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# ON THE REDUCTIONS AND HAMILTONIAN STRUCTURES OF $N$-WAVE TYPE EQUATIONS

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**Abstract**. The reductions of the integrable $N$-wave type equations solvable by the inverse scattering method with the generalized Zakharov–Shabat system $L$ and related to some simple Lie algebra $\mathfrak{g}$ are analyzed. Special attention is paid to the $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$-reductions including ones that can be embedded also in the Weyl group of $\mathfrak{g}$. The consequences of these restrictions on the properties of their Hamiltonian structures are analyzed on specific examples which find applications to nonlinear optics.

## 1. Introduction

It is well known that the $N$-wave equations [1–6]

$$\mathrm{i}[J, Q_t] - \mathrm{i}[I, Q_x] + [[I, Q], [J, Q]] = 0\,, \tag{1}$$

are solvable by the inverse scattering method (ISM) [4, 5] applied to the generalized system of Zakharov–Shabat type [4, 7, 8]:

$$L(\lambda)\Psi(x,t,\lambda) = \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + [J, Q(x,t)] - \lambda J\right)\Psi(x,t,\lambda) = 0, \quad J \in \mathfrak{h}\,, \tag{2}$$

$$Q(x,t) = \sum_{\alpha\in\Delta_+} (q_\alpha(x,t)E_\alpha + p_\alpha(x,t)E_{-\alpha}) \in \mathfrak{g}/\mathfrak{h}\,, \tag{3}$$

where $\mathfrak{h}$ is the Cartan subalgebra and $E_\alpha$ are the root vectors of the simple Lie algebra $\mathfrak{g}$. Indeed (1) can be written in the Lax form, or in other words, it is the compatibility condition

$$[L(\lambda), M(\lambda)] = 0\,, \tag{4}$$

where

$$M(\lambda)\Psi(x,t,\lambda) = \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + [I, Q(x,t)] - \lambda I\right)\Psi(x,t,\lambda) = 0, \quad I \in \mathfrak{h}\,. \tag{5}$$

Here and below $r = \operatorname{rank}\mathfrak{g}$, $\Delta_+$ is the set of positive roots of $\mathfrak{g}$ and $\vec{a}, \vec{b} \in \mathbb{E}^r$ are vectors corresponding to the Cartan elements $J, I \in \mathfrak{h}$. The inverse scattering problem for (2) with real valued $J$ [1] was reduced to a Riemann–Hilbert problem for the (matrix-valued) fundamental analytic solution of (2) [4, 7]; the action-angle variables for the $N$-wave equations were obtained in the preprint [1] and rederived later in [9]. However, often the reduction conditions require that $J$ be complex-valued. Then the solution of the corresponding inverse scattering problem for (2) becomes more difficult [10, 11].

The interpretation of the ISM as a generalized Fourier transform and the expansions over the "squared solutions" of (2) were derived in [8] for real $J$ and in [11] for complex $J$. They were used also to prove that all $N$-wave type equations are Hamiltonian and possess a hierarchy of Hamiltonian structures [8, 11] $\{H^{(k)}, \Omega^{(k)}\}$, $k = 0, \pm 1, \pm 2, \ldots$. The simplest Hamiltonian formulation of (1) is given by $\{H^{(0)} = H_0 + H_{\mathrm{int}},\ \Omega^{(0)}\}$ where

$$H_0 = \frac{c_0}{2\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}x \, \langle Q, [I, Q_x] \rangle \;, \tag{6}$$

$$H_{\mathrm{int}} = \frac{c_0}{3} \int_{-\infty}^{\infty} \mathrm{d}x \, \langle [J, Q], [Q, [I, Q]] \rangle \;, \tag{7}$$

$\langle \cdot, \cdot \rangle$ is the Killing form and the symplectic form $\Omega^{(0)}$ is equivalent to a canonical one

$$\Omega^{(0)} = \frac{\mathrm{i}c_0}{2} \int_{-\infty}^{\infty} \mathrm{d}x \, \left\langle [J, \delta Q(x,t)] \mathop{\wedge}_{,} \delta Q(x,t) \right\rangle \,. \tag{8}$$

The constant $c_0$ will be fixed up below. Physically each cubic term in $H_{\mathrm{int}}$ depends on a triple of positive roots such that $\alpha_i = \alpha_j + \alpha_k$ and shows how the wave of mode $i$ decays into $j$-th and $k$-th waves. In other words we assign to each positive root $\alpha$ an wave with an wave number $k_\alpha$ and a frequency $\omega_\alpha$ which are preserved in the elementary decays, i. e.

$$k_{\alpha_i} = k_{\alpha_j} + k_{\alpha_k}\,, \quad \omega_{\alpha_i} = \omega_{\alpha_j} + \omega_{\alpha_k}\,.$$

We shall show how one can exhibit new examples of integrable $N$-wave type interactions some of which have applications to physics. The integrability of a rich family of $N$-wave type equations and their importance as universal model of wave-wave interactions was demonstrated in [12]. Our approach allows to enrich still further this family.

Our studies are based on the reduction group $G_R$ introduced by Mikhailov [13] and further developed in [14–16]. More recently the $\mathbb{Z}_2$ and $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ reductions of the $N$-wave type equations were investigated [17–20]. In [18, 19] we point out that $G_R$ can be embedded in the group of automorphisms of $\mathfrak{g}$ in several different ways which may lead to inequivalent reductions of the $N$-wave equations.

