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SOME EXAMPLES RELATED TO THE DELIGNE–SIMPSON PROBLEM*

VLADIMIR P. KOSTOV

Université de Nice, Laboratoire de Mathématiques,
Parc Valrose, 06108 Nice Cedex 2, France

Abstract. We consider the variety of $(p+1)$-tuples of matrices M_j from given conjugacy classes $C_j \subset GL(n, \mathbb{C})$ such that $M_1 \cdots M_{p+1} = I$. This variety is connected with the *Deligne–Simpson problem: give necessary and sufficient conditions on the choice of the conjugacy classes* $C_j \subset GL(n, \mathbb{C})$ *so that there exist irreducible* $(p+1)$*-tuples of matrices* $M_j \in C_j$ *whose product equals* I. The matrices M_j are interpreted as monodromy operators of regular linear systems on Riemann's sphere. We consider among others cases when the dimension of the variety is higher than the expected one due to the presence of $(p+1)$-tuples with non-trivial centralizers.

1. Introduction

1.1. The Deligne–Simpson Problem

In the present paper we consider some examples related to the **Deligne–Simpson Problem** (DSP) which is formulated like this:

Give necessary and sufficient conditions upon the choice of the $p+1$ *conjugacy classes* $c_j \subset gl(n, \mathbb{C})$*, resp.* $C_j \subset GL(n, \mathbb{C})$*, so that there exist irreducible* $(p+1)$*-tuples of matrices* $A_j \in c_j$*,* $A_1 + \cdots + A_{p+1} = 0$*, resp. of matrices* $M_j \in C_j$*,* $M_1 \cdots M_{p+1} = I$*.*

By definition, the **weak DSP** is the DSP in which the requirement of irreducibility is replaced by the weaker requirement the centralizer of the $(p+1)$-tuple of matrices to be trivial.

The matrices A_j, resp. M_j, are interpreted as *matrices-residua* of *Fuchsian* systems on Riemann's sphere (i. e. linear systems of ordinary differential equa-

*To the memory of my mother.

tions with logarithmic poles), resp. as *monodromy operators* of *regular* systems on Riemann's sphere (i. e. linear systems of ordinary differential equations with moderate growth rate of the solutions at the poles). Fuchsian systems are a particular case of regular ones. By definition, the monodromy operators generate the **monodromy group** of a regular system.

In the multiplicative version (i. e. for matrices M_j) the classes C_j are interpreted as **local monodromies** around the poles and the problem admits the interpretation:

For what $(p+1)$-tuples of local monodromies do there exist monodromy groups with such local monodromies.

Remarks:

1) Suppose that A_j denotes a matrix-residuum and that M_j denotes the corresponding monodromy operator of a Fuchsian system. Then in the absence of non-zero integer differences between the eigenvalues of A_j the operator M_j is conjugate to $\exp(2\pi \mathrm{i} A_j)$.

2) In what follows the sum of the matrices A_j is always presumed to be 0 and the product of the matrices M_j is always presumed to be I.

1.2. The Aim of This Paper

For a conjugacy class C in $GL(n,\mathbb{C})$ or $gl(n,\mathbb{C})$ denote by $d(C)$ its dimension (which is always even). Set $d_j := d(c_j)$ (resp. $d(C_j)$).

For fixed conjugacy classes C_j consider the variety

$$\mathcal{V} = \{(M_1, \dots, M_{p+1});\ M_j \in C_j\ ,\ M_1 \cdots M_{p+1} = I\}\,.$$

This variety might contain $(p+1)$-tuples with non-trivial centralizers as well as with trivial ones. It might contain only the former or only the latter.

Proposition 1.1. *At a $(p+1)$-tuple with trivial centralizer the variety $\mathcal{V}$ is smooth and of dimension $d_1 + \cdots + d_{p+1} - n^2 + 1$.*

Remark. *The proposition is proved at the end of the subsection. A similar statement is true for the matrices A_j.*

For generic eigenvalues (the precise definition is given in the next section) the variety $\mathcal{V}$ contains only irreducible $(p+1)$-tuples and its dimension remains the same when the eigenvalues of the conjugacy classes are changed but not the Jordan normal forms which they define. We call its dimension for generic eigenvalues the **expected** one.

The aim of the present paper is to consider some examples of varieties $\mathcal{V}$ for non-generic eigenvalues. In the first and in the fifth of them (see Sections 3 and 8) $\dim \mathcal{V}$ is higher than the expected one. In the first example we discuss the stratified structure of $\mathcal{V}$ and we show that $\mathcal{V}$ consists only of $(p+1)$-tuples with non-trivial centralizers. The latter fact is true for the fifth example as well.

In the second example (see Section 5) the eigenvalues are not generic and the variety $\mathcal{V}$ contains at the same time $(p+1)$-tuples with trivial and ones with non-trivial centralizers. The dimension of $\mathcal{V}$ is the expected one.

In the third example (see Section 6) the variety $\mathcal{V}$ contains no $(p+1)$-tuples with trivial centralizers but its dimension equals the expected one.

In the fourth example (see Section 7) there is coexistence in $\mathcal{V}$ of $(p+1)$-tuples with trivial centralizers and of $(p+1)$-tuples with non-trivial ones. The dimension of $\mathcal{V}$ at the former (i. e. the expected dimension) is lower than the dimension at the latter.

In the first and third examples the closure of $\mathcal{V}$ (topological and algebraic) contains also $(p+1)$-tuples in which some of the matrices M_j belong not to C_j but to their closures, i. e. the eigenvalues are the necessary ones but the Jordan structure is "less generic".

Similar examples exist for matrices A_j as well. Before beginning with the examples we recall some known facts in the next section.

Proof: (Proposition 1.1) It suffices to prove the proposition in the case when $C_j \subset SL(n,\mathbb{C})$. The variety $\mathcal{V}$ is the intersection in $C_1 \times \cdots \times C_p \times SL(n,\mathbb{C})$ of the graph of the mapping

$$C_1 \times \cdots \times C_p \to SL(n,\mathbb{C})\,, \quad (M_1,\ldots,M_p) \mapsto (M_1\cdots M_p)^{-1}$$

and of the variety $\mathcal{C} = C_1 \times \cdots \times C_{p+1}$. To prove that $\mathcal{V}$ is smooth it suffices to prove that the intersection is transversal, i. e. the sum of the tangent spaces to the graph (which is the space $\{\sum_{j=1}^{p}[M_j,X_j], X_j \in sl(n,\mathbb{C})\}$) and the one to $\mathcal{C}$ (it equals $\{[M_{p+1},X_{p+1}], X_{p+1} \in sl(n,\mathbb{C})\}$) is $sl(n,\mathbb{C})$. This follows from

Proposition 1.2. *The $(p+1)$-tuple of matrices $R_j \in gl(n,\mathbb{C})$ is with trivial centralizer if and only if the map $(gl(n,\mathbb{C}))^{p+1} \to sl(n,\mathbb{C})$, $(X_1,\ldots,X_{p+1}) \mapsto \sum_{j=1}^{p+1}[R_j,X_j]$ is surjective.*

