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# ON THE BIANCHI IDENTITIES IN A GENERALIZED WEYL SPACE*

GÜLÇIN ÇIVI

*Faculty of Science and Letters, Istanbul Technical University*
*80626 Maslak-Istanbul, Turkey*

**Abstract**. In this paper, we show that the first Bianchi identity is valid for a generalized Weyl space having a semi-symmetric $E$-connection and that the second Bianchi identity is satisfied for a recurrent generalized Weyl space provided that the recurrence vector $\psi_l$ and the Vranceanu vector $\Omega_l$ are related by $\psi_l = \dfrac{2}{n-1}\Omega_l$.

## 1. Introduction

An $n$-dimensional differentiable manifold $W_n^*$ having an asymmetric connection $\nabla^*$ and asymmetric conformal metric tensor $g^*$ preserved by $\nabla^*$ is called a **generalized Weyl space** [1]. For a such a space, in local coordinates, we have the compatibility condition

$$\nabla_k^* g_{ij}^* - 2T_k^* g_{ij}^* = 0\,, \tag{1.1}$$

where $T_k^*$ are the components of a covariant vector field called the complementary vector field of the generalized Weyl space.

The coefficients $L_{jk}^i$ of the connection $\nabla^*$ are obtained from the compatibility condition as [2]

$$L_{jk}^i = \Gamma_{jk}^i + \frac{1}{2}\Big[\Omega_{kl}^h g_{(jh)}^* + \Omega_{jl}^h g_{(hk)}^* + \Omega_{jk}^h g_{(hl)}^*\Big] g^{*(li)} \tag{1.2}$$

or, putting

$$Q_{jk}^i = \frac{1}{2}\Big[\Omega_{kl}^h g_{(jh)}^* + \Omega_{jl}^h g_{(hk)}^* + \Omega_{jk}^h g_{(hl)}^*\Big] g^{*(li)} \tag{1.3}$$

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we have

$$L^i_{jk} = \Gamma^i_{jk} + Q^i_{jk} \tag{1.4}$$

where $\Gamma^i_{jk}$ and $\Omega^i_{jk}$ are, respectively, the coefficients of a Weyl connection and the torsion tensor of $W_n^*$ given by

$$\Gamma^i_{jk} = \left\{ \begin{array}{c} i \\ jk \end{array} \right\} - (\delta^i_j T_k + \delta^i_k T_j - g^{li} g_{jk} T_l)\,, \tag{1.5}$$

and

$$\Omega^i_{jk} = L^i_{jk} - L^i_{kj} = 2L^i_{[jk]}\,. \tag{1.6}$$

According to Norden [3], if under a renormalization of the fundamental tensor $g$ of the form $\tilde{g} = \lambda^2 g$, an object $A$ admitting a transformation of the form $\tilde{A} = \lambda^p A$ is called a **satellite with weight** $\{p\}$ of the tensor $g$. The **prolonged covariant derivative** of the satellite $A$ relative to the symmetric connection $\nabla$, denoted by $\dot{\nabla} A$, is defined by [4]

$$\dot{\nabla}_k A = \nabla_k A - pT_k A\,. \tag{1.7}$$

The prolonged covariant derivative of the satellite $A$ relative to $\nabla^*$ will be denoted by $\dot{\nabla}^*_k A$, is defined by

$$\dot{\nabla}^*_k A = \nabla^*_k A - pT^*_k A\,. \tag{1.8}$$

## 2. Bianchi Identities

Let $v^i$ be the contravariant components of vector field $\boldsymbol{v}$ in $W_n^*$. For the second order covariant derivative of $\boldsymbol{v}$ relative to $\nabla^*$ we have

$$\begin{aligned} \nabla^*_k \nabla^*_l v^i = {} & \partial_k \partial_l v^i + (\partial_k L^i_{hl}) v^h + (\partial_k v^h) L^i_{hl} \\ & + L^i_{jk} \partial_l v^j + L^i_{jk} L^j_{hl} v^h - L^j_{lk} (\partial_j v^i) - L^j_{lk} L^i_{hj} v^h\,. \end{aligned} \tag{2.1}$$