## 2. Preliminaries

The main idea underlying Mikhailov's reduction group [13] is to impose algebraic restrictions on the Lax operators $L$ and $M$ which will be automatically compatible with the corresponding equations of motion (4). Due to the purely Lie-algebraic nature of the Lax representation (4) this is most naturally done by imbedding $G_R$ as a subgroup of $\operatorname{Aut}\mathfrak{g}$ — the group of automorphisms of $\mathfrak{g}$. Obviously to each reduction imposed on $L$ and $M$ there will correspond a reduction of the space of fundamental solutions $\mathfrak{S}_\Psi \equiv \{\Psi(x,t,\lambda)\}$ of (2) and (5).

Some of the simplest $\mathbb{Z}_2$-reductions of $N$-wave systems (see [2–4]) are related to outer automorphisms of $\mathfrak{g}$ and $\mathfrak{G}$, namely:

$$C_1(\Psi(x,t,\lambda)) = A_1 \Psi^\dagger(x,t,\kappa_1(\lambda)) A_1^{-1} = \tilde{\Psi}^{-1}(x,t,\lambda)\,, \quad \kappa_1(\lambda) = \pm\lambda^*, \tag{9}$$

where $A_1$ belongs to the Cartan subgroup of the group $\mathfrak{G}$:

$$A_1 = \exp(\mathrm{i}\pi H_1)\,, \tag{10}$$

and $H_1 \in \mathfrak{h}$ is such that $\alpha(H_1) \in \mathbb{Z}$ for all roots $\alpha \in \Delta$ in the root system $\Delta$ of $\mathfrak{g}$. The reduction condition relates the fundamental solution $\Psi(x,t,\lambda) \in \mathfrak{G}$ to a fundamental solution $\tilde{\Psi}(x,t,\lambda)$ of (2) and (5) which in general differs from $\Psi(x,t,\lambda)$.

Another class of $\mathbb{Z}_2$ reductions are related to outer automorphisms, e. g.:

$$C_2\left(\Psi(x,t,\lambda)\right) = A_2 \Psi^{\mathrm{T}}(x,t,\kappa_2(\lambda)) A_2^{-1} = \tilde{\Psi}^{-1}(x,t,\lambda), \quad \kappa_2(\lambda) = \pm\lambda\,, \tag{11}$$

where $A_2$ is again of the form (10). The best known examples of NLEE obtained with the reduction (11) are the sine-Gordon and the MKdV equations which are related to $\mathfrak{g} \simeq sl(2)$. For higher rank algebras such reductions to our knowledge have not been studied. Generically reductions of type (11) lead to degeneration of the canonical Hamiltonian structure, i. e. $\Omega^{(0)} \equiv 0$; then we need to use some of their higher Hamiltonian structures (see [8, 11]).

One may use also reductions with inner automorphisms like:

$$C_3(\Psi(x,t,\lambda)) = A_3\Psi^*(x,t,\kappa_1(\lambda))A_3^{-1} = \tilde{\Psi}(x,t,\lambda)\,, \qquad (12)$$

$$C_4(\Psi(x,t,\lambda)) = A_4\Psi(x,t,\kappa_2(\lambda))A_4^{-1} = \tilde{\Psi}(x,t,\lambda)\,. \qquad (13)$$

Since our aim is to preserve the form of the Lax pair we limit ourselves by automorphisms preserving the Cartan subalgebra $\mathfrak{h}$. This conditions is obviously fulfilled if $A_k$, $k = 1,\dots,4$ is in the form (10). Another possibility is to choose $A_1,\dots,A_4$ so that they correspond to a Weyl group automorphisms.

The reduction group $G_R$ is a finite group which preserves the Lax representation (4), i. e. for each $g_k \in G_R$

$$C_k(L(\Gamma_k(\lambda))) = \eta_k L(\lambda)\,, \quad C_k(M(\Gamma_k(\lambda))) = \eta_k M(\lambda)\,. \qquad (14)$$

$G_R$ must have two realizations: (i) $G_R \subset \operatorname{Aut}\mathfrak{g}$ and $C_k \in \operatorname{Aut}\mathfrak{g}$; (ii) $G_R \subset \operatorname{Conf}\mathbb{C}$, i. e. $\Gamma_k(\lambda)$ are conformal mappings of the complex $\lambda$-plane. Below we consider specially the cases $G_R \simeq \mathbb{Z}_2$ or $G_R \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$.

The automorphisms $C_k$, $k = 1,\dots,4$ listed above lead to the following reductions for the matrix-valued functions

$$U(x,t,\lambda) = [J, Q(x,t)] - \lambda J, \quad V(x,t,\lambda) = [I, Q(x,t)] - \lambda I\,, \qquad (15)$$

of the Lax representation:

$$\begin{aligned}
C_1(U^\dagger(\kappa_1(\lambda))) &= U(\lambda)\,, & C_1(V^\dagger(\kappa_1(\lambda))) &= V(\lambda)\,,\\
C_2(U^T(\kappa_2(\lambda))) &= -U(\lambda)\,, & C_2(V^T(\kappa_2(\lambda))) &= -V(\lambda)\,,\\
C_3(U^*(\kappa_1(\lambda))) &= -U(\lambda)\,, & C_3(V^*(\kappa_1(\lambda))) &= -V(\lambda)\,,\\
C_4(U(\kappa_2(\lambda))) &= U(\lambda)\,, & C_4(V(\kappa_2(\lambda))) &= V(\lambda)\,.
\end{aligned} \qquad (16)$$