The dimension of $\mathcal{V}$ is the one of $C_1 \times \cdots \times C_p$, i. e. $d_1+\cdots+d_p$, diminished by the codimension of $\mathcal{C}$ in $C_1 \times \cdots \times C_p \times SL(n,\mathbb{C})$, i. e. by n^2-1-d_{p+1}. Hence, $\dim \mathcal{V} = d_1+\cdots+d_{p+1}-n^2+1$. □

Proof: (Proposition 1.2) The map is not surjective exactly if the image of every map $X_j \mapsto [R_j,X_j]$ belongs to one and the same linear subspace of $sl(n,\mathbb{C})$, i. e. one has $\mathrm{Tr}(D[R_j,X_j]) = 0$ for some matrix $0 \neq D \in sl(n,\mathbb{C})$ for $j = 1,\ldots,p+1$ and identically in the entries of X_j. One has $\mathrm{Tr}(D[R_j,X_j]) =$

$\mathrm{Tr}([D, R_j]X_j)$ which implies that $[D, R_j] = 0$ for all j — a contradiction with the triviality of the centralizer. $\square$

2. Some Known Facts

We expose here some facts which are given in some more detail in [2]. For a matrix Y from the conjugacy class C in $GL(n, \mathbb{C})$ or $gl(n, \mathbb{C})$ set $r(C) := \min_{\lambda\in\mathbb{C}} \mathrm{rank}(Y - \lambda I)$. The integer $n - r(C)$ is the maximal number of Jordan blocks of $J(Y)$ with one and the same eigenvalue. Set $r_j := r(c_j)$ (resp. $r(C_j)$). The quantities $r(C)$ and $d(C)$ depend only on the Jordan normal form of Y.

Definition 2.1. *A **Jordan normal form** (JNF) of size n is a family $J^n = \{b_{i,l}\}$ ($i \in I_l$, $I_l = \{1, \ldots, s_l\}$, $l \in L$) of positive integers $b_{i,l}$ whose sum is n. The index l is the one of an eigenvalue and the index i is the one of a Jordan block with the l-th eigenvalue; all eigenvalues are presumed distinct. An $n{\times}n$-matrix Y has the JNF J^n (notation $J(Y) = J^n$) if to its distinct eigenvalues λ_l, $l \in L$, there belong Jordan blocks of sizes $b_{i,l}$. We usually assume that for each fixed l the numbers $b_{i,l}$ form a non-increasing sequence.*

Proposition 2.1. (C. Simpson, see [3]) *The following couple of inequalities is a necessary condition for the existence of irreducible $(p+1)$-tuples of matrices M_j:*

$$d_1 + \cdots + d_{p+1} \geq 2n^2 - 2 \qquad \text{for all } j\,, \tag{α_n}$$

$$r_1 + \cdots + \hat{r}_j + \cdots + r_{p+1} \geq n\,. \tag{β_n}$$

Remark. *The conditions are necessary for the existence of irreducible $(p+1)$-tuples of matrices A_j as well.*

We presume that there holds the following evident necessary condition

$$\sum \mathrm{Tr}(c_j) = 0\,, \text{ resp. } \prod \det(C_j) = 1\,.$$

In terms of the eigenvalues $\lambda_{k,j}$ (resp. $\sigma_{k,j}$) of the matrices from c_j (resp. C_j) repeated with their multiplicities, this condition reads

$$\sum_{k=1}^{n}\sum_{j=1}^{p+1} \lambda_{k,j} = 0\,, \text{ resp. } \prod_{k=1}^{n}\prod_{j=1}^{p+1} \sigma_{k,j} = 1\,.$$

An equality of the kind

$$\sum_{j=1}^{p+1}\sum_{k\in\Phi_j} \lambda_{k,j} = 0\,, \text{ resp. } \prod_{j=1}^{p+1}\prod_{k\in\Phi_j} \sigma_{k,j} = 1$$

is called a **non-genericity relation**; the sets Φ_j contain the same number $< n$ of indices for all j. Eigenvalues satisfying none of these relations are called **generic**. Reducible $(p+1)$-tuples exist only for non-generic eigenvalues; indeed, the eigenvalues of each diagonal block of a block upper-triangular $(p+1)$-tuple satisfy some non-genericity relation.

Definition 2.2. *Denote by $\{J_j^n\}$ a $(p+1)$-tuple of JNFs, $j = 1, \ldots, p+1$. We say that the DSP is* ***solvable*** *(resp. that it is* ***weakly solvable*** *or, equivalently, that the weak DSP is solvable) for a given $\{J_j^n\}$ and for given eigenvalues if there exists an irreducible $(p+1)$-tuple (resp. a $(p+1)$-tuple with a trivial centralizer) of matrices M_j or of matrices A_j, with $J(M_j) = J_j^n$ or $J(A_j) = J_j^n$ and with the given eigenvalues. By definition, the DSP is solvable for $n = 1$. Solvability of the DSP implies its weak solvability, i. e. solvability of the weak DSP.*

For a given JNF $J^n = \{b_{i,l}\}$ define its *corresponding* diagonal JNF J'^n. A diagonal JNF is a partition of n defined by the multiplicities of the eigenvalues. For each l $\{b_{i,l}\}$ is a partition of $\sum_{i \in I_l} b_{i,l}$ and J'^n is the disjoint sum of the dual partitions. We say that two JNFs of one and the same size correspond to one another if they correspond to one and the same diagonal JNF.

Proposition 2.2.
1) One has $r(J^n) = r(J'^n)$ and $d(J^n) = d(J'^n)$.
2) To each diagonal JNF there corresponds a unique JNF with a single eigenvalue.

Example. *To the JNF $\{\{4,3,3\},\{3,2\}\}$ of size 15 (two eigenvalues, with respectively three Jordan blocks, of sizes 4, 3, 3 and with two Jordan blocks, of sizes 3, 2) there corresponds the diagonal JNF with multiplicities of the eigenvalues equal to 3, 3, 3, 2, 2, 1, 1. Indeed, the partition of* 10 *dual to 4, 3, 3 is 3, 3, 3, 1; the partition of* 5 *dual to 3, 2 is 2, 2, 1. After this we arrange the multiplicities in decreasing order.*
To the two above JNFs there corresponds the JNF with a single eigenvalue with sizes of the Jordan blocks equal to 7, 5, 3. Indeed, 7, 5, 3 is the partition of 15 dual to 3, 3, 3, 2, 2, 1, 1.

For a given $\{J_j^n\}$ with $n > 1$, which satisfies condition (β_n) and doesn't satisfy condition

$$(r_1 + \cdots + r_{p+1}) \geq 2n \qquad (\omega_n)$$

set $n_1 = r_1 + \cdots + r_{p+1} - n$. Hence, $n_1 < n$ and $n - n_1 \leq n - r_j$. Define the $(p+1)$-tuple $\{J_j^{n_1}\}$ as follows: to obtain the JNF $J_j^{n_1}$ from J_j^n one chooses one of the eigenvalues of J_j^n with greatest number $n - r_j$ of Jordan blocks, then decreases by 1 the sizes of the $n - n_1$ *smallest* Jordan blocks with this eigenvalue and deletes the Jordan blocks of size 0.

Definition 2.3. *The quantity* $\kappa = 2n^2 - \sum_{j=1}^{p+1} d_j$ *defined for a* $(p+1)$*-tuple of conjugacy classes is called the* ***index of rigidity***.

