Chapter 4

Oscillatory integrals with convexity

As discussed in Section 1.2, in the case of homogeneous m^{th} order strictly hyperbolic operators, geometric properties of the characteristic roots play the fundamental role in determining the $L^p - L^q$ decay; in particular, if the characteristic roots satisfy the convexity condition of Definition 1.2.1, then the decay is, in general, more rapid than when they do not. We will show that a similar improvement can be obtained for operators with lower order terms when a suitable 'convexity condition' holds. In Section 4.3, we shall extend this notion of the convexity condition to functions $\tau : \mathbb{R}^n \to \mathbb{R}$ and prove a decay estimate for an oscillatory integral (related to the solution representation for a strictly hyperbolic operator) with phase function τ.

First, we give a general result for oscillatory integrals and show how the concept of functions of "convex type" allows its application to derive the time decay.

4.1 Estimates for oscillatory integrals

The following theorem is central in proving results involving convexity conditions. In some sense, it bridges the gap between the man der Corput Lemma and the method of stationary phase, in that the former is used when there is no convexity but gives a weaker result, while the latter can be used when a stronger condition than simply convexity holds and gives a better result. Here, we state and prove a result that has no reference to convexity; however, in the following section, we show how convexity (in some sense) enables this result to be used in applications. An earlier version of this result has appeared in [Ruzh07], with applications to equations with time dependent homogeneous symbols in [MR09]. For completeness we also include a more detailed proof here.

Theorem 4.1.1. *Consider the oscillatory integral*

$$I(\lambda,\nu)=\int_{\mathbb{R}^N} e^{i\lambda\Phi(y,\nu)}A(y,\nu)g(y)\,dy\,, \tag{4.1.1}$$

where $N\in\mathbb{N}$, $I:[0,\infty)\times\mathcal{N}\to\mathbb{C}$, $\mathcal{N}$ *is any set of parameters* ν, *and*

(I1) *there exists a bounded open set* $U\subset\mathbb{R}^N$ *such that* $g\in C_0^\infty(U)$;

(I2) $\Phi(y,\nu)$ *is a complex-valued function such that* $\operatorname{Im}\Phi(y,\nu)\geq 0$ *for all* $y\in U$, $\nu\in\mathcal{N}$;

(I3) *for some fixed* $z\in\mathbb{R}^N$, *some* $\delta>0$, *and some* $\gamma\in\mathbb{N}$, $\gamma\geq 2$, *the function*

$$F(\rho,\omega,\nu):=\Phi(\rho\omega+z,\nu)$$

satisfies

$$|\partial_\rho F(\rho,\omega,\nu)|\geq C\rho^{\gamma-1} \text{ and } |\partial_\rho^m F(\rho,\omega,\nu)|\leq C_m\rho^{1-m}|\partial_\rho F(\rho,\omega,\nu)|$$

for all $(\omega,\nu)\in\mathbb{S}^{N-1}\times\mathcal{N}$, *all integers* $1\leq m\leq[N/\gamma]+1$ *and all* $\rho>0$, *for which* $\rho\omega+z\in U$;

(I4) *for each multi-index* α *such that* $|\alpha|\leq\left[\frac{N}{\gamma}\right]+1$, *there exists a constant* $C_\alpha>0$ *such that* $|\partial_y^\alpha A(y,\nu)|\leq C_\alpha$ *for all* $y\in U$, $\nu\in\mathcal{N}$.

Then there exists a constant $C=C_{N,\gamma}>0$ *such that*

$$|I(\lambda,\nu)|\leq C(1+\lambda)^{-\frac{N}{\gamma}} \quad \text{for all } \lambda\in[0,\infty),\ \nu\in\mathcal{N}\,. \tag{4.1.2}$$

Constant C in (4.1.2) is independent of λ and ν.

Remark 4.1.2. *This theorem extends to the case where* $A(y,\nu)$ *is replaced by* $A(y,\nu')$, *where* ν' *may be independent of the variable* ν *appearing in the phase function* $\Phi(y,\nu)$; *these parameters do not have to be related in any way, provided the estimates in hypotheses* (I2) *and* (I4) *hold uniformly in the appropriate parameters. We will simply unite both sets of parameters and call this union* ν *again.*

Proof. It is clear that (4.1.2) holds for $0\leq\lambda\leq 1$ since $|I(\lambda,\nu)|$ is bounded for such λ.

Now, consider the case where $\lambda \geq 1$. Set $y = \rho\omega + z$, where $\omega \in \mathbb{S}^{N-1}$ (using the convention that $\mathbb{S}^0 = \{-1, 1\}$), $\rho > 0$ and $z \in \mathbb{R}^N$ is some fixed point; then

$$I(\lambda, \nu) = \int_{\mathbb{S}^{N-1}} \int_0^\infty e^{i\lambda\Phi(\rho\omega+z,\nu)} A(\rho\omega + z, \nu) g(\rho\omega + z) \rho^{N-1} \, d\rho \, d\omega \, .$$

By the compactness of $\mathbb{S}^{N-1}$, it suffices to prove (4.1.2) for the inner integral.

Choose a function $\chi \in C_0^\infty([0, \infty))$, $0 \leq \chi(s) \leq 1$ for all s, which is identically 1 on $0 \leq s \leq \frac{1}{2}$ and is zero when $s \geq 1$; then, writing $F(\rho, \omega, \nu) = \Phi(\rho\omega + z, \nu)$, we split the inner integral into the sum of the two integrals

$$I_1(\lambda, \nu, \omega, z) = \int_0^\infty e^{i\lambda F(\rho,\omega,\nu)} A(\rho\omega + z, \nu) g(\rho\omega + z) \chi(\lambda^{\frac{1}{\gamma}} \rho) \rho^{N-1} \, d\rho \, ,$$
$$I_2(\lambda, \nu, \omega, z) = \int_0^\infty e^{i\lambda F(\rho,\omega,\nu)} A(\rho\omega + z, \nu) g(\rho\omega + z) (1 - \chi)(\lambda^{\frac{1}{\gamma}} \rho) \rho^{N-1} \, d\rho \, .$$

Let us first look at $I_1 = I_1(\lambda, \nu, \omega, z)$; since $\chi(\lambda^{\frac{1}{\gamma}}\rho)$ is zero for $\lambda^{\frac{1}{\gamma}}\rho \geq 1$, we have, by the change of variables $\widetilde{\rho} = \lambda^{\frac{1}{\gamma}}\rho$,

$$\begin{aligned} |I_1| &\leq C \int_0^\infty \chi(\lambda^{\frac{1}{\gamma}}\rho) \rho^{N-1} \, d\rho = C \int_0^\infty (\widetilde{\rho})^{N-1} \lambda^{-\frac{N-1}{\gamma}} \chi(\widetilde{\rho}) \lambda^{-\frac{1}{\gamma}} \, d\widetilde{\rho} \\ &\leq C \lambda^{-\frac{N}{\gamma}} \int_0^1 (\widetilde{\rho})^{N-1} \, d\widetilde{\rho} = C \lambda^{-\frac{N}{\gamma}} \, , \end{aligned}$$

where we have used $|e^{i\lambda F(\rho,\omega,\nu)}| \leq 1$ since $\operatorname{Im} F(\rho, \omega, \nu) \geq 0$ for all ρ, ω, ν by hypothesis (I2); this is the desired estimate for $|I_1|$.