Interchanging the indices $k$ and $l$ in (2.1) we obtain

$$\begin{aligned} \nabla^*_l \nabla^*_k v^i = {} & \partial_l \partial_k v^i + (\partial_l L^i_{hk}) v^h + (\partial_l v^h) L^i_{hk} + L^i_{jl} \partial_k v^j \\ & + L^i_{jl} L^j_{hk} v^h - L^j_{kl} (\partial_j v^i) - L^j_{kl} L^i_{hj} v^h\,. \end{aligned} \tag{2.2}$$

Substracting (2.2) from (2.1) we get

$$\nabla^*_k \nabla^*_l v^i - \nabla^*_l \nabla^*_k v^i = L^i_{hkl} v^h + \Omega^j_{kl} \nabla^*_j v^i\,, \tag{2.3}$$

where

$$L^l_{ijk} = \partial_j L^l_{ik} - \partial_k L^l_{ij} + L^h_{ik} L^l_{hj} - L^h_{ij} L^l_{hk}\,. \tag{2.4}$$

This is the curvature tensor corresponding to the connection $\nabla^*$.
By cyclic permutation of $i, j$ and $k$ in (2.4)

$$L^l_{jki} = \partial_k L^l_{ji} - \partial_i L^l_{jk} + L^h_{ji} L^l_{hk} - L^h_{jk} L^l_{hi}\,, \tag{2.5}$$

$$\mathrm{L}^l_{kij} = \partial_i L^l_{kj} - \partial_j L^l_{ki} + L^h_{kj} L^l_{hi} - L^h_{ki} L^l_{hj} \tag{2.6}$$

and summing (2.4), (2.5) and (2.6) side by side we obtain

$$\begin{aligned} L^l_{ijk} + L^l_{jki} + L^l_{kij} &= \partial_k \Omega^l_{ji} + \partial_i \Omega^l_{kj} + \partial_j \Omega^l_{ik} \\ &\quad + L^l_{hk}\Omega^h_{ji} + L^l_{hi}\Omega^h_{kj} + L^l_{hj}\Omega^h_{ik} \end{aligned} \tag{2.7}$$

showing that the first Bianchi identity is not satisfied in general.
If the connection $\nabla^*$ is semi-symmetric, i. e. if

$$\Omega^i_{jk} = \frac{1}{n-1}(\delta^i_j \Omega_k - \delta^i_k \Omega_j)\,, \tag{2.8}$$

the identity (2.7) reduces to

$$\begin{aligned} L^l_{ijk} + L^l_{jki} + L^l_{kij} = \frac{1}{n-1}\Big[&\delta^l_j(\Omega_{i,k} - \Omega_{k,i}) + \delta^l_k(\Omega_{j,i} - \Omega_{i,j}) \\ &+ \delta^l_i(\Omega_{k,j} - \Omega_{j,k})\Big]\,, \qquad n \neq 1 \end{aligned} \tag{2.9}$$

where $\Omega_j = \Omega^i_{ij}$ is the Vranceanu vector of the connection $\nabla^*$ and $\Omega_{i,k} = \dfrac{\partial \Omega_i}{\partial u^k}$.
From this we obtain the

**Theorem 2.1.** *If the Vranceanu vector is a gradient then the first Bianchi identity is satisfied.*

**Definition.** The connection $\nabla^*$ is said to be a $E$-connection if the condition

$$\nabla^*_k \Omega_i - \nabla^*_i \Omega_k = 0 \tag{2.10}$$

holds [5].

**Theorem 2.2.** *For a generalized Weyl space having a semi-symmetric $E$-connection the first Bianchi identity is satisfied.*

**Proof:** The covariant derivative of the Vranceanu vector $\Omega_i$ with respect to the coordinates $u^k$ is

$$\nabla^*_k \Omega_i = \frac{\partial \Omega_i}{\partial u^k} - L^h_{ik}\Omega_h\,. \tag{2.11}$$

Substracting from (2.11) the equation is obtained by interchanging the indices $i$ and $k$ we find that

$$\nabla_k^* \Omega_i - \nabla_i^* \Omega_k = \frac{\partial \Omega_i}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^i} + 2\Omega_{ki}^h \Omega_h \,. \tag{2.12}$$

On the other hand, for the generalized Weyl space $W_n^*$ with semi-symmetric $E$-connection (2.12) is reduced to

$$\frac{\partial \Omega_i}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^i} = 0 \,. \tag{2.13}$$

Using (2.9), from (2.13) we get

$$L_{ijk}^l + L_{jki}^l + L_{kij}^l = 0$$

so the proof is completed. □

**Corollary 2.1.** *For a generalized Weyl space having a semi-symmetric connection, the Vranceanu vector and the complementary vector are related by*

$$L_{[ik]} - nT_{[i,k]} = \frac{n-2}{n-1}\Omega_{[i,k]} \qquad (n \neq 1)$$

*where* $L_{ik}$ *denotes the Ricci tensor of* $W_n^*$.