### 2.1. Cartan–Weyl Basis and Weyl Group

Here we fix up the notations, the normalization conditions for the Cartan–Weyl generators of $\mathfrak{g}$ and their commutation relations, see [21]:

$$\begin{gathered}
[h_k, E_\alpha] = (\alpha, e_k)E_\alpha\,, \qquad [E_\alpha, E_{-\alpha}] = H_\alpha\,,\\
[E_\alpha, E_\beta] = \begin{cases} N_{\alpha,\beta}E_{\alpha+\beta} & \text{for } \alpha+\beta \in \Delta\\ 0 & \text{for } \alpha+\beta \notin \Delta\cup\{0\}\,. \end{cases}
\end{gathered} \qquad (17)$$

If $J$ is a regular real element in $\mathfrak{h}$ then we may use it to introduce an ordering in $\Delta$ by saying that the root $\alpha \in \Delta_+$ is positive (negative) if $(\alpha, \vec{J}) > 0$ ($(\alpha, \vec{J}) < 0$ respectively). The normalization of the basis is determined by:

$$\begin{aligned} E_{-\alpha} &= E_\alpha^T\,, \qquad \langle E_{-\alpha}, E_\alpha\rangle = \frac{2}{(\alpha,\alpha)}\,, \\ N_{-\alpha,-\beta} &= -N_{\alpha,\beta}\,, \qquad N_{\alpha,\beta} = \pm(p+1)\,, \end{aligned} \tag{18}$$

where the integer $p \geq 0$ is such that $\alpha + s\beta \in \Delta$ for all $s = 1, \dots, p$ and $\alpha + (p+1)\beta \notin \Delta$. The root system $\Delta$ of $\mathfrak{g}$ is invariant with respect to the Weyl reflections $S_\alpha$; on the vectors $\vec{y} \in \mathbb{E}^r$ they act as

$$S_\alpha \vec{y} = \vec{y} - \frac{2(\alpha, \vec{y})}{(\alpha,\alpha)}\alpha, \qquad \alpha \in \Delta\,. \tag{19}$$

$S_\alpha$ generate the Weyl group $W_\mathfrak{g}$ and act on the Cartan–Weyl basis by:

$$\begin{aligned} S_\alpha(H_\beta) &\equiv A_\alpha H_\beta A_\alpha^{-1} = H_{S_\alpha\beta}\,, \\ S_\alpha(E_\beta) &\equiv A_\alpha E_\beta A_\alpha^{-1} = n_{\alpha,\beta} E_{S_\alpha\beta}\,, \qquad n_{\alpha,\beta} = \pm 1\,. \end{aligned} \tag{20}$$

In fact $W_\mathfrak{g}$ is the group of inner automorphisms of $\mathfrak{g}$ preserving the Cartan subalgebra $\mathfrak{h}$. The same property is possessed also by $\mathrm{Ad}_\mathfrak{h}$ automorphisms: choosing $C = \exp(\mathrm{i}\pi H_{\vec{c}})$ we get from (17):

$$CH_\alpha C^{-1} = H_\alpha\,, \qquad CE_\alpha C^{-1} = e^{2\pi \mathrm{i}(\alpha,\vec{c})/2} E_\alpha\,, \tag{21}$$

where $\vec{c} \in \mathbb{E}^r$ is the vector corresponding to $H_{\vec{c}} \in \mathfrak{h}$. Then the condition $C^2 = \mathbb{1}$ means that $(\alpha, \vec{c}) \in \mathbb{Z}$ for all $\alpha \in \Delta$.

## 3. Scattering Data and the $\mathbb{Z}_2$-reductions

In order to determine the scattering data of the Lax operator (2) we start with the Jost solutions

$$\lim_{x\to\infty} \psi(x,\lambda) e^{\mathrm{i}\lambda J x} = \mathbb{1}\,, \qquad \lim_{x\to-\infty} \phi(x,\lambda) e^{\mathrm{i}\lambda J x} = \mathbb{1}\,, \tag{22}$$

and the scattering matrix

$$T(\lambda) = (\psi(x,\lambda))^{-1}\phi(x,\lambda)\,. \tag{23}$$

Here we limit ourselves with the simplest nontrivial case when $J$ has real and pair-wise different eigenvalues, i. e. when $(a, \alpha_j) > 0$ for $j = 1, \dots, r$, see [8]. Since the classical papers of Zakharov and Shabat [7, 22] the most efficient way to solve the inverse scattering problem for $L(\lambda)$ is to construct the **fundamental analytic solutions** (FAS) $\chi^\pm(x,\lambda)$ of (2) and then to make use of the equivalent Riemann–Hilbert problem (RHP). To do this we have to use the Gauss decomposition of $T(\lambda)$:

$$T(\lambda) = T^-(\lambda)D^+(\lambda)\hat{S}^+(\lambda) = T^+(\lambda)D^-(\lambda)\hat{S}^-(\lambda)\,, \tag{24}$$

where 'hat' above denotes the inverse matrix $\hat{S} \equiv S^{-1}$ and

$$S^\pm(\lambda) = \exp \sum_{\alpha\in\Delta_+} s_\pm^{\pm\alpha}(\lambda)E_{\pm\alpha}\,, \quad T^\pm(\lambda) = \exp \sum_{\alpha\in\Delta_+} t_\pm^{\pm\alpha}(\lambda)E_{\pm\alpha}\,, \tag{25}$$

