightarrow \hat{\mathcal{M}}_{\mathcal{L}}^0 \longrightarrow \mathcal{M}_{\mathcal{L}},$$

where $\hat{\mathcal{M}}_{\mathcal{L}}^0 = \{(A, \psi) : \psi \neq 0 \text{ and } (A, \psi) \text{ satisfies (SW)}\} / \mathcal{G}^0$ and \mathcal{G}^0 is the subgroup of \mathcal{G} consisting of those elements that act trivially on some fixed fiber of the Spin^c(4)-bundle.

One can show that, for a fixed, generic metric on M there are only a finite number of spin^c -structures \mathcal{L} for which $SW(M, \mathcal{L}) \neq 0$.

4. The Conjecture

Roughly, Witten conjectured that, for manifolds of “simple type”, the Seiberg–Witten invariants contain all of the topological information in the Donaldson invariants.

Remark: We will not record the precise definition of “simple type” (see [6]), but will only note that there are no known counterexamples to the conjecture that every manifold of the type we are considering is of simple type.

Somewhat more precisely, Kronheimer and Mrowka [6] proved that, if M is of simple type, then the Donaldson invariants of M are uniquely determined by a finite number of integral cohomology classes $K_1, \dots, K_S \in H^2(M, \mathbb{Z})$ (“basic classes”) and rational numbers a_1, \dots, a_S (“coefficients”). Witten conjectured that $SW(M, \mathcal{L}) \neq 0$ if and only if $c_1(L^0) = K \in H^2(M, \mathbb{Z})$ is a Kronheimer–Mrowka basic class and, in this case, $SW(M, \mathcal{L})$ is universally proportional to the corresponding coefficient a .

Yet more precisely, Witten’s conjecture can be summarized in what has been called “Witten’s Magical Formula” for the Donaldson series $\mathbf{D}_M(x)$. Recall (Section 1) that the Kronheimer–Mrowka formula for $\mathbf{D}_M(x)$ (proved in [6]) is

$$\mathbf{D}_M(x) = e^{1/2q_M(x,x)} \sum_{i=1}^S a_i e^{K_i(x)}.$$

Witten’s proposed alternative is

$$\mathbf{D}_M(x) = e^{1/2q_M(x,x)} \sum_{\mathcal{L} \in \Lambda} 2^{m(M)} SW(M, \mathcal{L}) e^{c_1(L^0)(x)}$$

where $m(M) = 2 + \frac{1}{4}(7\chi(M) + 11\sigma(M))$ and Λ is the set of isomorphism classes of spin^c -structures \mathcal{L} for which $c_1(L^0) = 2\chi(M) + 3\sigma(M)$.

There are a number of attitudes one can adopt toward Witten’s conjecture. One can, of course, try to prove it. The conjecture has been verified for all of the known examples in which both sets of invariants have been calculated and much work has been devoted to constructing a rigorous proof of the general result (see [9], [5], and [13]). One could also argue that, whether or not Seiberg–Witten invariants are “equivalent” to the Donaldson invariants, they obviously contain a great deal of topological information that is much more easily accessed than that contained in Donaldson theory and should be used as an alternative tool. To a large extent this is the attitude that has been adopted by topologists and the results have been spectacular (see, for example, [7] and [12]). Another point

of view, however, that has not gotten as much attention is that the conjecture arose, after all, in physics, not mathematics and mathematicians generally have no idea how this came about. If quantum field theory is able to provide such profound insights into the deepest questions facing contemporary mathematics it would seem appropriate that some bridges be (re-)built between mathematics and physics. While some steps have been taken in this direction (see [2]) the task of making theoretical physics accessible to the general mathematical community is a formidable one and much remains to be done.

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