In order to estimate $I_2 = I_2(\lambda, \nu, \omega, z)$, let us first define the operator $L := (i\lambda \partial_\rho F(\rho, \omega, \nu))^{-1} \frac{\partial}{\partial \rho}$ and observe that

$$L(e^{i\lambda F(\rho,\omega,\nu)}) = e^{i\lambda F(\rho,\omega,\nu)} \, .$$

Denoting the adjoint of L by L^*, we have, for each $l \in \mathbb{N} \cup \{0\}$,

$$I_2 = \int_0^\infty e^{i\lambda F(\rho,\omega,\nu)} (L^*)^l [A(\rho\omega + z, \nu) g(\rho\omega + z) (1 - \chi)(\lambda^{\frac{1}{\gamma}}\rho) \rho^{N-1}] \, d\rho \, .$$

Now,

$$(L^*)^l = \left(\frac{i}{\lambda}\right)^l \sum C_{s_1,\ldots,s_p,p,r,l} \frac{\partial_\rho^{s_1} F \ldots \partial_\rho^{s_p} F}{(\partial_\rho F)^{l+p}} (\rho, \omega, \nu) \frac{\partial^r}{\partial \rho^r} \, ,$$

where the sum is over all integers $s_1, \dots, s_p, p, r \geq 0$ such that $s_1 + \cdots + s_p + r - p = l$. By Hypothesis (I3),

$$\left|\frac{\partial_\rho^{s_1} F \dots \partial_\rho^{s_p} F}{(\partial_\rho F)^{l+p}}(\rho, \omega, \nu)\right| \leq C\rho^{p-s_1-\dots-s_p-l\gamma+l} = C\rho^{r-l\gamma}.$$

Also, we claim that, for $r \leq [\frac{N}{\gamma}] + 1$,

$$\left|\frac{\partial^r}{\partial\rho^r}[A(\rho\omega + z, \nu)g(\rho\omega + z)(1-\chi)(\lambda^{\frac{1}{\gamma}}\rho)\rho^{N-1}]\right| \leq C_N\rho^{N-1-r}\widetilde{\chi}(\lambda, \rho), \quad (4.1.3)$$

where $\widetilde{\chi}(\lambda, \rho)$ is a smooth function in ρ which is zero for $\lambda^{\frac{1}{\gamma}}\rho < \frac{1}{2}$. Assuming this is true, we see that, for large enough l—it suffices to take $l = [\frac{N}{\gamma}] + 1$, i.e. $N - l\gamma < 0$—we have,

$$\begin{aligned}|I_2| \leq & C_N\lambda^{-l}\int_0^\infty \sum C_{s_1,\dots,s_p,p,r,l}\,\rho^{r-l\gamma}[\rho^{N-1-r}]\widetilde{\chi}(\lambda, \rho)\, d\rho \\ \leq & C_N\lambda^{-l}\int_{\frac{1}{2}\lambda^{-\frac{1}{\gamma}}}^\infty \rho^{N-1-l\gamma}\, d\rho = C_N\lambda^{-l}\Big[\frac{\rho^{N-l\gamma}}{N-l\gamma}\Big]_{\frac{1}{2}\lambda^{-\frac{1}{\gamma}}}^\infty = C_{N,\gamma}\lambda^{-\frac{N}{\gamma}};\end{aligned}$$

together with the estimate for $|I_1|$, this yields the desired estimate (4.1.2). Here we need $l > N/\gamma$, which means an application of $(L^*)^l$, or estimates on $\partial_\rho^\alpha F$ for $|\alpha| \leq l$. This gives a restriction on the number m of derivatives in (I3).

Finally, let us check (4.1.3). It holds because:

(i) $|\partial_\rho^r(\rho^{N-1})| \leq C_{r,N}\rho^{N-1-r}$ for all $r \in \mathbb{N}$.

(ii) For each $r \in \mathbb{N}$, $\partial_\rho^r[(1-\chi)(\lambda^{\frac{1}{\gamma}}\rho)] = -\lambda^{\frac{r}{\gamma}}(\partial_s^r\chi)(\lambda^{\frac{1}{\gamma}}\rho)$; now, $(\partial_s\chi)(\lambda^{\frac{1}{\gamma}}\rho)$ is supported on the set $\left\{(\lambda, \rho) \in (0, \infty) \times (0, \infty) : \frac{1}{2} < \lambda^{\frac{1}{\gamma}}\rho < 1\right\}$, so, in particular, on its support $\lambda^{\frac{1}{\gamma}} < \rho^{-1}$; therefore,

$$|\partial_\rho^r[(1-\chi)(\lambda^{\frac{1}{\gamma}}\rho)]| \leq C\rho^{-r}(\partial_s^r\chi)(\lambda^{\frac{1}{\gamma}}\rho) \quad \text{for all } r \in \mathbb{N},$$

and $(\partial_s^r\chi)(\lambda^{\frac{1}{\gamma}}\rho)$ is smooth in ρ and zero for $\lambda^{\frac{1}{\gamma}}\rho \leq \frac{1}{2}$.

(iii) By hypothesis (I4), $|\partial_\rho^r A(\rho\omega + z, \nu)| \leq C_r$ for each $r \leq [\frac{N}{\gamma}] + 1$ (this can be seen for $r = 1$ by noting that $\partial_\rho A(\rho\omega + z, \nu) = \omega \cdot \nabla_y A(y, \nu)\big|_{y=\rho\omega+z}$, and then for $r \geq 2$ by calculating the higher derivatives). Also, g is smooth in U, so, $|\partial_\rho^r[A(\rho\omega + z, \nu)g(\rho\omega + z)]| \leq C_r$ for $r \leq [\frac{N}{\gamma}] + 1$. Furthermore, by hypothesis (I1), there exists a constant $\rho_0 > 0$ so that

$g(\rho\omega + z) = 0$ for $\rho > \rho_0$; thus, $\partial_\rho^r[A(\rho\omega + z, \nu)g(\rho\omega + z)]$ is zero for $\rho > \rho_0$; hence, for $r \leq [\frac{N}{\gamma}] + 1$,

$$|\partial_\rho^r[A(\rho\omega + z, \nu)g(\rho\omega + z)]| \leq C_r\rho_0^r\rho^{-r} .$$

This completes the proof of the claim, and thus the theorem. □

4.2 Functions of convex type

Hypothesis (I3) of Theorem 4.1.1 is sufficient for the result of the theorem to hold; however, it is often difficult to check. For this reason, we now introduce the concept of a function of convex type—a condition that is far simpler to verify—and show that for such functions, (I3) automatically holds.

Definition 4.2.1. *Let $F = F(\rho, \upsilon) : [0, \infty) \times \Upsilon \to \mathbb{C}$ be a function that is smooth in ρ for each fixed $\upsilon \in \Upsilon$, where Υ is some parameter space. Write its M^{th} order Taylor expansion in ρ about 0 in the form*

$$F(\rho, \upsilon) = \sum_{j=0}^{M} a_j(\upsilon)\rho^j + R_M(\rho, \upsilon) , \tag{4.2.1}$$

where $R_M(\rho, \upsilon) = \int_0^\rho \partial_s^{M+1}F(s, \upsilon)\frac{(\rho-s)^M}{M!}\,ds$ is the M^{th} remainder term.

We say that F is a function of convex type γ *if, for some $\gamma \in \mathbb{N}$, $\gamma \geq 2$, and for some $\delta > 0$, we have*

(CT1) *$a_0(\upsilon) = a_1(\upsilon) = 0$ for all $\upsilon \in \Upsilon$ (i.e. the Taylor expansion of F starts from order ≥ 2);*

(CT2) *there exists a constant $C > 0$ such that $\sum_{j=2}^{\gamma}|a_j(\upsilon)| \geq C$ for all $\upsilon \in \Upsilon$;*

(CT3) *for each $\upsilon \in \Upsilon$, $|\partial_\rho F(\rho, \upsilon)|$ is increasing in ρ for $0 < \rho < \delta$;*

(CT4) *for each $k \in \mathbb{N}$, $\partial_\rho^k F(\rho, \upsilon)$ is bounded uniformly in $0 < \rho < \delta$, $\upsilon \in \Upsilon$.*

Remark 4.2.2. *Note that, if F is* real-valued*, then* (CT3) *implies that we have either $\partial_\rho^2 F(\rho, \upsilon) \geq 0$ for all $0 < \rho < \delta$, or $\partial_\rho^2 F(\rho, \upsilon) \leq 0$ for all $0 < \rho < \delta$—this is because $\partial_\rho F(0, \nu) = 0$. This is the connection with convexity, hence the name of such functions.*

Such functions have the following useful property:

Lemma 4.2.3. *Let $F(\rho, \upsilon)$ be a function of convex type γ. Then, for each sufficiently small $0 < \delta \leq 1$ there exist constants $C, C_m > 0$ such that*

$$|\partial_\rho F(\rho, \upsilon)| \geq C\rho^{\gamma-1} \tag{4.2.2}$$

$$\text{and } |\partial_\rho^m F(\rho, \upsilon)| \leq C_m \rho^{1-m} |\partial_\rho F(\rho, \upsilon)| \tag{4.2.3}$$

for all $0 < \rho < \delta$, $\upsilon \in \Upsilon$ and $m \in \mathbb{N}$.