$$D^+(\lambda) = \exp\sum_{j=1}^{r} \frac{2d_j^+(\lambda)}{(\alpha_j,\alpha_j)} H_j\,, \quad D^-(\lambda) = \exp\sum_{j=1}^{r} \frac{2d_j^-(\lambda)}{(\alpha_j,\alpha_j)} H_j^-\,, \tag{26}$$

$$H_j \equiv H_{\alpha_j}\,, \quad H_j^- = w_0(H_j)\,.$$

Here the superscript $+$ (or $-$) in $D^\pm(\lambda)$ shows that $D_j^+(\lambda)$ (or $D_j^-(\lambda)$) are analytic functions of $\lambda$ for $\operatorname{Im}\lambda > 0$ (or $\operatorname{Im}\lambda < 0$) respectively and $w_0$ is the Weyl reflection that maps the highest weight $\omega_j^+$ in $R(\omega_j^+)$ into the lowest weight $\omega_j^-$ of $R(\omega_j^+)$ (see [21] for details). Then we can prove that

$$\chi^\pm(x,\lambda) = \phi(x,\lambda)S^\pm(\lambda) = \psi(x,\lambda)T^\mp(\lambda)D^\pm(\lambda) \tag{27}$$

are fundamental analytic solutions (FAS) of (2) for $\operatorname{Im}\lambda \gtrless 0$. On the real axis $\chi^+(x,\lambda)$ and $\chi^-(x,\lambda)$ are linearly related by

$$\chi^+(x,\lambda) = \chi^-(x,\lambda)G_0(\lambda)\,, \quad G_0(\lambda) = S^+(\lambda)\hat{S}^-(\lambda)\,, \tag{28}$$

and the sewing function $G_0(\lambda)$ may be considered as a minimal set of scattering data provided the Lax operator (2) has no discrete eigenvalues. The presence of discrete eigenvalues $\lambda_k^\pm$ means that some of the functions

$$D_j^\pm(\lambda) = \langle \omega_j^\pm \mid D^\pm(\lambda) \mid \omega_j^\pm\rangle = \exp\left(d_j^\pm(\lambda)\right)\,,$$

where $\omega_j^+$ are the fundamental weights of $\mathfrak{g}$ and $\omega_j^- = w_0(\omega_j^+)$, will have zeroes and poles at $\lambda_k^\pm$, for more details see [23, 19]. Equation (28) can be easily rewritten in the form:

$$\xi^+(x,\lambda) = \xi^-(x,\lambda)G(x,\lambda)\,, \quad G(x,\lambda) = \mathrm{e}^{-\mathrm{i}\lambda Jx}G_0(\lambda)\,\mathrm{e}^{\mathrm{i}\lambda Jx}\,. \tag{29}$$

Then (29) together with

$$\lim_{\lambda\to\infty} \xi^\pm(x,\lambda) = \mathbb{1} \tag{30}$$

can be considered as a RHP with canonical normalization condition.

The solution $\xi^+(x,\lambda)$, $\xi^-(x,\lambda)$ to (29), (30) is called regular if $\xi^+(x,\lambda)$ and $\xi^-(x,\lambda)$ are nondegenerate and non-singular functions of $\lambda$ for all $\operatorname{Im}\lambda > 0$

and $\operatorname{Im}\lambda < 0$ respectively. To the class of regular solutions of RHP there correspond Lax operators (2) without discrete eigenvalues. The presence of discrete eigenvalues $\lambda_k^{\pm}$ leads to singular solutions of the RHP; their explicit construction can be done by the Zakharov–Shabat dressing method [22], for the case of orthogonal algebras see also [19].

If the potential $Q(x,t)$ of the Lax operator (2) satisfies the $N$-wave equation (1) then $S^{\pm}(t,\lambda)$ and $T^{\pm}(t,\lambda)$ satisfy the linear evolution equations

$$\mathrm{i}\frac{\mathrm{d}S^{\pm}}{\mathrm{d}t} - \lambda[I, S^{\pm}(t,\lambda)] = 0\,, \qquad \mathrm{i}\frac{\mathrm{d}T^{\pm}}{\mathrm{d}t} - \lambda[I, T^{\pm}(t,\lambda)] = 0\,, \tag{31}$$

while the functions $D^{\pm}(\lambda)$ are time-independent. In other words $D_j^{\pm}(\lambda)$ can be considered as the generating functions of the integrals of motion of (1).

Each reduction on $L$ imposes restriction also on the scattering data. If $L$ satisfies (14) then the scattering matrix will satisfy

$$C_k\left(T(\Gamma_k(\lambda)) = T(\lambda)\,, \qquad \lambda \in \mathbb{R}\,. \tag{32}$$