It is introduced by Katz in [1]. For irreducible representations it takes the values $2, 0, -2, -4, \ldots$. Indeed, every conjugacy class is of even dimension and there holds condition (α_n). If for an irreducible $(p+1)$-tuple one has $\kappa = 2$, then the $(p+1)$-tuple is called *rigid*. Such irreducible $(p+1)$-tuples are unique up to conjugacy (see [1] and [3]).

Lemma 2.1. *The index of rigidity is invariant for the construction* $\{J_j^n\} \mapsto \{J_j^{n_1}\}$.

Theorem 2.1. *Let* $n > 1$. *The DSP is solvable for the conjugacy classes* C_j *or* c_j *(with generic eigenvalues, defining the JNFs* J_j^n *and satisfying condition* (β_n)*) if and only if either* $\{J_j^n\}$ *satisfies condition* (ω_n) *or the construction* $\{J_j^n\} \mapsto \{J_j^{n_1}\}$ *iterated as long as it is defined stops at a* $(p+1)$*-tuple* $\{J_j^{n'}\}$ *either with* $n' = 1$ *or satisfying condition* $(\omega_{n'})$.

Remarks:

1) The conditions of the theorem are necessary for the weak solvability of the DSP for any eigenvalues.

2) A posteriori one knows that the theorem does not depend on the choice(s) of eigenvalue(s) made when defining the construction $\{J_j^n\} \mapsto \{J_j^{n_1}\}$.

3. An Example with Index of Rigidity Equal to 2

3.1. Description of the Example

Denote by J^*, J^{**} two quadruples of JNFs J_j of size 4, $j = 1, \ldots, 4$, in both of which J_1, J_2 and J_3 are diagonal, each with two eigenvalues of multiplicity 2; in J^* the JNF J_4 is with a single eigenvalue to which there correspond three Jordan blocks, of sizes 2, 1, 1; in J^{**} the JNF J_4 is diagonal, with two eigenvalues, of multiplicities 3 and 1. The JNFs J_4 from the two quadruples correspond to each other.

Hence, both J^* and J^{**} satisfy the conditions of Theorem 2.1 (to be checked by the reader). They are both with index of rigidity 2. In both cases (of matrices A_j or M_j) the quadruple J^{**} admits generic eigenvalues and, hence, there exist irreducible quadruples of matrices A_j or M_j with such respective JNFs.

Definition 3.1. *Suppose that the greatest common divisor of the multiplicities of all eigenvalues of the matrices M_j or A_j equals $q > 1$. In the case of matrices M_j denote by ξ the product of all eigenvalues with multiplicities decreased q times. Hence, ξ is a root of unity of order q: $\xi = \exp(2\pi\mathrm{i}l/q)$, $l \in \mathbb{N}$. Denote by m the greatest common divisor of l and q. Hence, for $m > 1$ the eigenvalues satisfy the non-genericity relation (called* basic*) their product with multiplicities divided m times to equal 1. In the case of matrices A_j the basic non-genericity relation is the sum of all eigenvalues with multiplicities decreased q times to equal 0. Eigenvalues satisfying only the basic non-genericity relation and its corollaries are called* ***relatively generic****.*

The quadruple J^* does not admit generic but only relatively generic eigenvalues in the case of matrices A_j because one has $q = 2$.

The quadruple J^* admits generic eigenvalues in the case of matrices M_j. Indeed, such is the set of eigenvalues of the four matrices $(\mathrm{e}, \mathrm{e}^{-1})$, $(\sqrt{2}, 1/\sqrt{2})$, $(3, 1/3)$, i. In this case $q = 2$ and the product of all eigenvalues with multiplicities decreased twice equals -1. This is not a non-genericity relation. If the eigenvalue of the fourth matrix is changed from i to -1, then the eigenvalues will not be generic — their product when the multiplicities are decreased twice equals 1. This is the basic non-genericity relation. In this case the eigenvalues are relatively generic but not generic.

In our example we consider conjugacy classes C_j defining the quadruple of JNFs J^*, with relatively generic but not generic eigenvalues. Observe that the expected dimension of $\mathcal{V}$ both in the case of J^* and of J^{**} equals $8 + 8 + 8 + 6 - 15 = 15$.

3.2. The Stratified Structure of the Variety $\mathcal{V}$ from the Example

The variety $\mathcal{V}$ from the example contains at least the following two strata denoted by $\mathcal{U}$ and $\mathcal{W}$. The stratum $\mathcal{U}$ consists of all quadruples defining representations which are direct sums of two irreducible representations, i. e. up to conjugacy one has (for $(M_1, M_2, M_3, M_4) \in \mathcal{U}$)

$$M_j = \begin{pmatrix} N_j & 0 \\ 0 & P_j \end{pmatrix}, \quad N_j, P_j \in GL(2, \mathbb{C}) \tag{1}$$

where the matrices N_j (resp. P_j) are diagonal for $j = 1, 2, 3$. Their quadruples are with generic eigenvalues and for $j = 4$ the eigenvalues equal -1, P_4 is conjugate to a Jordan block of size 2 while N_4 is scalar. The existence of irreducible quadruples of matrices N_j and P_j is guaranteed by Theorem 2.1.

Remark. *The matrices N_j (resp. P_j) define an irreducible rigid representation (resp. an irreducible representation of zero index of rigidity).*

Proposition 3.1.

1) The variety of matrices N_j (resp. P_j) as above is smooth, irreducible and of dimension 3 (resp. 5).

2) The variety of quadruples of diagonalizable matrices $M_j \in GL(2,\mathbb{C})$ each with two distinct eigenvalues (the eigenvalues of the quadruple being generic) is smooth, irreducible and of dimension 5.

All propositions from this subsection are proved in Section 4.

The stratum $\mathcal{W}$ consists of all quadruples defining semi-direct sums of two equivalent rigid representations. Up to conjugacy one has (for $(M_1, M_2, M_3, M_4) \in \mathcal{W}$)

$$M_j = \begin{pmatrix} N_j & R_j \\ 0 & N_j \end{pmatrix}, \quad N_j, R_j \in GL(2,\mathbb{C}) \tag{2}$$

with N_j as above. The blocks R_j are such that for $j = 1, 2, 3$ the matrices M_j are diagonalizable while M_4 has JNF J_4 (i. e. $\operatorname{rank} R_4 = 1$).

The absence of other possible types of representations is guaranteed by the following theorem which follows from Theorem 1.1.2 from [1]. The theorem and its proof were suggested by Ofer Gabber.

Theorem 3.1. *For fixed conjugacy classes with index of rigidity 2 there cannot coexist irreducible and reducible $(p+1)$-tuples of matrices M_j.*

The theorem is proved in the Appendix. It follows from the theorem that there can exist only reducible quadruples of matrices M_j in the example under consideration.

Proposition 3.2. *One has $\mathcal{V} = \mathcal{U} \cup \mathcal{W}$.*

Proposition 3.3.

1) In a quadruple (2) the matrix R_4 is nilpotent of rank 1 and for $j = 1, 2, 3$ one has $R_j = [N_j, Z_j]$ with $Z_j \in sl(2,\mathbb{C})$.