**Proof:** If, in (2.9) a contraction on $l$ and $j$ is made we have

$$L_{ilk}^l + L_{lki}^l + L_{kil}^l = \frac{n-2}{n-1}\left(\Omega_{i,k} - \Omega_{k,i}\right) \tag{2.14}$$

from which we get

$$L_{[ik]} - nT_{[i,k]}^* = \frac{n-2}{n-1}\Omega_{[i,k]} \,, \qquad \left(T_{i,k}^* = \frac{\partial T_i^*}{\partial u^k} \,, \quad \Omega_{i,k} = \frac{\partial \Omega_i}{\partial u^k}\right)$$

where we have used the facts that

$$L_{ilk}^l = L_{ik} \,, \qquad L_{kil}^l = -L_{ki}$$

so the proof is completed. □

The prolonged covariant derivative of the curvature tensor $L_{ijk}^h$, of weight $\{0\}$, is

$$\begin{aligned} \dot{\nabla}_l^* L_{ijk}^h &= \nabla_l^* L_{ijk}^h \\ &= \partial_l L_{ijk}^h + L_{ml}^h L_{ijk}^m - L_{il}^m L_{mjk}^h - L_{jl}^m L_{imk}^h - L_{kl}^m L_{ijm}^h \,. \end{aligned} \tag{2.15}$$

If the indices $j$, $k$ and $l$ are changed cyclically in (2.15) the equations

$$\dot{\nabla}_j^* L_{ikl}^h = \nabla_j^* L_{ikl}^h = \partial_j L_{ikl}^h + L_{mj}^h L_{ikl}^m - L_{ij}^m L_{mkl}^h - L_{kj}^m L_{iml}^h - L_{lj}^m L_{ikm}^h \tag{2.16}$$

and

$$\dot{\nabla}_k^* L_{ilj}^h = \nabla_k^* L_{ilj}^h = \partial_k L_{ilj}^h + L_{mk}^h L_{ilj}^m - L_{ik}^m L_{mlj}^h - L_{lk}^m L_{imj}^h - L_{jk}^m L_{ilm}^h \quad (2.17)$$

are obtained respectively.

Summing (2.15), (2.16) and (2.17) we get

$$\dot{\nabla}_l^* L_{ijk}^h + \dot{\nabla}_j^* L_{ikl}^h + \dot{\nabla}_k^* L_{ilj}^h = \Omega_{lj}^m L_{imk}^h + \Omega_{jk}^m L_{iml}^h + \Omega_{kl}^m L_{imj}^h \, . \quad (2.18)$$

This shows that the second Bianchi identity is not valid in $W_n^*$.

The generalized Weyl space $W_n^*$ is called recurrent if its curvature tensor $L_{ijk}^l$ satisfies the condition

$$\dot{\nabla}_l^* L_{ijk}^h = \psi_l L_{ijk}^h \quad (2.19)$$

where $\psi_l$ is a 1-form called the recurrence vector of $W_n^*$.

Let $W_n^*$ be a recurrent generalized Weyl space having a semi-symmetric connection. Then, using (2.8) and (2.19), the identity (2.18) is reduced to

$$a_l L_{ijk}^h + a_j L_{ikl}^h + a_k L_{ilj}^h = 0 \, . \quad (2.20)$$

where

$$a_l = \psi_l - \frac{2}{n-1} \Omega_l \, .$$

Thus we proved the

**Theorem 2.3.** *For a generalized recurrent Weyl space having a semi-symmetric connection the second Bianchi identity is satisfied provided that the recurrence vector* $\psi_l$ *and the Vranceanu vector* $\Omega_l$ *are related by* $\psi_l = \dfrac{2}{n-1}\Omega_l$, $\psi_l \neq \Omega_l$.

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