Equation (32) is valid only for real values of $\lambda$. If the reduction is of the form (9), (11) and (12) then for the FAS and for the Gauss factors $S^{\pm}(\lambda)$, $T^{\pm}(\lambda)$ and $D^{\pm}(\lambda)$ we will get:

$$\begin{aligned} S^{+}(\lambda) &= A_1\left(\hat{S}^{-}(\lambda^*)\right)^{\dagger} A_1^{-1}\,, & T^{+}(\lambda) &= A_1\left(\hat{T}^{-}(\lambda^*)\right)^{\dagger\dagger} A_1^{-1}\,, \\ D^{+}(\lambda) &= A_1\left(\hat{D}^{-}(\lambda^*)\right)^{\dagger} A_1^{-1}\,, & F(\lambda) &= A_1\left(F(\lambda^*)\right)^{\dagger} A_1^{-1}\,, \end{aligned} \tag{33}$$

$$\begin{aligned} S^{+}(\lambda) &= A_2 S^{-}(-\lambda) A_2^{-1}\,, & T^{+}(\lambda) &= A_2 T^{-}(-\lambda) A_2^{-1}\,, \\ D^{+}(\lambda) &= A_2 D^{-}(-\lambda) A_2^{-1}\,, & F(\lambda) &= A_2 F(-\lambda) A_2^{-1}\,, \end{aligned} \tag{34}$$

$$\begin{aligned} S^{\pm}(\lambda) &= A_3\left(S^{\pm}(-\lambda^*)\right)^{*} A_3^{-1}\,, & T^{\pm}(\lambda) &= A_3\left(T^{\pm}(-\lambda^*)\right)^{T} A_3^{-1}\,, \\ D^{\pm}(\lambda) &= A_3\left(D^{\pm}(-\lambda^*)\right)^{*} A_3^{-1}\,, & F(\lambda) &= A_3\left(F(-\lambda^*)\right)^{*} A_3^{-1}\,, \end{aligned} \tag{35}$$

where $A_1$ and $A_3$ are assumed to be elements of the Cartan subgroup of $\mathfrak{G}$ while $A_2$ corresponds to the $w_0$ element in the Weyl group.

We will also make use of the integral representations for $d_j^{\pm}(\lambda)$ allowing one to reconstruct them as analytic functions in their regions of analyticity $\mathbb{C}_{\pm}$. In the case of absence of discrete eigenvalues we have [8, 11]:

$$\mathcal{D}_j(\lambda) = \frac{\mathrm{i}}{2\pi}\int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\mu}{\mu-\lambda} \ln\langle\omega_j^{+}|\hat{T}^{+}(\mu)T^{-}(\mu)|\omega_j^{+}\rangle\,, \tag{36}$$

where $|\omega_j^{+}\rangle$ is the highest weight vector in the corresponding fundamental representation $R(\omega_j^{+})$ of $\mathfrak{g}$. The function $\mathcal{D}_j(\lambda)$ as a fraction-analytic function of

$\lambda$ is equal to:

$$\mathcal{D}_j(\lambda) = \begin{cases} d_j^+(\lambda), & \text{for } \lambda \in \mathbb{C}_+ \\ (d_j^+(\lambda) - d_{j'}^-(\lambda))/2, & \text{for } \lambda \in \mathbb{R}, \\ -d_{j'}^-(\lambda), & \text{for } \lambda \in \mathbb{C}_-, \end{cases} \tag{37}$$

where $d_j^{\pm}(\lambda)$ were introduced in (26) and the index $j'$ is related to $j$ by $w_0(\alpha_j) = -\alpha_{j'}$. The functions $\mathcal{D}_j(\lambda)$ can be viewed also as generating functions of the integrals of motion. Indeed, if we expand

$$\mathcal{D}_j(\lambda) = \sum_{k=1}^{\infty} \mathcal{D}_{j,k}\lambda^{-k}, \tag{38}$$

and take into account that $D^{\pm}(\lambda)$ are time independent we find that $\mathrm{d}\mathcal{D}_{j,k}/\mathrm{d}t = 0$ for all $k = 1, \ldots, \infty$ and $j = 1, \ldots, r$. Moreover it can be checked that $\mathcal{D}_{j,k}$ expressed as functionals of $q(x,t)$ has kernel that is local in $q$, i. e. depends only on $q$ and its derivatives with respect to $x$.

From (36) and (33–35) we easily obtain the effect of the reductions on the set of integrals of motion:

$$\mathcal{D}_j(\lambda) = -\mathcal{D}_j^*(\lambda^*), \quad \text{i. e.} \quad \mathcal{D}_{j,k} = -\mathcal{D}_{j,k}^*, \tag{39}$$

$$\mathcal{D}_j(\lambda) = -\mathcal{D}_j(-\lambda), \quad \text{i. e.} \quad \mathcal{D}_{j,k} = (-1)^{k+1}\mathcal{D}_{j,k}, \tag{40}$$

$$\mathcal{D}_j(\lambda) = \mathcal{D}_j^*(-\lambda^*), \quad \text{i. e.} \quad \mathcal{D}_{j,k} = (-1)^k\mathcal{D}_{j,k}^*. \tag{41}$$

for the reductions (33), (34) and (35) respectively.

In particular from (40) it follows that al integrals of motion with even $k$ become degenerate, i. e. $\mathcal{D}_{j,2k} = 0$. The reduction (39) means that the integrals $\mathcal{D}_{j,k}$ become purely imaginary. Finally, if we have chosen the reduction (35) from (41) it follows that $\mathcal{D}_{j,2k}$ are real while $\mathcal{D}_{j,2k+1}$ are purely imaginary.

We finish this section with a few comments on the simplest local integrals of motion. To this end we write down the first two types of integrals of motion $\mathcal{D}_{j,1}$ and $\mathcal{D}_{j,2}$ as functionals of the potential $Q$ of (2). Skipping the details (see [8]) we get:

$$\mathcal{D}_{j,1} = -\frac{\mathrm{i}}{4}\int_{-\infty}^{\infty} \mathrm{d}x\, \langle [J,Q], [H_j^{\vee}, Q]\rangle, \tag{42}$$

$$\mathcal{D}_{j,2} = -\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}x\, \langle Q, [H_j^{\vee}, Q_x]\rangle - \frac{\mathrm{i}}{3}\int_{-\infty}^{\infty} \mathrm{d}x\, \langle [J,Q], [Q, [H_j^{\vee}, Q]]\rangle, \tag{43}$$

where $H_j^{\vee} = 2H_{\omega_j}/(\alpha_j, \alpha_j)$.