2) If the matrices N_1, N_2, N_3 are fixed, then for every nilpotent rank 1 matrix R_4 there exists a quadruple of matrices (2).

Proposition 3.4. *The centralizers in $SL(4,\mathbb{C})$ of the quadruples (1) and (2) are both of dimension 1. They consist respectively of the matrices*

$$\begin{pmatrix} \alpha I & 0 \\ 0 & \pm\alpha^{-1} I \end{pmatrix} \text{ and } \begin{pmatrix} \delta I & \beta I \\ 0 & \delta I \end{pmatrix}, \quad \alpha \in \mathbb{C}^*, \ \beta \in \mathbb{C}, \ \delta^4 = 1$$

Proposition 3.5. *The stratum $\mathcal{W}$ belongs to the closure of the stratum $\mathcal{U}$.*

Proposition 3.6. *The stratum $\mathcal{W}$ is an irreducible smooth variety of dimension 15.*

Proposition 3.7. *The stratum $\mathcal{U}$ is an irreducible smooth variety of dimension 16.*

Remarks:

The closure of the variety $\mathcal{W}$ (hence, the one of $\mathcal{U}$ as well) contains the variety $\mathcal{Y}$ of quadruples which up to conjugacy are of the form (2) with $R_j = 0$ for all j. For such quadruples
1) the matrix M_4 is scalar;
2) they define direct sums of two equivalent irreducible rigid representations.
There exist no irreducible such quadruples of matrices M_j or A_j because the conditions of Theorem 2.1 are not fulfilled (neither the necessary condition (α_n)).

Proposition 3.8. *The variety $\mathcal{Y}$ is smooth and irreducible. One has* $\dim \mathcal{Y} = 12$.

4. Proofs of the Propositions

Proof of Proposition 3.1:

1^0. The variety of quadruples of matrices N_j is obtained by conjugating one such quadruple by matrices from $SL(2, \mathbb{C})$ (indeed, rigid $(p+1)$-tuples are unique up to conjugacy, see [1] and [3]). This proves the connectedness. The smoothness and the dimension follow from Proposition 1.1.

2^0. Denote by C_j^* the conjugacy class of the matrix P_j. Prove that the variety Π of quadruples of matrices P_j is connected. Denote by δ the product $\det P_1 \det P_2$. By varying the matrices P_1 and P_2 (resp. P_3 and P_4) one can obtain as their product P_1P_2 (resp. as $P_4^{-1}P_3^{-1}$) any matrix from the set $\Delta(\delta)$ of 2×2-matrices with determinant equal to δ. The set $\Delta(\delta)$ being connected so is the variety Π because $\Pi = \{(P_1, P_2, P_3, P_4) | P_j \in C_j^*, P_1P_2 = P_4^{-1}P_3^{-1}\}$.

3^0. The eigenvalues of the matrices P_j being generic, the variety Π contains no reducible quadruples. Hence, the variety Π is smooth, one has $\dim \Pi = 5$, see Proposition 1.1.

4^0. Part 2) is proved by analogy with 2^0 and 3^0. $\square$

Proof of Proposition 3.2:

1^0. A quadruple from $\mathcal{V}$ is block upper-triangular up to conjugacy. The eigenvalues being relatively generic, the diagonal blocks can be only of size 2 and the restrictions of the matrices M_j to them can be with conjugacy classes like in the cases of quadruples of matrices N_j or P_j.

2^0. Show that if one of the diagonal blocks is a quadruple of matrices N_j and the other one of matrices P_j, then this is a direct sum conjugate to a quadruple (1). Indeed, for the representations P and N defined by the quadruples of matrices P_j and N_j one has $\mathrm{Ext}^1(P,N) = \mathrm{Ext}^1(N,P) = 0$ (to be checked directly). This implies that a block upper-triangular quadruple of matrices M_j with diagonal blocks N_j and P_j is conjugate to its restriction to the two diagonal blocks, i. e. the quadruple is a point from $\mathcal{U}$. On the other hand, if both diagonal blocks equal N_j, then the quadruple is like in (2).

Hence, only quadruples like the ones from $\mathcal{U}$ and $\mathcal{W}$ can exist in $\mathcal{V}$. □

Proof of Proposition 3.3:

1^0. The blocks R_1, R_2, R_3 must be of the form $R_j = [N_j, Z_j]$ for some matrices $Z_j \in gl(n,\mathbb{C})$. Indeed, it suffices to prove this under the assumption that N_j is diagonal: $N_j = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, $\lambda \neq \mu$. Set $R_j = \begin{pmatrix} g & h \\ f & s \end{pmatrix}$. One must have $g = s = 0$, otherwise M_j will not be diagonalizable. But then $R_j = [N_j, Z_j]$ with $Z_j = \begin{pmatrix} 0 & h/(\lambda-\mu) \\ f/(\mu-\lambda) & 0 \end{pmatrix}$.

On the other hand, if for $j = 1,2,3$ one has $R_j = [N_j, Z_j]$, then the matrices M_1, M_2, M_3 have the necessary JNFs — one has

$$M_j = \begin{pmatrix} I & Z_j \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} N_j & 0 \\ 0 & N_j \end{pmatrix} \begin{pmatrix} I & Z_j \\ 0 & I \end{pmatrix} .$$

2^0. If one has $\operatorname{rank} R_4 = 0$, then $R_4 = 0$ and M_4 must be scalar, i. e. $M_4 \notin C_4$. If $\operatorname{rank} R_4 = 2$, then $\operatorname{rank}(M_4+I) = 2$ and again $M_4 \notin C_4$. Hence, $\operatorname{rank} R_4 = 1$. This leaves two possibilities — either R_4 has two distinct eigenvalues one of which is 0 or it is nilpotent.

3^0. The condition $M_1 \cdots M_4 = I$ restricted to the right upper block and to each of the diagonal blocks reads respectively

$$R_1N_2N_3N_4 + N_1R_2N_3N_4 + N_1N_2R_3N_4 + N_1N_2N_3R_4 = 0\,,$$
$$N_1N_2N_3 = -I\,.$$

Hence, the first of these two equalities takes the form

$$-R_1 - N_1R_2(N_1)^{-1} - (N_1N_2)R_3(N_1N_2)^{-1} - R_4 = 0\,.$$

As $R_j = [N_j, Z_j]$, $j = 1,2,3$, see 1^0, one has

$$\mathrm{Tr}\, R_1 = \mathrm{Tr}\, R_2 = \mathrm{Tr}(N_1R_2(N_1)^{-1}) = \mathrm{Tr}\, R_3 = \mathrm{Tr}((N_1N_2)R_3(N_1N_2)^{-1}) = 0\,.$$

Hence, $\mathrm{Tr}\, R_4 = 0$. This means that R_4 is nilpotent, of rank 1. This proves 1).