Remark 4.2.4. *A version of this lemma appeared in [Sug94] for* analytic *functions without dependence on υ and is based on Lemmas 3, 4 and 5 of Randol [Ran69] (which also appeared in Beals [Bea82], Lemmas 3.2, 3.3). Lemma 4.2.3 extends it to functions that are only smooth and which depend on an additional parameter, which will be necessary of our analysis. A limited regularity version of this lemma appeared in [Ruzh07]. The proof of lemma given here is based on estimating the remainder rather than on using the Cauchy's integral formula for analytic functions.*

Proof. First, let us note that, for $0 < \rho \leq 1$ we have, by (CT2),

$$\pi(\rho, \upsilon) := \sum_{j=2}^{\gamma} j|a_j(\upsilon)|\rho^{j-1} \geq C\rho^{\gamma-1}\,. \tag{4.2.4}$$

Thus, in order to prove (4.2.2), it suffices to show

$$|\partial_\rho F(\rho, \upsilon)| \geq C\pi(\rho, \upsilon) \quad \text{for all } 0 < \rho < \delta,\ \upsilon \in \Upsilon\,; \tag{4.2.5}$$

For $1 \leq m \leq \gamma$, we have, using (4.2.1),

$$\partial_\rho^m F(\rho, \upsilon) = \sum_{k=0}^{\gamma-m} \frac{(k+m)!}{k!} a_{k+m}(\upsilon)\rho^k + R_{m,\gamma-m}(\rho, \upsilon)\,, \tag{4.2.6}$$

where $R_{m,\gamma-m}(\rho, \upsilon) = \int_0^\rho \partial_\rho^{\gamma+1} F(s, \upsilon) \frac{(\rho-s)^{\gamma-m}}{(\gamma-m)!}\, ds$ is the remainder term of the $(\gamma - m)^{\text{th}}$ Taylor expansion of $\partial_\rho^m F(\rho, \upsilon)$. By (CT4) and (4.2.4), we see

$$|R_{m,\gamma-m}(\rho, \upsilon)| \leq C_{\gamma,m}\rho^{\gamma+1-m} \leq C_{\gamma,m}\pi(\rho, \upsilon)\rho^{2-m} \quad \text{for } 0 < \rho < \delta\,. \tag{4.2.7}$$

Hence, for $0 < \rho < \delta$,

$$\begin{aligned}
|\partial_\rho F(\rho, \upsilon)| &= \Big|\sum_{k=0}^{\gamma-1}(k+1)a_{k+1}(\upsilon)\rho^k + R_{1,\gamma-1}(\rho, \upsilon)\Big| \\
&\geq \Big|\sum_{j=2}^{\gamma} ja_j(\upsilon)\rho^{j-1}\Big| - \Big|R_{1,\gamma-1}(\rho, \upsilon)\Big| \geq \Big|\sum_{j=2}^{\gamma} ja_j(\upsilon)\rho^{j-1}\Big| - C_\gamma\pi(\rho, \upsilon)\rho\,.
\end{aligned}$$

Now, by (CT3), $|\partial_\rho F(\rho,\upsilon)|$ is increasing in ρ for each $\upsilon \in \Upsilon$ and, by (CT1), $\partial_\rho F(0,\upsilon) = 0$; therefore,

$$\begin{aligned}|\partial_\rho F(\rho,\upsilon)| &= \max_{0\le\sigma\le\rho}|\partial_\rho F(\sigma,\upsilon)| \\ &\ge \max_{0\le\sigma\le\rho}\Big|\sum_{j=2}^{\gamma} j a_j(\upsilon)\sigma^{j-1}\Big| - \max_{0\le\sigma\le\rho} C_\gamma \pi(\sigma,\upsilon)\sigma \\ &= \max_{0\le\bar\sigma\le1}\Big|\sum_{j=2}^{\gamma} j a_j(\upsilon)\rho^{j-1}\bar\sigma^{j-1}\Big| - C_\gamma\pi(\rho,\upsilon)\rho\,,\end{aligned}$$

since $\pi(\sigma,\upsilon)\sigma = \sum_{j=2}^{\gamma} j|a_j(\upsilon)|\sigma^j$ clearly achieves its maximum on $0 \le \sigma \le \rho$ at $\sigma = \rho$. Noting that

$$\max_{0\le\bar\sigma\le1}\Big|\sum_{j=1}^{L} z_j\bar\sigma^{j-1}\Big| \quad \text{and} \quad \sum_{j=1}^{L}|z_j|$$

are norms on $\mathbb{C}^L$ and, hence, are equivalent, we immediately get

$$\begin{aligned}|\partial_\rho F(\rho,\upsilon)| \ge& C\sum_{j=2}^{\gamma} j|a_j(\upsilon)|\rho^{j-1} - C_\gamma\pi(\rho,\upsilon)\rho \\ \ge&(C - C_\gamma\delta)\pi(\rho,\upsilon) = C_{\gamma,\delta}\pi(\rho,\upsilon)\,,\end{aligned}$$

which completes the proof of (4.2.5).

To prove (4.2.3), we consider the cases $1 \le m \le \gamma$ and $m > \gamma$ separately.

For $m > \gamma$, we have, by (CT4),

$$|\partial_\rho^m F(\rho,\upsilon)| \le C_m \le C_{m,\delta}\rho^{\gamma+1-m} \quad \text{for } 0 < \rho < \delta\,,$$

since $\gamma + 1 - m \le 0$, and, thus, $\rho^{\gamma+1-m} \ge \delta^{\gamma+1-m} > 0$; so, by (4.2.2), we have

$$|\partial_\rho^m F(\rho,\upsilon)| \le C_{m,\delta}\rho^{2-m}|\partial_\rho F(\rho,\upsilon)| \quad \text{for } 0 < \rho < \delta,\ m > \gamma\,. \tag{4.2.8}$$

For $1 \le m \le \gamma$, we have the representation (4.2.6). It is clear that

$$\Big|\sum_{k=0}^{\gamma-m}\frac{(k+m)!}{k!}\,a_{k+m}(\upsilon)\rho^k\Big| \le C_m\pi(\rho,\upsilon)\rho^{1-m}\,,$$

which, together with (4.2.7) and (4.2.5), yields

$$|\partial_\rho^m F(\rho,\upsilon)| \le C_{m,\delta}\rho^{1-m}|\partial_\rho F(\rho,\upsilon)| \quad \text{for } 0 < \rho < \delta,\ 1 \le m \le \gamma\,.$$

This, together with (4.2.8), completes the proof of (4.2.3) and, thus, the lemma. $\square$

This lemma means we have the following alternative version of Theorem 4.1.1.

Corollary 4.2.5. *Hypothesis* (I3) *of Theorem 4.1.1 may be replaced by:*

(I3′) *for some fixed* $z \in \mathbb{R}^N$, *the function* $F(\rho, \omega, \nu) := \Phi(\rho\omega + z, \nu)$ *is a function of convex type* γ, *for some* $\gamma \in \mathbb{N}$, *in the sense of Definition 4.2.1 with* $(\omega, \nu) \in \mathbb{S}^{N-1} \times \mathcal{N} \equiv \Upsilon$.