The fact that $\mathcal{D}_{j,1}$ are integrals of motion for $j = 1, \dots, r$, can be considered as natural analog of the Manley–Rowe relations [1, 3]. In the case when the reduction is of the type (9), i. e. $p_\alpha = s_\alpha q_\alpha^*$ then (42) is equivalent to

$$\sum_{\alpha>0} \frac{2(\vec{a},\alpha)(\omega_j,\alpha)}{(\alpha,\alpha)} \int_{-\infty}^{\infty} \mathrm{d}x\, s_\alpha |q_\alpha(x)|^2 = \mathrm{const}\,, \tag{44}$$

and can be interpreted as relations between the densities $|q_\alpha|^2$ of the 'particles' of type $\alpha$. For the other types of reductions such interpretation is not so obvious.
The integrals of motion $\mathcal{D}_{j,2}$ are related directly to the Hamiltonian of the $N$-wave equations (1), namely:

$$H_{N-w} = -\sum_{j=1}^{r} \frac{2(\alpha_j,\vec{b})}{(\alpha_j,\alpha_j)} \mathcal{D}_{j,2} = \frac{1}{2\mathrm{i}} \left\langle\left\langle \dot{\mathcal{D}}(\lambda)\,,\, F(\lambda) \right\rangle\right\rangle_0\,, \tag{45}$$

where $\dot{\mathcal{D}}(\lambda) = \mathrm{d}\mathcal{D}/\mathrm{d}\lambda$ and $F(\lambda) = \lambda I$ is the dispersion law of the $N$-wave equation (1). In (45) we used just one of the hierarchy of scalar products in the Kac–Moody algebra (see [24]) $\widehat{\mathfrak{g}} \equiv \mathfrak{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$:

$$\langle\langle X(\lambda), Y(\lambda)\rangle\rangle_k = \operatorname{Res} \lambda^{k+1} \left\langle \hat{D}^+(\lambda) X(\lambda), Y(\lambda) \right\rangle, \quad X(\lambda), Y(\lambda) \in \widehat{\mathfrak{g}}. \tag{46}$$

## 4. Example: $N$-wave Systems Related to $B_2$-algebra

Let us illustrate these general results by an example related to the $B_2$ algebra. This algebra has two simple roots $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2$, and two more positive roots: $\alpha_1 + \alpha_2 = e_1$ and $\alpha_1 + 2\alpha_2 = e_1 + e_2 = \alpha_{\max}$. When they come as indices, e. g. in $q_\alpha$, we will replace them by sequences of two integers: $\alpha \to kn$ if $\alpha = k\alpha_1 + n\alpha_2$; if $\alpha = -(k\alpha_1 + n\alpha_2)$ we will use $\overline{kn}$.
The reduction $KU^\dagger(\lambda^*)K^{-1} = U(\lambda)$ where $K$ is an element of the Cartan subgroup with $K = \operatorname{diag}(s_1, s_2, 1, s_2, s_1)$ and $s_k = \pm 1$, $k = 1, 2$, extracts the real forms of $B_2 \simeq so(5)$. So $a_i = a_i^*$, $i = 1, 2$ and $q_\alpha$ must satisfy:

$$p_{10} = -s_2 s_1 q_{10}^*\,, \quad p_{01} = -s_2 q_{01}^*\,, \quad p_{11} = -s_1 q_{11}^*\,, \quad p_{12} = -s_1 s_2 q_{12}^*\,. \tag{47}$$

Thus we get 4-wave system which is described by the Hamiltonian $H = H_0 + H_{\mathrm{int}}$ with:

$$\begin{aligned} H_0 = \frac{\mathrm{i}}{2} \int_{-\infty}^{\infty} \mathrm{d}x\, \Big[ &(b_1 - b_2)(q_{10} q_{10,x}^* - q_{10,x} q_{10}^*) + 2b_2 (q_{01} q_{01,x}^* - q_{01,x} q_{01}^*) \\ &+ 2b_1 (q_{11} q_{11,x}^* - q_{11,x} q_{11}^*) + (b_1 + b_2)(q_{12} q_{12,x}^* - q_{12,x} q_{12}^*) \Big] \end{aligned} \tag{48}$$

$$H_{\mathrm{int}} = 2\kappa s_1 \int\limits_{-\infty}^{\infty} \mathrm{d}x \, \left[ s_2(q_{12}q_{11}^* q_{01}^* + q_{12}^* q_{11} q_{01}) + (q_{11}q_{01}^* q_{10}^* + q_{11}^* q_{01} q_{10}) \right] \,,$$

where $\kappa = a_1 b_2 - a_2 b_1$, and the symplectic 2-form:

$$\begin{aligned} \Omega^{(0)} = \mathrm{i} \int\limits_{-\infty}^{\infty} \mathrm{d}x \, [(a_1 - a_2)\delta q_{10} \wedge \delta q_{10}^* + 2a_2 \delta q_{01} \wedge \delta q_{01}^* \\ + 2a_1 \delta q_{11} \wedge \delta q_{11}^* + (a_1 + a_2)\delta q_{12} \wedge \delta q_{12}^*] \,, \end{aligned} \tag{49}$$

The corresponding wave-decay diagram is shown in Fig. 1.