4^0. To prove 2) one has to recall that $R_j = [N_j, Z_j]$ for $j = 1, 2, 3$, see 1^0, and that each matrix from $sl(2, \mathbb{C})$ can be represented as $\sum_{j=1}^{3}[N_j, Z_j]$, see Proposition 1.2. Hence, for every nilpotent R_4 one can find matrices Z_j such that for $j = 1, 2, 3$ one has $R_j = [N_j, Z_j]$, i. e. $M_j \in C_j$ and $M_1M_2M_3M_4 = I$. □

Proof of Proposition 3.4:

1^0. Denote by $F = \begin{pmatrix} U & V \\ W & Y \end{pmatrix}$ a matrix from the centralizer of the quadruple. In the case of a quadruple (1) the commutation relations read:

$$[U, N_j] = [Y, P_j] = 0\,,\; N_jV = VP_j\,,\; WN_j = P_jW\,.$$

The representations defined by the matrices N_j and P_j being non-equivalent, these relations imply $V = W = 0$. The irreducibility of the quadruples of matrices N_j and P_j and Schur's lemma imply that U and Y are scalar. Hence, $U = \alpha I$, $Y = \xi I$ with $\alpha^2\xi^2 = 1$, i. e. $\xi = \pm\alpha^{-1}$.

2^0. In the case of a quadruple (2) the matrix algebra $\mathcal{A}$ generated by the matrices M_j contains the matrix $M_4 + I$ and its left and right products by matrices from the algebra $\mathcal{B}$ generated by M_1, M_2 and M_3. As $\mathcal{B}$ contains matrices of the form $\begin{pmatrix} T & * \\ 0 & T \end{pmatrix}$ for any $T \in gl(2, \mathbb{C})$ (the Burnside theorem), the algebra $\mathcal{A}$ contains all matrices of the form $\begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix}$ with $Q \in gl(2, \mathbb{C})$.

The commutation relations imply that $WQ = 0$, hence, $W = 0$, and $UQ = QY$ for any Q, i. e. $U = Y = \delta I$. Finally, one has $[N_j, V] = 0$ which implies that $V = \beta I$ (use Schur's lemma).

One must have $\delta^4 = 1$ because $F \in SL(4, \mathbb{C})$. □

Proof of Proposition 3.5:

1^0. One can deform the matrices M_j from a quadruple from $\mathcal{W}$ as follows. The deformation parameter is denoted by $\varepsilon \in (\mathbb{C}, 0)$ and the deformed matrices by M_j'. Assume that $N_4 = -I$, $R_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (one can achieve this by conjugation of the quadruple with a block-diagonal matrix). Set $M_4' = M_4 + \varepsilon(E_{1,2} + w(\varepsilon)E_{1,3})$; the matrix $E_{k,j}$ by definition has a single non-zero entry equal to 1 in position (k, j); $w(\varepsilon)$ is an unknown germ of an analytic function.

2^0. For $j = 1, 2, 3$ set $M_j' = (I + \varepsilon X_j(\varepsilon))^{-1}M_j(I + \varepsilon X_j(\varepsilon))$ where $X_j = \begin{pmatrix} U_j & V_j \\ 0 & 0 \end{pmatrix}$. Set $X_j(0) = X_j^0$. One must have $M_1'M_2'M_3'M_4' = I$ which in first approximation w.r.t. ε reads

$$\begin{aligned}[][M_1, X_1^0]M_2M_3M_4 + M_1[M_2, X_2^0]M_3M_4 + M_1M_2[M_3, X_3^0]M_4 \\ + M_1M_2M_3(E_{1,2} + w(0)E_{1,3}) = 0\,.\end{aligned} \tag{3}$$

3^0. Set $U_j^0 = U_j(0)$, $U^0 = (U_1^0, U_2^0, U_3^0)$, $V_j^0 = V_j(0)$, $V^0 = (V_1^0, V_2^0, V_3^0)$, $w^0 = w(0)$. Equation (3) restricted to the left upper block reads:

$$\mathcal{G}(U^0) := [N_1, U_1^0]N_2N_3 + N_1[N_2, U_2^0]N_3 + N_1N_2[N_3, U_3^0] = N_1N_2N_3E_{1,2}$$

(because $N_4 = -I$). Making use of $N_1N_2N_3 = -I$ one finds

$$[N_1, U_1^0N_1^{-1}] + [N_1N_2N_1^{-1}, N_1U_2^0N_2^{-1}N_1^{-1}] + [N_3, N_3^{-1}U_3^0] = E_{1,2} \tag{4}$$

The triple of matrices N_1, N_2, N_3 is irreducible, hence, so is the triple N_1, $N_1N_2N_1^{-1}$, N_3. By Proposition 1.2, one can find matrices U_j^0 satisfying equation (4).

4^0. Equation (3) restricted to the right lower block is of the form $0 = 0$, i. e. it gives no condition at all upon U_j^0, V_j^0 and w^0. Its restriction to the right upper block reads:

$$\mathcal{F}(V^0, U^0, w^0) := \mathcal{G}(V^0) + \mathcal{H}(U^0) - w^0E_{1,3} = 0 \tag{5}$$

where $\mathcal{H}$ is some linear form in the entries of the matrices U_j^0. Hence, if U_j^0 are found such that (4) holds, then one can find w^0 such that $\mathrm{Tr}(\mathcal{H}(U_1^0, U_2^0, U_3^0)) = w^0$. After this one can find matrices V_j^0 such that (5) hold.

5^0. The map $(U^0, V^0, w^0) \mapsto (\mathcal{G}(U^0), \mathcal{F}(V^0, U^0, w^0))$ is surjective onto the space of 2×4-matrices. By the implicit function theorem one can find germs of matrices U_j, V_j and a germ of a function w holomorphic in ε at 0 such that $M_1' \cdots M_4' = I$.

Fix $\varepsilon \neq 0$. The quadruple of matrices M_j' is block upper-triangular with diagonal blocks having the properties of P_j and N_j (P_j is above). Moreover, each of the matrices M_j' is conjugate to the block-diagonal matrix whose restriction to the two diagonal blocks is the same as the one of M_j' (to be checked directly). By Proposition 3.2, up to conjugacy the quadruple of matrices is like the one from (1). □

Proof of Proposition 3.6:

1^0. Proof of the irreducibility. The variety $\mathcal{W}$ is obtained by conjugating with matrices from $SL(4, \mathbb{C})$ the quadruples of matrices of the form (2) with R_4 nilpotent of rank 1. The orbit of R_4 is an irreducible variety which implies the irreducibility of $\mathcal{W}$.

2^0. Fix the blocks N_j of a quadruple (2). The variety $\mathcal{S}$ of such quadruples defined modulo conjugacy is of dimension 1. Indeed, the orbit of R_4 is of dimension 2. The only conjugations that preserve the form of the quadruple and its restrictions to the two diagonal blocks are with matrices of the form $\begin{pmatrix} aI & V \\ 0 & bI \end{pmatrix}$, $ab \neq 0$, $V \in gl(2, \mathbb{C})$; this is proved in 4^0. If one requires the matrix to be from $SL(4, \mathbb{C})$, this means that $b = \pm 1/a$ and factoring out these conjugations decreases the dimension by 1. Indeed, such a conjugation changes R_4 to bR_4/a, the presence of V does not affect the block R_4.

3^0. To obtain the variety $\mathcal{H}$ of all quadruples defining semi-direct sums like (2) one has to conjugate the quadruples from $\mathcal{S}$ by matrices from $SL(4, \mathbb{C})$. This increases the dimension by 14 (not by 15 because the centralizer of such a quadruple is non-trivial, of dimension 1, see Proposition 3.4). Hence, $\dim \mathcal{H} = 15$.