4.3 Convexity condition for real-valued phases

Using the results of the previous two sections, we can now prove a series of results for which a so-called convexity condition holds; here we recall Definitions 2.2.3 and 2.2.4 from Section 2 and prove the basic result for real-valued functions. We recall that a smooth function $\tau : \mathbb{R}^n \to \mathbb{R}$ is said to satisfy the *convexity condition* if Σ_λ is convex for each $\lambda \in \mathbb{R}$ (and the empty set is considered to be convex). The *maximal order of contact* of a hypersurface Σ is defined as follows. Let $\sigma \in \Sigma$, and denote the tangent plane at σ by T_σ. Let P be a plane containing the normal to Σ at σ and denote the order of the contact between the line $T_\sigma \cap P$ and the curve $\Sigma \cap P$ by $\gamma(\Sigma; \sigma, P)$. Then we set

$$\gamma(\Sigma) := \sup_{\sigma \in \Sigma} \sup_{P} \gamma(\Sigma; \sigma, P)\,.$$

In the proof of Theorem 2.2.6 we will need a Besov space version of the estimate for the kernel. For this, let us introduce some useful notation for a family of cut-off functions $g_R \in C_0^\infty(\mathbb{R}^n)$, $R \in [0, \infty)$: these functions will correspond to the cut-offs to annuli in the frequency space and we need to trace the dependence on the parameter R. Suppose $g \in C_0^\infty(\mathbb{R}^n)$ is such that, for some constants $c_0, c_1 \geq 0$, it is supported in the set

$$\{\xi : c_0 < |\xi| < c_1\}\ ,$$

and let $g_0 \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ be another (arbitrary) compactly supported function. Then, for $R \geq 0$, set

$$g_R(\xi) := \begin{cases} g(\xi/R) & \text{if } R \geq 1, \\ g_0(\xi) & \text{if } 0 \leq R < 1. \end{cases} \tag{4.3.1}$$

Now we can prove the main convexity theorem:

Theorem 4.3.1. *Suppose* $\tau : \mathbb{R}^n \to \mathbb{R}$ *satisfies the convexity condition. Set* $\gamma := \sup_{\lambda > 0} \gamma(\Sigma_\lambda(\tau))$ *and assume this is finite. Let* $a(\xi)$ *be a symbol of order* $\frac{n-1}{\gamma} - n$ *of type* $(1, 0)$ *on* $\mathbb{R}^n$; *furthermore, on* $\operatorname{supp} a$, *we assume*:

(i) *for all multi-indices α there exists a constant $C_\alpha > 0$ such that*

$$|\partial_\xi^\alpha \tau(\xi)| \leq C_\alpha (1 + |\xi|)^{1-|\alpha|};$$

(ii) *there exist constants $M, C > 0$ such that for all $|\xi| \geq M$ we have $|\tau(\xi)| \geq C|\xi|$;*

(iii) *there exists a constant $C_0 > 0$ such that $|\partial_\omega \tau(\lambda\omega)| \geq C_0$ for all $\omega \in \mathbb{S}^{n-1}$, $\lambda > 0$; in particular, $|\nabla\tau(\xi)| \geq C_0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$;*

(iv) *there exists a constant $R_1 > 0$ such that, for all $\lambda > 0$,*

$$\frac{1}{\lambda}\Sigma_\lambda(\tau) \equiv \frac{1}{\lambda}\{\xi \in \mathbb{R}^n : \ \tau(\xi) = \lambda\} \subset B_{R_1}(0)\,.$$

Then, the following estimate holds for all $R \geq 0$, $x \in \mathbb{R}^n$, $t > 1$:

$$\Big|\int_{\mathbb{R}^n} e^{i(x\cdot\xi + \tau(\xi)t)} a(\xi) g_R(\xi)\, d\xi\Big| \leq C t^{-\frac{n-1}{\gamma}}\,, \tag{4.3.2}$$

where $g_R(\xi)$ is as given in (4.3.1) and $C > 0$ is independent of R.

Remark 4.3.2. *For an integral of this type with some specific compactly supported function, $\chi \in C_0^\infty(\mathbb{R}^n)$ say, in place of g_R, we can just use the result for $R = 0$. In this way we obtain Corollary 2.2.7.*

Proof. We may assume throughout, without loss of generality, that either $\tau(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$ or $\tau(\xi) \leq 0$ for all $\xi \in \mathbb{R}^n$. Indeed, hypothesis (ii) and the continuity of τ ensure that either $\tau(\xi)$ is positive for all $|\xi| \geq M$ or negative for all $|\xi| \geq M$. In the case where $\tau(\xi)$ is positive for all $|\xi| \geq M$, set

$$\tau_+(\xi) := \tau(\xi) + \min(0, \inf_{|\xi|<M} \tau(\xi)) \geq 0 \ \text{ for all } \xi \in \mathbb{R}^n.$$

Now, $\tau(\xi) - \tau_+(\xi)$ is a constant (in particular, it is independent of ξ) and $|e^{i[\tau(\xi)-\tau_+(\xi)]t}| = 1$, so it suffices to show

$$\Big|\int_{\mathbb{R}^n} e^{i(x\cdot\xi + \tau_+(\xi)t)} a(\xi) g_R(\xi)\, d\xi\Big| \leq C t^{-\frac{n-1}{\gamma}}\,.$$

In the case where $\tau(\xi)$ is negative for $|\xi| \geq M$, set $\widetilde{\tau}(\xi) := -\tau(\xi)$ and by similar reasoning to above, it is sufficient to show

$$\Big|\int_{\mathbb{R}^n} e^{i(x\cdot\xi - \widetilde{\tau}_+(\xi)t)} a(\xi) g_R(\xi)\, d\xi\Big| \leq C t^{-\frac{n-1}{\gamma}}\,,$$

where $-\widetilde{\tau}_+(\xi) \leq 0$ for all $\xi \in \mathbb{R}^n$.

We begin by dividing the integral into two parts: near to the wave-front set, i.e. points where $\nabla_\xi[x \cdot \xi + \tau(\xi)t] = 0$, and away from such points. To this end, we introduce a cut-off function $\kappa \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \kappa(y) \leq 1$, which is identically 1 in the ball of radius $r > 0$ (which will be fixed below) centred at the origin, $B_r(0)$, and identically 0 outside the ball of radius $2r$, $B_{2r}(0)$. Then we estimate the following two integrals separately:

$$\begin{aligned} I_1(t,x) &:= \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) g_R(\xi) \kappa\big(t^{-1}x + \nabla\tau(\xi)\big)\, d\xi\,, \\ I_2(t,x) &:= \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) g_R(\xi) (1-\kappa)\big(t^{-1}x + \nabla\tau(\xi)\big)\, d\xi\,. \end{aligned}$$

For $I_2(t,x)$ we have the following result:

Lemma 4.3.3. *Suppose $a(\xi)$ is a symbol of order $j \in \mathbb{R}$. Then, for each $l \in \mathbb{N}$ with $l > n + j$, we have, for all $t > 0$,*

$$|I_2(t,x)| \leq C_{r,l} t^{-l}\,, \tag{4.3.3}$$

where the constants $C_{r,l} > 0$ are independent of R.

Proof. In the support of $(1-\kappa)(t^{-1}x+\nabla\tau(\xi))$, we have $|x+t\nabla\tau(\xi)| \geq rt > 0$, so we can write

$$\frac{(x+t\nabla\tau(\xi))}{i|x+t\nabla\tau(\xi)|^2} \cdot \nabla_\xi\big(e^{i(x\cdot\xi+\tau(\xi)t)}\big) = e^{i(x\cdot\xi+\tau(\xi)t)}\,;$$

therefore, denoting the adjoint to $P \equiv \frac{(x+t\nabla\tau(\xi))}{i|x+t\nabla\tau(\xi)|^2} \cdot \nabla_\xi$ by P^*, we get

$$I_2(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} (P^*)^l \big[a(\xi) g_R(\xi)(1-\kappa)\big(t^{-1}x + \nabla\tau(\xi)\big)\big]\, d\xi$$

for each $l \in \mathbb{N}$. We claim that for each l there exists some constant $C_{r,l} > 0$ independent of R so that, when $t > 1$, we have

$$(P^*)^l \big[a(\xi) g_R(\xi)(1-\kappa)\big(t^{-1}x + \nabla\tau(\xi)\big)\big] \leq C_{r,l} t^{-l} (1+|\xi|)^{j-l}\,; \tag{4.3.4}$$

assuming this, we obtain,

$$|I_2(t,x)| \leq C_{r,l} t^{-l} \int_{\mathbb{R}^n} \frac{1}{(1+|\xi|)^{l-j}}\, d\xi\,.$$

Noting that $\int_{\mathbb{R}^n} \frac{1}{(1+|\xi|)^{l-j}}\, d\xi$ converges for $l - j > n$ yields the desired estimate (4.3.3).