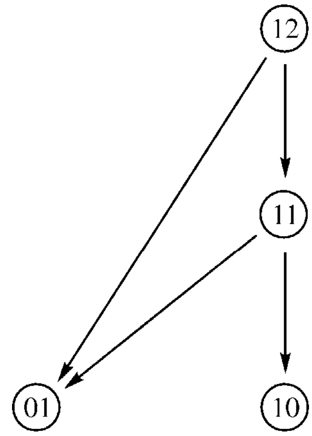


**Figure 1.** Wave-decay diagram for the $so(5)$ algebra To each positive root of the algebra $\underline{kn} \equiv k\alpha_1 + n\alpha_2$ we put in correspondence a wave of type $\underline{kn}$. If the positive root $\underline{kn} = \underline{k'n'} + \underline{k''n''}$ can be represented as a sum of two other positive roots, we say that the wave $\underline{kn}$ decays into the waves $\underline{k'n'}$ and $\underline{k''n''}$.

The particular case $s_1 = s_2 = 1$ leads to $N$-wave equations on the compact real form $so(5,0) \simeq so(5,\mathbb{R})$ of the $B_2$-algebra, see also [19, 25]. The choices $s_1 = -s_2 = -1$ and $s_1 = s_2 = -1$ lead to $N$-wave equations on the noncompact real forms $so(2,3)$ and $so(1,4)$ respectively.

Let us apply a second $\mathbb{Z}_2$-reduction to the already reduced system of the previous subsection. We take it in the form $w_0(U(-\lambda)) = U(\lambda)$ which gives $a_i = a_i^*$, $b_i = b_i^*$ and:

$$q_{10}^* = -s_1 s_2 q_{10} \,, \quad q_{01}^* = -s_2 q_{01} \,, \quad q_{11}^* = -s_1 q_{11} \,, \quad q_{12}^* = -s_1 s_2 q_{12} \,. \tag{50}$$

This gives the following 4-wave system for 4 real-valued functions:

$$\begin{aligned} \mathrm{i}(a_1 - a_2)q_{10,t} - \mathrm{i}(b_1 - b_2)q_{10,x} + 2\kappa q_{11} q_{01} = 0 \,, \\ \mathrm{i}a_2 q_{01,t} - \mathrm{i}b_2 q_{01,x} + \kappa(q_{11}q_{12} + q_{11}q_{10}) = 0 \,, \\ \mathrm{i}a_1 q_{11,t} - \mathrm{i}b_1 q_{11,x} + \kappa(q_{12}q_{01} - q_{10}q_{01}) = 0 \,, \\ \mathrm{i}(a_1 + a_2)q_{12,t} - \mathrm{i}(b_1 + b_2)q_{12,x} - 2\kappa q_{11} q_{01} = 0 \,. \end{aligned} \tag{51}$$

Since $w_0(J) = -J$ the Hamiltonian structure $\{H^{(0)}, \Omega^{(0)}\}$ becomes degenerated and we must consider the next Hamiltonian structure in the hierarchy.

It is known that the $j$-type discrete eigenvalues of $L$ are located at the zeroes $\lambda_k^\pm \in \mathbb{C}_\pm$ of the functions $D_j^\pm(\lambda)$ [8, 19]. If we assume that $L$ has only two eigenvalues $\lambda_1^\pm$, of type $j$ then we can write

$$D_j^+(\lambda) = \frac{\lambda - \lambda_1^+}{\lambda - \lambda_1^-} \tilde{D}_j^+(\lambda) , \qquad D_j^-(\lambda) = \frac{\lambda - \lambda_1^-}{\lambda - \lambda_1^+} \tilde{D}_j^-(\lambda) , \tag{52}$$

where $\tilde{D}_j^\pm(\lambda)$ have no zeroes in $\mathbb{C}_\pm$. Then the first reduction which is of the type (33) ensures that the eigenvalues must be pair-wise complex conjugate, i. e. $\lambda_1^- = (\lambda_1^+)^*$. The second reduction of the type (34) leads to $\lambda_1^- = -\lambda_1^+$. Therefore if $L$ has only two eigenvalues of type $j$ and both reductions are imposed this means that $\lambda_1^\pm = \pm \mathrm{i}\zeta_1$ where $\zeta_1 > 0$ is a positive real number. However, if $L$ has two pairs of eigenvalues $\lambda_k^\pm$, $k = 1, 2$ there is another nontrivial way to satisfy both reductions simultaneously:

$$\lambda_1^\pm = \mu_1 \pm \mathrm{i}\zeta_1 , \qquad \lambda_2^\pm = -\mu_1 \pm \mathrm{i}\zeta_1 ,$$

where $\mu_1$, $\zeta_1$ are real positive numbers. Therefore when both reductions are effective the operator $L$ may have two different types of eigenvalue configurations and to each such configuration there corresponds a reflectionless potential for $L$ and soliton solution for the $N$-wave system.

Such configurations have been well known for the sine-Gordon equation [4, 5] where we have: (i) topological solitons related to the purely imaginary eigenvalues $\pm \mathrm{i}\zeta_k$ and (ii) the breathers related to the quadruplets of eigenvalues.