4^0. Denote by G a matrix the conjugation with which preserves the block upper-triangular form of the quadruple and the blocks N_j. If $G = \begin{pmatrix} U & V \\ W & Y \end{pmatrix}$, then the condition the quadruple to remain block upper-triangular implies that $[W, N_j] = 0$, i. e. $W = hI$. The condition the diagonal blocks of M_4 to remain the same implies $[N_4, Y] - WR_4 = R_4 W + [N_4, U] = 0$. As $N_4 = -I$, one has $[N_4, Y] = [N_4, U] = 0$, i. e. $W = 0$.

The conditions $[N_j, U] = [N_j, Y] = 0$ imply that $U = aI$, $Y = bI$. $\square$

Proof of Proposition 3.7:

1^0. The varieties of quadruples of matrices N_j or P_j, see Proposition 3.1, are smooth, irreducible and of dimensions respectively 3 and 5. Hence, the variety $\mathcal{P}$ of quadruples of matrices M_j like in (1) is smooth, irreducible and of dimension 8.

2^0. The variety $\mathcal{U}$ is of dimension $8+15-7 = 16$. Here "8" stands for "$\dim \mathcal{P}$", "15" stands for "$\dim SL(4, \mathbb{C})$" and 7 is the dimension of the subgroup of $SL(4, \mathbb{C})$ of block-diagonal matrices with blocks 2×2 conjugation with which preserves the block-diagonal form of quadruple (1) (infinitesimal conjugations *only* with such matrices preserve the block-diagonal form of quadruple (1)); this subgroup contains the centralizer of quadruple (1), see Proposition 3.4. $\square$

Proof of Proposition 3.8:

The variety $\mathcal{Y}$ is the orbit of one quadruple of the form (2) with $R_j = 0$, $j = 1, \ldots, 4$, under conjugation by $SL(4, \mathbb{C})$ (recall that the matrices N_j define a rigid representation, i. e. unique up to conjugacy). Hence, $\mathcal{Y}$ is irreducible and smooth.

To obtain $\dim \mathcal{Y}$ one has to subtract from $15 = \dim SL(4,\mathbb{C})$ the dimension of the centralizer in $SL(4,\mathbb{C})$ of the above quadruple. The latter equals 3 — the centralizer is the set of all matrices of the form $\begin{pmatrix} \alpha I & \beta I \\ \delta I & \eta I \end{pmatrix}$ with $\alpha\eta - \delta\beta = \pm 1$. □

5. Another Example with Index of Rigidity 2

Consider the variety $\mathcal{V}$ in the case when $p = 2$, $n = 4$, the three conjugacy classes are diagonalizable and have eigenvalues (a,a,b,c), (f,f,g,h) and (u,u,v,w) (different letters denote different eigenvalues). The index of rigidity equals 2 (to be checked directly).

The eigenvalues are presumed to satisfy the only non-genericity relation $abfguv = 1$. Hence, for such conjugacy classes there exist irreducible triples of diagonalizable matrices $L_j \in gl(2,\mathbb{C})$ (resp. $B_j \in gl(2,\mathbb{C})$) with eigenvalues (a,b); (f,g); (u,v) (resp. (a,c); (f,h); (u,w)) such that $L_1L_2L_3 = I$ (resp. $B_1B_2B_3 = I$). This follows from Theorem 2.1. Hence, there exist triples of block-diagonal matrices M_j with diagonal blocks equal to L_j and B_j. Denote by $\mathcal{D}$ the variety of such triples. By Theorem 3.1, irreducible triples of matrices M_j do not exist.

There do exist, however, triples with trivial centralizers which are block upper-triangular: $M_j = \begin{pmatrix} L_j & T_j \\ 0 & B_j \end{pmatrix}$ where $T_j = L_jY_j - Y_jB_j$ for some $Y_j \in gl(2,\mathbb{C})$ because M_j is conjugate to $\begin{pmatrix} L_j & 0 \\ 0 & B_j \end{pmatrix}$. The condition $M_1M_2M_3 = I$ restricted to the right upper block reads:

$$T_1B_2B_3 + L_1T_2B_3 + L_1L_2T_3 = 0 \qquad (*)$$

Thus the triple of matrices T_j belongs to the space

$$\mathcal{T} = \{(T_1,T_2,T_3)\,;\, T_j = L_jY_j - Y_jB_j\,,\ Y_j \in gl(2,\mathbb{C})\,,$$
$$T_1B_2B_3 + L_1T_2B_3 + L_1L_2T_3 = 0\}\,.$$

One has $\dim \mathcal{T} = 5$.

Indeed, the conditions $T_j = L_jY_j - Y_jB_j$ imply that each matrix T_j belongs to the image of the map $(.) \mapsto L_j(.) - (.)B_j$ which is a subspace of $gl(2,\mathbb{C})$ of dimension 3. Condition (*) is equivalent to four linearly independent equations (we let the reader prove their linear independence using the non-equivalence of the representations defined by the matrices L_j and B_j).

Consider the space

$$\mathcal{Q} = \{(T_1,T_2,T_3)\,;\, T_j = L_jY - YB_j\,,\ Y \in gl(2,\mathbb{C})\}\,.$$

For such matrices T_j there holds $(*)$, therefore $\mathcal{Q} \subset \mathcal{T}$. The space $\mathcal{Q}$ is the space of right upper blocks of triples of block upper-triangular matrices M_j which are obtained from block-diagonal ones from $\mathcal{D}$ by conjugation with matrices of the form $\begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}$.

One has $\dim \mathcal{Q} = 4$.

Indeed, for no matrix from $gl(2, \mathbb{C})$ does one have $L_j Y - Y B_j = 0$ for $j = 1, 2, 3$ because the triples of matrices L_j and B_j define non-equivalent representations.

Hence, $\dim(\mathcal{T}/\mathcal{Q}) = 1$. Choose the triple of matrices Y_j to span the factorspace $(\mathcal{T}/\mathcal{Q})$. Hence, the centralizer $\mathcal{Z}$ of the triple of matrices M_j will be trivial. Indeed, let $Z = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in \mathcal{Z}$. Hence, $RL_j = B_j R$ for $j = 1, 2, 3$ (commutation relations restricted to the left lower block), i. e. $R = 0$ because the matrices L_j and B_j define non-equivalent representations.

One must have $[P, L_j] = [S, B_j] = 0$ (commutation relations restricted to the diagonal blocks), i. e. $P = aI$, $B = bI$. But then one must have (commutation relations restricted to the right upper block) $(a - b)T_j = L_j Q - Q B_j$ which means that $a = b$ (otherwise $(T_1, T_2, T_3) \in \mathcal{Q}$), hence, $L_j Q - Q B_j = 0$ for $j = 1, 2, 3$, i. e. $Q = 0$. Hence, $Z = aI$.

Remarks:

1) It is clear that the variety $\mathcal{D}$ belongs to the closure of $\mathcal{V} \backslash \mathcal{D}$ — the triple of matrices $M_j = \begin{pmatrix} L_j & \varepsilon T_j \\ 0 & B_j \end{pmatrix}$ belongs to $\mathcal{V} \backslash \mathcal{D}$ for $\varepsilon \neq 0$, for $\varepsilon = 0$ it belongs to $\mathcal{D}$.