It remains to prove (4.3.4). Let $f \equiv f(\xi; x, t)$ be a function that is zero for $|x + t\nabla\tau(\xi)| \leq rt$ and is continuously differentiable with respect to ξ; then,

$$P^* f = \nabla_\xi \cdot \left[\frac{(x + t\nabla\tau(\xi))}{i|x + t\nabla\tau(\xi)|^2} f\right] = \frac{t\Delta\tau(\xi)}{i|x + t\nabla\tau(\xi)|^2} f + \frac{(x + t\nabla\tau(\xi))}{i|x + t\nabla\tau(\xi)|^2} \cdot \nabla_\xi f \\ - \frac{2t(x + t\nabla\tau(\xi)) \cdot [\nabla^2\tau(\xi) \cdot (x + t\nabla\tau(\xi))]}{i|x + t\nabla\tau(\xi)|^4} f\,. \quad (4.3.5)$$

Hence, using $|x+t\nabla\tau(\xi)| \geq rt$ (hypothesis on f) and $|\partial^\alpha\tau(\xi)| \leq C(1+|\xi|)^{1-|\alpha|}$ (hypothesis (i)), we have

$$|P^* f| \leq C_r t^{-1}[(1 + |\xi|)^{-1}|f| + |\nabla_\xi f|]\,. \quad (4.3.6)$$

Now, for all multi-indices α and for all $\xi \in \mathbb{R}^n$, we get

- $|\partial^\alpha a(\xi)| \leq C_\alpha(1 + |\xi|)^{j-|\alpha|}$ for all $\xi \in \mathbb{R}^n$ as $a \in S^j_{1,0}(\mathbb{R}^n)$;
- $|\partial^\alpha_\xi\big[(1 - \kappa)\big(t^{-1}x + \nabla\tau(\xi)\big)\big]| \leq C_\alpha(1 + |\xi|)^{-|\alpha|}$, for all $\xi \in \mathbb{R}^n$—here we have used hypothesis (i) once more. Also, it is zero for each α when $|x + t\nabla\tau(\xi)| \leq rt$ by the definition of κ.

Furthermore, $|\partial^\alpha g_R(\xi)| = |\partial^\alpha g_0(\xi)| \leq C_\alpha(1 + |\xi|)^{-|\alpha|}$ for $0 \leq R < 1$, since $C_0^\infty(\mathbb{R}^n \setminus \{0\}) \subset S^0_{1,0}(\mathbb{R}^n)$. For $R \geq 1$, we have:

$$\partial^\alpha g_R(\xi) = \partial^\alpha[g(\xi/R)] = R^{-|\alpha|}(\partial^\alpha g)(\xi/R) \text{ and } g \in S^0_{1,0}(\mathbb{R}^n) \\ \Longrightarrow\ |\partial^\alpha g_R(\xi)| \leq C_\alpha R^{-|\alpha|}(1 + |\xi/R|)^{-|\alpha|} \leq C_\alpha(1 + |\xi|)^{-|\alpha|}\,.$$

Therefore,

$$|\partial^\alpha g_R(\xi)| \leq C_\alpha(1 + |\xi|)^{-|\alpha|} \text{ for all } \xi \in \mathbb{R}^n \text{ and multi-indices } \alpha\,, \quad (4.3.7)$$

where the $C_\alpha > 0$ are independent of R.

Hence, by (4.3.6), we obtain

$$\big|P^*[a(\xi)g_R(\xi)(1 - \kappa)\big(t^{-1}x + \nabla\tau(\xi)\big)]\big| \leq C_r t^{-1}(1 + |\xi|)^{j-1}\,.$$

To prove (4.3.4) for $l \geq 2$ we do induction on l. Note that

$$|(P^*)^l f| \leq C_r t^{-1}[(1 + |\xi|)^{-1}|(P^*)^{l-1} f| + |\nabla_\xi\{(P^*)^{l-1} f\}|]\,.$$

The first term satisfies the desired estimate by the inductive hypothesis. For the second term, repeated application of the properties of $a(\xi)$, $g(\xi)$ and $(1 - \kappa)(t^{-1}x + \nabla\tau(\xi))$ noted above to inductively estimate derivatives of $(P^*)^{l'} f$, $1 \leq l' \leq l - 2$ yields the desired estimate. This completes the proof of the lemma. $\square$

This lemma, with $j = \frac{n-1}{\gamma} - n$, means that it suffices to prove (4.3.2) for $I_1(t,x)$, where $|t^{-1}x + \nabla\tau(\xi)| < 2r$.

Let $\{\Psi_\ell(\xi)\}_{\ell=1}^{L}$ be a partition of unity in $\mathbb{R}^n$ where $\Psi_\ell(\xi) \in C^\infty(\mathbb{R}^n)$ is supported in a narrow (the breadth will be fixed below) open cone K_ℓ, $\ell = 1, \ldots, L$; let us assume that K_1 contains the point $e_n = (0, \ldots, 0, 1)$ (if necessary, relabel the cones to ensure this) and also that each K_ℓ, $\ell = 1, \ldots, L$, can be mapped onto K_1 by rotation. Then, it suffices to estimate

$$I_1'(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) g_R(\xi) \Psi_1(\xi) \kappa\big(t^{-1}x + \nabla\tau(\xi)\big)\, d\xi\,, \tag{4.3.8}$$

since the properties of $\tau(\xi)$, $a(\xi)$, $g_R(\xi)$ and $\kappa(t^{-1}x+\nabla\tau(\xi))$ used throughout are invariant under rotation.

By hypothesis (iii), the level sets $\Sigma_\lambda = \{\xi \in \mathbb{R}^n : \tau(\xi) = \lambda\}$ are all non-degenerate (or empty). Furthermore, the Implicit Function Theorem allows us to parameterise the intersection of the surface $\Sigma'_\lambda \equiv \frac{1}{\lambda}\Sigma_\lambda$ and the cone K_1:

$$K_1 \cap \Sigma'_\lambda = \{(y, h_\lambda(y)) : y \in U\}\ ;$$

here $U \subset \mathbb{R}^{n-1}$ is a bounded open set for which $p(U) = \mathbb{S}^{n-1} \cap K_1$ where $p(y) = (y, \sqrt{1-|y|^2})$, and $h_\lambda : U \to \mathbb{R}$ is a smooth function for each $\lambda > 0$; in particular, each h_λ is concave due to $\tau(\xi)$ satisfying the convexity condition, i.e. Σ'_λ is convex for each $\lambda \in \mathbb{R}$. Then, in the case that $\tau(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$, the cone K_1 is parameterised by

$$K_1 = \{(\lambda y, \lambda h_\lambda(y)) : \lambda > 0,\ y \in U\}\ ,$$

and when $\tau(\xi) \leq 0$ for all $\xi \in \mathbb{R}^n$,

$$K_1 = \{(\lambda y, \lambda h_\lambda(y)) : \lambda < 0,\ y \in U\}\ .$$

Now, let $\underline{\mathbf{n}} : K_1 \cap \Sigma'_\lambda \to \mathbb{S}^{n-1}$ be the Gauss map,

$$\underline{\mathbf{n}}(\zeta) = \frac{\nabla\tau(\zeta)}{|\nabla\tau(\zeta)|}\,.$$

By the definition of $\kappa(t^{-1}x + \nabla\tau(\xi))$, we have

$$|t^{-1}x - (-\nabla\tau(\xi_\lambda))| < 2r$$

for each $\xi_\lambda \in K_1 \cap \Sigma'_\lambda$ that is also in the support of the integrand of (4.3.8). Hence, provided $r > 0$ is taken sufficiently small, the convexity of Σ'_λ ensures that the points $t^{-1}x/|t^{-1}x|$ and $-\underline{\mathbf{n}}(\xi_\lambda)$ are close enough so that there exists $z(\lambda) \in U$ (for each $\xi_\lambda \in K_1 \cap \Sigma'_\lambda$) satisfying

$$\underline{\mathbf{n}}\big(z(\lambda), h_\lambda(z(\lambda))\big) = -t^{-1}x/|t^{-1}x| = -x/|x| \in \mathbb{S}^{n-1}\,.$$