## 5. Hierarchy of Hamiltonian Structures of $N$-wave Equations and Reductions

The generic $N$-wave interactions (i. e., prior to any reductions) possess a hierarchy of Hamiltonian structures which is generated by the so-called generating (or recursion) operator $\Lambda = (\Lambda_+ + \Lambda_-)/2$ [8]:

$$\begin{aligned} \Lambda_\pm Z(x) = \mathrm{ad}_J^{-1} \biggl( \mathrm{i} \frac{\mathrm{d}Z}{\mathrm{d}x} &+ P_0 \cdot ([q(x), Z(x)]) \\ &+ \mathrm{i}\, [q(x), I_\pm \,(\mathbb{1} - P_0)\, [q(y), Z(y)]] \biggr), \end{aligned}$$

$$P_0 S \equiv \mathrm{ad}_J^{-1} \cdot \mathrm{ad}_J \cdot S , \quad (I_\pm S)(x) \equiv \int_{\pm\infty}^{x} \mathrm{d}y\, S(y) , \tag{53}$$

where $q(x,t) = [J, Q(x,t)]$. The hierarchy of symplectic forms is given by:

$$\Omega^{(k)} = \frac{\mathrm{i}}{2} \int\limits_{-\infty}^{\infty} \mathrm{d}x \left\langle [J, \delta Q(x,t)] \underset{,}{\wedge} \Lambda^k \delta Q(x,t) \right\rangle , \tag{54}$$

Using the completeness relation for the "squared" solutions which is directly related to the spectral decomposition of $\Lambda$ we can recalculate $\Omega^{(k)}$ in terms of the scattering data of $L$ with the result [8]:

$$\begin{aligned} \Omega^{(k)} &= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \mathrm{d}\lambda\, \lambda^k \left( \Omega_0^+(\lambda) - \Omega_0^-(\lambda) \right) , \\ \Omega_0^{\pm}(\lambda) &= \left\langle \hat{D}^{\pm}(\lambda) \hat{T}^{\mp}(\lambda) \delta T^{\mp}(\lambda) D^{\pm}(\lambda) \underset{,}{\wedge} \hat{S}^{\pm}(\lambda) \delta S^{\pm}(\lambda) \right\rangle . \end{aligned} \tag{55}$$

Therefore the kernels of $\Omega^{(k)}$ differs only by the factor $\lambda^k$; i. e., all of them can be cast into canonical form simultaneously. This is quite compatible with the results of [1, 2, 9] for the action-angle variables.

Again it is not difficult to find how the reductions influence $\Omega^{(k)}$. Using the invariance of the Killing form, from (55) and (33–35) we get:

$$\Omega_0^+(\lambda) = \left( \Omega_0^-(\lambda^*) \right)^* , \tag{56}$$

$$\Omega_0^+(\lambda) = \Omega_0^-(-\lambda), \tag{57}$$

$$\Omega_0^{\pm}(\lambda) = \left( \Omega_0^{\pm}(-\lambda^*) \right)^* . \tag{58}$$

Then for $\Omega^{(k)}$ from (33), (34) and (35) we obtain:

$$\Omega^{(k)} = -\left( \Omega^{(k)} \right)^* , \tag{59}$$

$$\Omega^{(k)} = (-1)^{k+1} \Omega^{(k)} , \tag{60}$$

$$\Omega^{(k)} = (-1)^k \left( \Omega^{(k)} \right)^* . \tag{61}$$

respectively. Like for the integrals $\mathcal{D}_{j,k}$ we find that the reductions (33) and (35) mean that each $\Omega^{(k)}$ can be made real with a proper choice of the constant $c_0$ in (8).

Let us now briefly analyze the reduction (34) which may lead to degeneracies. We already mentioned that $\mathcal{D}_{j,2k} = 0$, see (40); in addition from (60) it follows that $\Omega^{(2k)} \equiv 0$. In particular this means that the canonical 2-form $\Omega^{(0)}$ is also degenerated, so the $N$-wave equations with the reduction (34) do not allow

Hamiltonian formulation with canonical Poisson brackets. However they still possess a hierarchy of Hamiltonian structures:

$$\Omega^{(k)}\left(\frac{\mathrm{d}q}{\mathrm{d}t},\cdot\right) = \nabla_q H^{(k+1)}\,, \tag{62}$$

where $\nabla_q H^{(k+1)} = \Lambda \nabla_q H^{(k)}$; by definition $\nabla_q H = (\delta H)/(\delta q^T(x,t))$. Thus we find that while the choices $\{\Omega^{(2k)}, H^{(2k)}\}$ for the $N$-wave equations are degenerated, the choices $\{\Omega^{(2k+1)}, H^{(2k+1)}\}$ provide us with correct nondegenerated (though non-canonical) Hamiltonian structures, see [8, 11, 13].

## 6. Conclusion

Here we have analyzed how can be imposed one or two $\mathbb{Z}_2$-reductions on the $N$-wave type equations related to the simple Lie algebras and what will be the consequences of these reductions to the Hamiltonian structures and to the structure of their soliton solutions. A list of all nontrivial $\mathbb{Z}_2$-reductions for the low-rank simple Lie algebras (rank less than 4) can be found in [18]. The reductions that lead to a real forms of $\mathfrak{g}$ are discussed in [20]. The classification of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-reductions is under investigation. We note also that the explicit construction of the dressing factors for the symplectic and orthogonal algebras requires modifications of the Zakharov–Shabat dressing method [19]. This leads to new types of reflectionless potentials and soliton solutions.

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