2) The variety $\mathcal{V}$ is connected, hence, irreducible. This follows from $(\mathcal{T}/\mathcal{Q})$ being a linear space ($\mathcal{V}$ is obtained by conjugating block upper-triangular triples with $(T_1, T_2, T_3) \in (\mathcal{T}/\mathcal{Q})$ and with fixed diagonal blocks by matrices from $SL(4, \mathbb{C})$).

6. A Third Example with Index of Rigidity 2

Let $n = 4$, $p = 2$. Use the notation from the previous section. Define the conjugacy classes C_j as follows: their eigenvalues equal (a, a, b, b), (f, f, g, g), (u, u, v, v), the eigenvalues are relatively generic but not generic (one has $abfguv = 1$). To each of the eigenvalues a, b and f there corresponds a single Jordan block of size 2, to each of the eigenvalues g, u, v there correspond two Jordan blocks of size 1. Hence, the index of rigidity equals 2.

The variety $\mathcal{V}$ contains triples of matrices which up to conjugacy are block upper-triangular with two diagonal blocks equal to L_j, see their definition in the previous section. By Theorem 3.1, $\mathcal{V}$ contains no irreducible triples. Hence, it contains none with trivial centralizer either because the matrices M_j from any such block upper-triangular triple commute with the matrix $E_{1,3}+E_{2,4}$; on the other hand, if a triple of matrices $M_j \in C_j$ is conjugated to a block upper-triangular form, then the diagonal blocks are of size 2 and up to conjugacy they equal L_j — this follows from the choice of the eigenvalues.

Proposition 6.1. *One has* $\dim \mathcal{V} = 15$ *which is the expected dimension.*

Remark. *The closure of the variety $\mathcal{V}$ contains the varieties in which at least one of the two Jordan normal forms $J(M_1)$ and $J(M_2)$ contains instead of some Jordan block(s) of size 2 two Jordan blocks of size 1. We leave the details for the reader. One can prove that $\mathcal{V}$ is irreducible.*

Proof of Proposition 6.1:

1^0. Suppose that one has $M_j = \begin{pmatrix} L_j & T_j \\ 0 & L_j \end{pmatrix}$ with $L_1 = \mathrm{diag}(a,b)$, $T_1 = \mathrm{diag}(1,1)$. Fix L_2 and L_3. Then the couple of blocks (T_2,T_3) belongs to a space of dimension 1.

Indeed, one has $T_3 = [L_3, Z_3]$ in order M_3 to be diagonalizable and the dimension of the image of the map $Z_3 \mapsto [L_3,Z_3]$ in $gl(2,\mathbb{C})$ equals 2.

The block T_2 belongs to an affine space of dimension 2. Indeed, one has $T_2 = S + [L_2, Z_2]$, where the dimension of the image of the map $Z_2 \mapsto [L_2,Z_2]$ equals 2 and the matrix S is defined as follows. Set $L_2 = H^{-1}\,\mathrm{diag}(f,g)H$. Then $S = \xi H^{-1}E_{1,3}H$ where ξ satisfies the condition

$$\mathrm{Tr}(L_2L_3 + L_1SL_3) = 0 \qquad (**)$$

(If by chance this condition gives $\xi = 0$, then one has to choose two diagonal entries of T_1 other than $(1,1)$ so that $\xi \neq 0$, otherwise M_2 will be diagonalizable.)

2^0. The coefficient ξ satisfies condition (**) for the following reason. The condition $M_1M_2M_3 = I$ implies that $\mathcal{H} := T_1L_2L_3 + L_1T_2L_3 + L_1L_2T_3 = 0$. In particular, $\mathrm{Tr}\,\mathcal{H} = 0$. As

$$L_1L_2L_3 = I\,, \qquad T_1 = I\,,$$
$$\mathrm{Tr}(L_1L_2T_3) = \mathrm{Tr}(L_1L_2L_3Z_3 - L_1L_2Z_3L_3) = \mathrm{Tr}(Z_3 - L_3^{-1}Z_3L_3) = 0$$

and $\mathrm{Tr}(L_1[L_2,Z_2]L_3) = \mathrm{Tr}(L_3^{-1}Z_2L_3 - L_1Z_2L_1^{-1}) = 0$, one has $\mathrm{Tr}(L_2L_3 + L_1SL_3) = 0$.

3^0. From the dimension $2+2$ of the space to which the couple (T_2,T_3) belongs one has to subtract 3 because the equation $\mathcal{H} = 0$ (after one has chosen ξ so that $\mathrm{Tr}\,\mathcal{H} = 0$) imposes 3 conditions.

4^0. The centralizer $\mathcal{Z}$ of the triple of matrices M_j in $SL(4,\mathbb{C})$ is generated by the matrix $E_{1,3} + E_{2,4}$. Moreover, any matrix from $SL(4,\mathbb{C})$ the conjugation with which preserves the form of the triple belongs to $\mathcal{Z}$. This can be proved by a direct computation which we leave for the reader.

5^0. To find the dimension of $\mathcal{V}$ one has to conjugate the block upper-triangular triples from 1^0 whose variety is of dimension 1 by matrices from $SL(n,\mathbb{C})/\mathcal{Z}$. The latter variety is of dimension 14. Hence, $\dim \mathcal{V} = 15$. □

7. A Fourth Example with Index of Rigidity 2

Let $n = p = 3$ and let the conjugacy classes C_j define diagonal but non-scalar JNFs the eigenvalues being equal respectively to $(a,1,1)$, $(b,1,1)$, $(c,1,1)$, $(d,1,1)$, with $abcd = 1$. Hence, the index of rigidity is 0. There exist reducible such quadruples of matrices M_j with trivial centralizers. Example:

$$M_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} b & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} c & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_4 = \begin{pmatrix} d & -1/bc & -1/c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(the reader is invited to check the triviality of the centralizer oneself). Denote by $\mathcal{T}$ the stratum of $\mathcal{V}$ of quadruples with trivial centralizers. Hence, $\dim \mathcal{T} = 8$ (Proposition 1.1). By Theorem 3.1, there exist no irreducible quadruples of matrices $M_j \in C_j$.

On the other hand, there exist quadruples defining direct sums of an irreducible representation of rank 2 and of a one-dimensional one. Example:

$$M_1 = \begin{pmatrix} a & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} b & -1/a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} c & 0 & 0 \\ -1/d & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_4 = \begin{pmatrix} d & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Denote by $\mathcal{S}$ the stratum of $\mathcal{V}$ of quadruples defining such direct sums.

One has $\dim \mathcal{S} = 9$.

Indeed, the subvariety $\mathcal{S}' \subset \mathcal{S}$ of block-diagonal such quadruples is of dimension 5 (Proposition 1.1). Hence, $\mathcal{S}$ is obtained from $\mathcal{S}'$ by conjugating with matrices from $SL(3,\mathbb{C})$ ($\dim SL(3,\mathbb{C}) = 8$) and one has to factor out the conjugation with block-diagonal matrices whose subgroup is of dimension 4. Thus $\dim \mathcal{S} = 5 + 8 - 4 = 9$.