Also, $(-\nabla_y h_\lambda(y), 1)$ is normal to Σ'_λ at $(y, h_\lambda(y))$, so, writing $x = (x', x_n)$, we have

$$-\frac{x}{|x|} = \frac{(-\nabla_y h_\lambda(z(\lambda)), 1)}{|(-\nabla_y h_\lambda(z(\lambda)), 1)|} \implies -\frac{x_n}{|x|} = \frac{1}{|(-\nabla_y h_\lambda(z(\lambda)), 1)|}$$
$$\text{and } -\frac{x'}{|x|} = \frac{-\nabla_y h_\lambda(z(\lambda))}{|(-\nabla_y h_\lambda(z(\lambda)), 1)|} = \frac{x_n \nabla_y h_\lambda(z(\lambda))}{|x|};$$

therefore, $-x' = x_n \nabla_y h_\lambda(z(\lambda))$. We claim that x_n is away from 0 provided the breadth of the cone K_1 is chosen to be sufficiently narrow, so

$$\frac{x'}{x_n} = -\nabla_y h_\lambda(z(\lambda)). \tag{4.3.9}$$

To prove this claim, first recall that $\Sigma'_\lambda \subset B_{R_1}(0)$ for all $\lambda > 0$ (hypothesis (iv)) and note that $\partial_{\xi_n}\tau(\xi)$ is absolutely continuous on $\overline{B_{R_1}(0)}$ (it is continuous in $\mathbb{R}^n$): taking $C_0 > 0$ as in hypothesis (iii), we get that

$$\begin{gathered}\text{there exists } \delta > 0 \text{ so that } |\eta^1 - \eta^2| < \delta, \ \text{ where } \eta^1, \eta^2 \in \overline{B_{R_1}(0)}, \\ \text{implies } |\partial_{\xi_n}\tau(\eta^1) - \partial_{\xi_n}\tau(\eta^2)| < C_0/4.\end{gathered} \tag{4.3.10}$$

Then, fix the breadth of K_1 so that the maximal shortest distance from a point $\xi \in K_1 \cap (\bigcup_{\lambda>0} \Sigma'_\lambda)$ to the ray $\{\mu e_n : \mu > 0\}$ is less than this δ, i.e.

$$\sup\left\{\inf_{\mu>0}|\xi - \mu e_n| : \xi \in K_1 \cap \big(\bigcup_{\lambda>0} \Sigma'_\lambda\big)\right\} < \delta.$$

Now, observe that for any $\xi^0 \in \mathbb{R}^n$, $\mu > 0$, we have

$$\left|\tfrac{x_n}{t}\right| \geq |\partial_{\xi_n}\tau(\mu e_n)| - |\partial_{\xi_n}\tau(\xi^0) - \partial_{\xi_n}\tau(\mu e_n)| - |\tfrac{x_n}{t} + \partial_{\xi_n}\tau(\xi^0)|.$$

Choose $\xi^0 \in K_1 \cap \Sigma'_\lambda \cap \operatorname{supp}[\kappa(t^{-1}x + \nabla\tau(\xi))]$ and $\mu > 0$ so that $|\xi^0 - \mu e_n| < \delta$ and, hence,

$$|\partial_{\xi_n}\tau(\xi^0) - \partial_{\xi_n}\tau(\mu e_n)| < C_0/4;$$

also, by hypothesis (iii), $|\partial_{\xi_n}\tau(\mu e_n)| \geq C_0$, so

$$|t^{-1}x_n| \geq 3C_0/4 - 2r.$$

Taking r sufficiently small, less than $C_0/8$ say, (ensuring $r > 0$ satisfies the earlier condition also) we get

$$|x_n| \geq ct > 0 \tag{4.3.11}$$

proving the claim.

Before estimating (4.3.8), we introduce some useful notation: by the definition of $g_R(\xi)$, (4.3.1), when $R \geq 1$

$$\xi \in \operatorname{supp} g_R \implies Rc_0 < |\xi| < Rc_1;$$

also, if $0 \leq R < 1$, then there exist constants $\widetilde{c}_0, \widetilde{c}_1 > 0$ so that $\widetilde{c}_0 < |\xi| < \widetilde{c}_1$ for $\xi \in \operatorname{supp} g_R$. Thus, by hypotheses (i) and (ii), there exist constants $c_0', c_1' > 0$ such that

$$\begin{cases} Rc_0' < |\tau(\xi)| < Rc_1' & \text{if } R \geq 1 \text{ and } \xi \in \operatorname{supp} g_R, \\ c_0' < |\tau(\xi)| < c_1' & \text{if } 0 \leq R < 1 \text{ and } \xi \in \operatorname{supp} g_R. \end{cases}$$

Let $G \in C_0^\infty(\mathbb{R})$ be identically one on the set $\{s \in \mathbb{R} : c_0' < s < c_1'\}$ and identically zero in a neighbourhood of the origin; writing $\mathcal{R} = \max(R, 1)$, this then satisfies

$$g_R(\xi) = g_R(\xi) G(\tau(\xi)/\mathcal{R}) .$$

Also, for simplicity, write

$$\widetilde{a}(\xi) \equiv \widetilde{a}_R(\xi) := a(\xi) g_R(\xi) \Psi_1(\xi) ; \tag{4.3.12}$$

this is a type (1,0) symbol of order $\frac{n-1}{\gamma} - n$ supported in the cone K_1, and the constants in the symbolic estimates are all independent of R as each $g_R(\xi)$, $R \geq 0$, is a symbol of order 0 with constants independent of R (see (4.3.7)).

We now turn to estimating (4.3.8). Using the change of variables $\xi \mapsto (\lambda y, \lambda h_\lambda(y))$ and equality (4.3.9), it becomes

$$\begin{aligned} I_1'(t,x) &= \int_0^\infty \int_U e^{i[\lambda x' \cdot y + \lambda x_n h_\lambda(y) + \tau(\lambda y, \lambda h_\lambda(y)) t]} a(\lambda y, \lambda h_\lambda(y)) \\ &\quad g_R(\lambda y, \lambda h_\lambda(y)) \Psi_1(\lambda y, \lambda h_\lambda(y)) \kappa\big(t^{-1} x + \nabla \tau(\lambda y, \lambda h_\lambda(y))\big) \frac{d\xi}{d(\lambda, y)} \, dy \, d\lambda \\ &= \int_0^\infty \int_U e^{i\lambda x_n [-\nabla_y h_\lambda(z(\lambda)) \cdot y + h_\lambda(y) + t x_n^{-1}]} \widetilde{a}(\lambda y, \lambda h_\lambda(y)) \\ &\qquad G(\lambda/\mathcal{R}) \kappa\big(t^{-1} x + \nabla \tau(\lambda y, \lambda h_\lambda(y))\big) \frac{d\xi}{d(\lambda, y)} \, dy \, d\lambda, \end{aligned} \tag{4.3.13}$$

where we have used $\tau(\lambda y, \lambda h_\lambda(y)) = \lambda$ (definition of Σ_λ) in the last line. Here, note that

$$\frac{d\xi}{d(\lambda, y)} = \begin{vmatrix} \lambda I & y \\ \lambda \nabla_y h_\lambda(y) & \partial_\lambda[\lambda h_\lambda(y)] \end{vmatrix} = \lambda^{n-1} (\partial_\lambda [\lambda h_\lambda(y)] - y \cdot \nabla_y h_\lambda(y)) ,$$

where I is the identity matrix. Differentiating $\tau(\lambda y, \lambda h_\lambda(y)) = \lambda$ with respect to λ in the first case and with respect to y in the second, gives

$$y \cdot \nabla_{\xi'} \tau(\lambda y, \lambda h_\lambda(y)) + \partial_\lambda[\lambda h_\lambda(y)] \partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y)) = 1\,,$$
$$\lambda \nabla_{\xi'} \tau(\lambda y, \lambda h_\lambda(y)) + \lambda \nabla_y h_\lambda(y) \partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y)) = 0\,.$$