Remarks:

1) Both strata $\mathcal{S}$ and $\mathcal{T}$ contain in their closures the variety of quadruples which are diagonal up to conjugacy, also the ones of quadruples defining direct sums of the one-dimensional representation 1, 1, 1, 1 with the semi-direct sums of the representations 1, 1, 1, 1 and a, b, c, d.

2) The stratum $\mathcal{T}$ *does not* lie in the closure of the stratum $\mathcal{S}$ (triviality of the centralizer is an "open" property).

3) One can show that at every point of $\mathcal{V}$ one has $\dim \mathcal{V} \leq 9$.

8. An Example with Zero Index of Rigidity

By Theorem 2.1, there exist irreducible quadruples of matrices A_j or M_j of size 2 in which each matrix has two distinct eigenvalues and the eigenvalues are generic. For such quadruples the index of rigidity equals 0 (to be checked directly).

Consider a quadruple of matrices (say, M_j; for matrices A_j one can give a similar example) of the form

$$M_j = \begin{pmatrix} B_j & 0 \\ 0 & G_j \end{pmatrix}$$

where each of the quadruples of matrices B_j and G_j is like above, with generic eigenvalues. Moreover, for each j the eigenvalues of B_j and C_j are the same but the quadruples of matrices B_j and G_j define non-equivalent representations. To choose them such is possible because the quadruples are not rigid.

Compute the dimension of the variety $\mathcal{M}$ of such quadruples of matrices M_j. The varieties $\mathcal{B}$ and $\mathcal{G}$ of quadruples of 2×2-matrices B_j or G_j are both of dimension 5 (see part 2) of Proposition 3.1).

Hence, $\dim \mathcal{M} = 10$. The variety $\mathcal{N}$ of quadruples of matrices M_j defining a direct sum of two representations of rank 2 with the properties of $\mathcal{B}$ and $\mathcal{G}$ is obtained by conjugating the quadruples from $\mathcal{M}$ by matrices from $SL(4, \mathbb{C})$. Infinitesimal conjugation by block-diagonal matrices from $SL(4, \mathbb{C})$ with two diagonal blocks of size 2 and only by such matrices preserves $\mathcal{M}$ (their subgroup is of dimension 7 in $SL(4, \mathbb{C})$). Hence, $\dim \mathcal{N} = 10 + 15 - 7 = 18$ where $15 = \dim SL(4, \mathbb{C})$.

The expected dimension of the variety $\mathcal{N}$ equals 17, see Proposition 1.1. In a subsequent paper the author intends to prove that for zero index of rigidity and for relatively generic but not generic eigenvalues the Deligne–Simpson problem is not weakly solvable. Hence, in the above example one has $\mathcal{V} = \mathcal{N}$ and the dimension of $\mathcal{V}$ is higher than the expected one.

Open Questions:

1) Is it true that for negative indices of rigidity the dimension of the variety of $(p+1)$-tuples with non-trivial centralizers is always smaller than the expected dimension of the variety of all $(p+1)$-tuples (of matrices M_j or A_j)?

2) Is it true that for negative indices of rigidity if the Jordan normal forms $J_1^n, \dots, J_{p+1}^n$ satisfy the conditions of Theorem 2.1, then the Deligne–Simpson problem is weakly solvable for any eigenvalues?

Appendix A. Proof of Theorem 3.1 (by Ofer Gabber)

1^0. We use arguments related to the ones from [1]. Suppose we are given the conjugacy classes $C_i \subset GL(n,\mathbb{C})$, $1 \le i \le p+1$, and we are interested in solutions of

$$M_1 \cdots M_{p+1} = \mathrm{id}\,, \quad M_i \in C_i \tag{1}$$

We say that a solution $M = (M_1, \dots, M_{p+1})$ is *rigid* if every solution M' in some neighbourhood of M is $GL(n,\mathbb{C})$-conjugate to M. Here "neighbourhood" can be taken in the classical or in the Zariski topology.

2^0. Consider distinct points $a_1, \dots, a_{p+1} \in \mathbb{P}^1_{\mathbb{C}}$ and set $U = \mathbb{P}^1_{\mathbb{C}} \backslash \{a_1, \dots, a_{p+1}\}$. Choose a base point $x_0 \in U$ and a standard set of generators $\gamma_i \in \pi_1(U, x_0)$ where γ_i is freely homotopic to a positive loop around a_i, $\gamma_1 \cdots \gamma_{p+1} = 1$ (using π_1 conventions as in Deligne's LNM 163).

Then a solution of (1) determines a local system L on U, $L_{x_0} \simeq \mathbb{C}^n$; the local monodromies are given by the matrices M_i.

3^0. Recall that if $f\colon X \to Y$ is an algebraic map of irreducible algebraic varieties, then every irreducible component of a fibre of f has dimension $\ge \dim(X) - \dim(Y)$.

Suppose we are given a rigid solution of (1). In particular, if δ_i is the value of the determinant on C_i, then $\prod \delta_i = 1$, so we have the product morphism

$$f\colon C_1 \times \cdots \times C_{p+1} \to SL(n,\mathbb{C})$$

and by assumption the $GL(n,\mathbb{C})$-orbit of $(M_1, \dots, M_{p+1})$ is dense in an irreducible component of $f^{-1}(\mathrm{id})$. The above orbit is also an $SL(n,\mathbb{C})$-orbit, so it is of dimension $\le n^2 - 1$.

4^0. Hence,

$$\sum_{i=1}^{p+1} d_i \le 2(n^2-1)\,.$$

Denote by j the inclusion of U in $\mathbb{P}^1_{\mathbb{C}}$ and by $\mathcal{Z}(M_i)$ the space of matrices commuting with M_i. Then $d_i = n^2 - \dim \mathcal{Z}(M_i)$ and by the Euler–Poincaré formula (cf. [1] p. 16) the above inequality is equivalent to

$$\chi(\mathbb{P}^1_{\mathbb{C}}, \mathrm{j}_* \underline{\mathrm{End}}(L)) \geq 2\,.$$

Now if F is a rank n irreducible local system with local monodromies in the prescribed conjugacy classes, then by the Euler–Poincaré formula

$$\chi(\mathbb{P}^1_{\mathbb{C}}, \mathrm{j}_* \underline{\mathrm{End}}(L)) = \chi(\mathbb{P}^1_{\mathbb{C}}, \mathrm{j}_* \underline{\mathrm{Hom}}(L, F)) \geq 2\,,$$

so one of the two cohomology groups $H^0(\mathbb{P}^1_{\mathbb{C}}, \mathrm{j}_* \underline{\mathrm{Hom}}(L, F)) \cong \mathrm{Hom}_U(L, F)$ or

$$H^2(\mathbb{P}^1_{\mathbb{C}}, \mathrm{j}_* \underline{\mathrm{Hom}}(L, F)) \cong H^2_c(U, \underline{\mathrm{Hom}}(L, F)) \cong \mathrm{Hom}_U(F, L)^v$$

is non-zero, which implies (as F is irreducible) that $F \simeq L$ (cp. [1], Theorem 1.1.2). Hence, if L is reducible, then F does not exist. $\square$

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