Substituting the second of these equalities into the first yields

$$\big(\partial_\lambda[\lambda h_\lambda(y)] - y \cdot \nabla_y h_\lambda(y)\big) \partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y)) = 1\,.$$

We claim that

$$|\partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y))| \geq C > 0\,. \tag{4.3.14}$$

To see this, first note that

$$|\partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y))| \geq |\partial_{\xi_n} \tau(\lambda \mu e_n)| - \big|\partial_{\xi_n} \tau(\lambda \mu e_n) - \partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y))\big|$$

where $\mu > 0$ is chosen as above so that $|\mu e_n - (y, h_\lambda(y))| \leq \delta$; now,

$$|\partial_{\xi_n} \tau(\lambda \mu e_n)| \geq C_0$$

by hypothesis (iii). Also, by the Mean Value Theorem, there exists $\bar{\xi}$ lying on the segment between $(\lambda y, \lambda h_\lambda(y))$ and $\lambda \mu e_n$ such that

$$|\partial_{\xi_n} \tau(\lambda \mu e_n) - \partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y))| \leq C |\nabla_\xi \partial_{\xi_n} \tau(\bar{\xi})| \lambda \delta \leq C |\bar{\xi}|^{-1} \lambda \delta \leq C\delta\,;$$

choosing $\delta > 0$ small enough (also ensuring it satisfies condition (4.3.10) above) completes the proof of the claim. Hence,

$$\Big|\frac{d\xi}{d(\lambda, y)}\Big| = \Big|\frac{\lambda^{n-1}}{\partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y))}\Big| \leq C \lambda^{n-1}\,. \tag{4.3.15}$$

Also, note that this Jacobian is bounded below away from zero because $|\partial_{\xi_n} \tau(\xi)| \leq C$ for all $\xi \in \mathbb{R}^n$ (hypothesis (i)), which means that the transformation above is valid in K_1.

Next, using the change of variables $\widetilde{\lambda} = \lambda x_n = \lambda \widetilde{x}_n t$ in (4.3.13), writing $h(\lambda, y) \equiv h_\lambda(y)$ and setting $\widetilde{x} := t^{-1} x$ (so $\widetilde{x}_n = t^{-1} x_n$), we obtain

$$\int_0^\infty \int_U e^{i\widetilde{\lambda}(-\nabla_y h\left(\frac{\widetilde{\lambda}}{\widetilde{x}_n t}, z\left(\frac{\widetilde{\lambda}}{\widetilde{x}_n t}\right)\right) \cdot y + h\left(\frac{\widetilde{\lambda}}{\widetilde{x}_n t}, y\right) + \widetilde{x}_n^{-1})} \widetilde{a}\Big(\frac{\widetilde{\lambda}}{\widetilde{x}_n t} y, \frac{\widetilde{\lambda}}{\widetilde{x}_n t} h\Big(\frac{\widetilde{\lambda}}{\widetilde{x}_n t}, y\Big)\Big)$$
$$G\Big(\frac{\widetilde{\lambda}}{\mathcal{R} \widetilde{x}_n t}\Big) \kappa\Big(\widetilde{x} + \nabla \tau\Big(\frac{\widetilde{\lambda}}{\widetilde{x}_n t} y, \frac{\widetilde{\lambda}}{\widetilde{x}_n t} h\Big(\frac{\widetilde{\lambda}}{\widetilde{x}_n t}, y\Big)\Big)\Big) \frac{d\xi}{d(\lambda, y)} t^{-1} \widetilde{x}_n^{-1}\, dy\, d\widetilde{\lambda}\,.$$

Therefore, using $\left|\frac{d\xi}{d(\lambda,y)}\right| \le C\widetilde{\lambda}^{n-1}|\widetilde{x}_n|^{-(n-1)}t^{-(n-1)}$ (by (4.3.15)) and recalling that $|\kappa(\eta)| \le 1$, we have,

$$|I_1'(t,x)| \le Ct^{-\frac{n-1}{\gamma}}|\widetilde{x}_n|^{-\frac{n-1}{\gamma}} \int_0^\infty \left| I\Big(\widetilde{\lambda}, \tfrac{\widetilde{\lambda}}{\widetilde{x}_n t}; z\Big(\tfrac{\widetilde{\lambda}}{\widetilde{x}_n t}\Big)\Big) G\Big(\tfrac{\widetilde{\lambda}}{\mathcal{R}\widetilde{x}_n t}\Big)\widetilde{\lambda}^{\frac{n-1}{\gamma}-1}\right| d\widetilde{\lambda}\,, \tag{4.3.16}$$

where,

$$I\Big(\widetilde{\lambda}, \tfrac{\widetilde{\lambda}}{\widetilde{x}_n t}; z\Big(\tfrac{\widetilde{\lambda}}{\widetilde{x}_n t}\Big)\Big) = \int_U e^{i\widetilde{\lambda}\left[h\left(\frac{\widetilde{\lambda}}{\widetilde{x}_n t},y\right) - h\left(\frac{\widetilde{\lambda}}{\widetilde{x}_n t},z\right) - (y-z)\cdot\nabla_y h\left(\frac{\widetilde{\lambda}}{\widetilde{x}_n t},z\right)\right]}$$

$$\widetilde{a}\Big(\tfrac{\widetilde{\lambda}}{\widetilde{x}_n t}y, \tfrac{\widetilde{\lambda}}{\widetilde{x}_n t}h\Big(\tfrac{\widetilde{\lambda}}{\widetilde{x}_n t}, y\Big)\Big)\Big(\tfrac{\widetilde{\lambda}}{t|\widetilde{x}_n|}\Big)^{n-\frac{n-1}{\gamma}}\, dy\,.$$

With Theorem 4.1.1 in mind, let us rewrite this in the form of (4.1.1):

$$I(\lambda,\mu;z) = \int_{\mathbb{R}^{n-1}} e^{i\lambda\Phi(y,\mu;z)} a_0(\mu y, \mu h_\mu(y)) b(y)\, dy\,,$$

with arbitrary $\lambda > 0$, $\mu > 0$ and $z \in \mathbb{R}^{n-1}$, where

- $\Phi(y,\mu;z) = h_\mu(y) - h_\mu(z) - (y-z)\cdot\nabla_y h_\mu(z)$;
- $a_0(\xi) := \widetilde{a}(\xi)|\xi|^{n-\frac{n-1}{\gamma}}$;
- $b \in C_0^\infty(\mathbb{R}^{n-1})$ with support contained in U.

We shall show that the following conditions (numbered as in Theorem 4.1.1 and Corollary 4.2.5) are satisfied by $I(\lambda,\mu;z)$:

(I1) there exists a bounded set $U \subset \mathbb{R}^{n-1}$ such that $b \in C_0^\infty(U)$;

(I2) $\operatorname{Im}\Phi(y,\mu;z) \ge 0$ for all $y \in U$, $\mu > 0$;

(I3′) $F(\rho,\omega,\mu;z) = \Phi(\rho\omega + z, \mu; z)$, $\omega \in \mathbb{S}^{n-2}$, $\rho > 0$, is a function of convex type γ (see Definition 4.2.1);

(I4) there exist constants C_α such that $|\partial_y^\alpha[a_0(\mu y, \mu h_\mu(y))]| \le C_\alpha$ for all $y \in U$, $\mu > 0$ and $|\alpha| \le [\frac{n-1}{\gamma}] + 1$.

Assuming for now that these hold, Theorem 4.1.1 (or, more precisely, Corollary 4.2.5) states that, for all $\lambda > 0$, $\mu > 0$,

$$|I(\lambda,\mu;z)| \le C(1+\lambda)^{-\frac{n-1}{\gamma}} \le C\lambda^{-\frac{n-1}{\gamma}}\,.$$

This, together with (4.3.16), gives

$$|I_1'(t,x)| \leq Ct^{-\frac{n-1}{\gamma}} |\widetilde{x}_n|^{-\frac{n-1}{\gamma}} \int_0^\infty \widetilde{\lambda}^{-\frac{n-1}{\gamma}} G\Big(\frac{\widetilde{\lambda}}{\mathcal{R}\widetilde{x}_n t}\Big) \widetilde{\lambda}^{\frac{n-1}{\gamma}-1}\, d\widetilde{\lambda}\,;$$

then, setting $\nu = \frac{\widetilde{\lambda}}{\mathcal{R}\widetilde{x}_n t}$, we have

$$\begin{aligned} |I_1'(t,x)| &\leq Ct^{-\frac{n-1}{\gamma}} |\widetilde{x}_n|^{-\frac{n-1}{\gamma}} \int_0^\infty (\mathcal{R}\widetilde{x}_n t\nu)^{-1} G(\nu) \mathcal{R}\widetilde{x}_n t\, d\nu \\ &= Ct^{-\frac{n-1}{\gamma}} |\widetilde{x}_n|^{-\frac{n-1}{\gamma}} \int_0^\infty \nu^{-1} G(\nu)\, d\nu \leq Ct^{-\frac{n-1}{\gamma}} \quad \text{for all } t > 1\,. \end{aligned}$$

Here we have used that G is identically zero in a neighbourhood of the origin and that it is compactly supported and also (4.3.11) ($|\widetilde{x}_n| \geq C > 0$); also, note the constant here is independent of R. Since this inequality holds for $I_1'(t,x)$, it also holds for $I_1(t,x)$; thus, together with Lemma 4.3.3, this proves the desired estimate (4.3.2), provided we show that the four properties (I1)–(I4) above hold.

Now, clearly (I1) holds automatically and (I2) is true since $h_\mu(y)$ is real-valued, so $\operatorname{Im}\Phi(y,\mu;z) = 0$ for all $y \in U$, $\mu > 0$.

For (I3′) and (I4), we need an auxiliary result about the boundedness of the derivatives of $h_\lambda(y)$:

Lemma 4.3.4. *All derivatives of $h_\lambda(y)$ with respect to y are bounded uniformly in y. That is, for each multi-index α there exists a constant $C_\alpha > 0$ such that*

$$|\partial_y^\alpha h_\lambda(y)| \leq C_\alpha \quad \textit{for all } y \in U,\ \lambda > 0\,.$$

Proof. By definition, $\tau(\lambda y, \lambda h_\lambda(y)) = \lambda$. So,

$$\begin{aligned} (\nabla_{\xi'}\tau)(\lambda y, \lambda h_\lambda(y)) + (\partial_{\xi_n}\tau)(\lambda y, \lambda h_\lambda(y)) \nabla_y h_\lambda(y) \\ = \lambda^{-1} \nabla_y [\tau(\lambda y, \lambda h_\lambda(y))] = 0\,, \end{aligned}$$

or, equivalently,

$$\nabla_y h_\lambda(y) = -\frac{(\nabla_{\xi'}\tau)(\lambda y, \lambda h_\lambda(y))}{(\partial_{\xi_n}\tau)(\lambda y, \lambda h_\lambda(y))}\,. \tag{4.3.17}$$

Hypothesis (i) ($|\partial_\xi^\alpha \tau(\xi)| \leq C_\alpha (1+|\xi|)^{1-|\alpha|}$ for all $\xi \in \mathbb{R}^n$) and (4.3.14) ($|\partial_{\xi_n}\tau(\lambda y, \lambda h_\lambda(y))| \geq C > 0$) then ensure that $|\nabla_y h_\lambda(y)| \leq C$ for all $y \in U$, $\lambda > 0$.

For higher derivatives, note that $|(y, h_\lambda(y))| \leq R_1$ by hypothesis (iv); so, using hypothesis (i) once more, for all multi-indices α, there exists a constant $C_\alpha > 0$ such that

$$|(\partial_\xi^\alpha \tau)(\lambda y, \lambda h_\lambda(y))| \leq C_\alpha \lambda^{1-|\alpha|}\,.$$

Then, differentiating (4.3.17), this ensures, by an inductive argument, that the desired result for higher derivatives of $h_\lambda(y)$ holds, proving the Lemma. $\square$

Returning to the proof of (I4), note that,

$$|\partial_\xi^\alpha a_0(\xi)| \leq C_\alpha (1+|\xi|)^{-|\alpha|} \text{ for all } \xi \in \mathbb{R}^n\,,$$

since, $\widetilde{a}(\xi)$ is a symbol of order $\frac{n-1}{\gamma} - n$ (see (4.3.12) for its definition). Together with Lemma 4.3.4, this ensures that $\partial_y^\alpha[a_0(\mu y, \mu h_\mu(y))$ is uniformly bounded for all $y \in U$, $\mu > 0$ and $|\alpha| \leq [\frac{n-1}{\gamma}] + 1$ as required.

Finally, we show (I3′): observe that for $|\rho| < \delta'$, some suitably small $\delta' > 0$,

$$\begin{aligned} F(\rho,\omega,\mu;z) &= h_\mu(\rho\omega+z) - h_\mu(z) - \rho\omega\cdot\nabla_y h_\mu(z) \\ &= \sum_{k=2}^{\gamma+1}\Big[\sum_{|\alpha|=k}\frac{1}{\alpha!}(\partial_y^\alpha h_\mu)(z)\omega^\alpha\Big]\rho^k + R_{\gamma+1}(\bar\rho,\omega,\mu;z)\rho^{\gamma+2}\,. \end{aligned}$$

So, $F(\rho,\omega,\mu;z)$ is a function of convex type γ if (using the numbering of Definition 4.2.1)

(CT2) $\sum_{k=2}^{\gamma+1}\Big|\sum_{|\alpha|=k}\frac{1}{\alpha!}(\partial_y^\alpha h_\mu)(z)\omega^\alpha\Big| \geq C > 0$ for all $\omega \in \mathbb{S}^{n-2}$, $\mu > 0$, $z \in \mathbb{R}^{n-1}$.

(CT3) $|\partial_\rho F(\rho,\omega,\mu;z)|$ is increasing in ρ for $0 < \rho < \delta$, for each $\omega \in \mathbb{S}^{n-2}$, $\mu > 0$;

(CT4) for each $k \in \mathbb{N}$, $\partial_\rho^k F(\rho,\omega,\mu;z)$ is bounded uniformly in $0 < \rho < \delta'$, $\omega \in \mathbb{S}^{n-2}$, $\mu > 0$.

Condition (CT4), follows straight from Lemma 4.3.4. The concavity of $h_\mu(y)$ means that

$$\partial_\rho^2 F(\rho,\omega,\mu;z) = \partial_\rho^2[h_\mu(\rho\omega+z)] = \omega^t \operatorname{Hess} h_\mu(\rho\omega+z)\omega \leq 0$$

for all $0 < \rho < \delta'$ and for each $\omega \in \mathbb{S}^{n-2}$, $\mu > 0$, $z \in \mathbb{R}^{n-1}$; coupled with the fact that $\partial_\rho F(0,\omega,\mu;z) = 0$, this ensures Condition (CT3) holds.

Lastly, recall that, by definition, $\gamma \geq \gamma(\Sigma_\lambda)$ for all $\lambda > 0$, which is the maximal order of contact between Σ_λ and its tangent plane; furthermore, γ is assumed to be finite; thus, for some $k \leq \gamma + 1 < \infty$, we have

$$\partial_\rho^k[h_\mu(z+\rho\omega)]\Big|_{\rho=0} \neq 0\,.$$

Now, $\partial_\rho^k[h_\mu(z+\rho\omega)]\big|_{\rho=0} = \sum_{|\alpha|=k} \frac{k!}{\alpha!}\partial_y^\alpha h_\mu(z)\omega^\alpha$, so for some $k \leq \gamma+1$, we have

$$k! \left| \sum_{|\alpha|=k} \frac{1}{\alpha!}\partial_y^\alpha h_\mu(z)\omega^\alpha \right| \geq C > 0$$

for all $\omega \in \mathbb{S}^{n-2}$. Thus, condition (CT2) holds.

This completes the proof of conditions (I1)–(I4), and, hence, Theorem 4.3.1. $\square$