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Nonparametric bounds for the probability of future prices based on option values

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Abstract: Interest in using option prices to estimate implied probabilities of stock values has emerged out of evidence suggesting the lognormal assumption of the Black Scholes model is no longer accurate. Most of the evidence relates to stock index option prices, especially since October 1987. The Black Scholes model assumes stock prices follow a geometric Brownian motion in continuous time - a lognormal distribution in discrete time. The standard deviation or volatility of the stock price process is the only unknown value in the formula so that implied standard deviations (volatilities) can be deduced from observed option prices. Prior to 1987, however, the implied volatility tended to curve upwards at far from at-the-money strike prices. Because of its shape, the relation came to be known as the "smile". The smile implies a fat-tailed underlying distribution, a long recognized feature of stock prices. Since the 1987 crash, the smile has deteriorated much farther from what it is supposed to look like under lognormality. Not flat and now not even a smile, it skews significantly to the left, indicating large probabilities of price decreases. This has led to recent proposals that focus on nonparametric estimates of the shape of the underlying distribution. A similar approach is followed here, but rather than estimating specific distributions, bounds are derived for the set of probability distributions that could have generated observed prices. These may be considered as either the first step toward identifying a single estimate, or as a nonparametric range of estimates for the underlying probabilities.

Key words: Option prices, nonparametric statistics, futures prices, robustness, Black Scholes model.

AMS subject classification: 90A09.

1 Introduction

Interest in using option prices to estimate implied probabilities of stock values has emerged out of evidence suggesting the lognormal assumption of the Black Scholes model is not very accurate. Most of the evidence relates to stock index option prices, especially since October 1987. The Black Scholes model assumes stock prices follow a geometric Brownian motion in continuous time–a lognormal distribution in discrete time. The standard deviation or volatility of the stock price process is the only unknown value in the formula so that implied standard deviations (volatilities) can be deduced from observed option prices.[1] Since European calls with different strike prices, but the same expiration date, are governed by the same probability distribution, they will have identical implied volatilities when the lognormal specification is valid. Prior to 1987, however, the implied volatility tended to curve upwards at far from at-the-money strike prices. Because of its shape, the relation came to be known as the "smile". The smile implies a fat-tailed underlying distribution, a long recognized feature of stock prices; see, e.g., Mandelbrot (1963) and Fama (1965). Since the 1987 crash, the smile has deteriorated much farther from what it is supposed to look like under lognormality. Not flat and now not even a smile, it skews significantly to the left, indicating large probabilities of price decreases - what Rubinstein (1994) calls, "crashophobia".

The initial response to understanding the smile was to generalize the geometric Brownian motion model by making volatility random, while maintaining lognormality. Stochastic volatility models generate smiles because they make the (unconditioned by volatility) underlying distribution fatter-tailed than lognormal. (This corresponds to the well known Monte Carlo trick for generating fat-tailed distributions: generate normal variates, but with different variances).

The recent evidence on implied volatilities has led to proposals that focus on estimating the entire shape of the underlying distribution; see, e.g., Shimko (1993) and Rubinstein (1994). These methods are nonparametric and do not presume lognormality. A similar approach will be pursued here, but rather than estimating specific distributions, bounds are derived for the set of probability distributions that could have generated observed prices. These may be considered as either the first step toward identifying a single estimate, or as a nonparametric range of estimates for the underlying probabilities.

[1] As the only free parameter, volatility stands for everything that affects option prices, but which is not in the model; see Figlewski (1989).

1.1 Convexity

The estimates are based on a connection between convex functions and cumulative probability distributions. To any cdf F there is the associated convex function:

$$g(K) = \int_{\infty}^{K} F(s)ds.$$

Convexity of g follows from the fact that its derivative is the nondecreasing cdf F.[2] The resulting convex function is not arbitrary as its derivative must also satisfy the boundary conditions of a cdf; $F(x) \rightarrow 1$ and 0 as $x \rightarrow \pm\infty$. Conversely, to a convex function g (satisfying the boundary conditions), there is the associated cdf that is its first derivative.[3]

This convexity correspondence arises with option valuation because arbitrage-free option prices are necessarily a convex function of strike prices. Hence there is always a probability distribution implicit in such option prices. Since convexity follows from arbitrage-free valuation alone, there exists an implied cdf given *any* specification of risk preferences and *any* stochastic process for the underlying stock price.

When investors are risk-neutral the implied probability distribution is identical to the cdf of the underlying stock price at expiration. This also occurs under assumptions, such as those in the Black Scholes model, where call values are determined independently of investor risk preferences. In the Black-Scholes model the underlying stock price is assumed to follow a geometric Brownian motion, a lognormal distribution in discrete time (or a binomial process that is Brownian motion in the limit). When prices follow such a process, call values are determined by arbitrage considerations alone - risk preferences do not matter - and the implied risk-neutral cdf is the same as the one that governs the stock at expiration.

The existence of implied probabilities however holds generally and does not require the lognormal specification. There is an implied distribution

[2]This does not require that g be differentiable or that F be continuous. When F corresponds to a discrete distribution, g is a polyhedral convex function. F can be recovered from g via the directional derivative, where the direction is determined by the left/right continuity convention adopted for cdfs. For properties of convex functions and their derivatives see Rockafellar (1970).

[3]This convexity/probability connection arises in unexpected places. One case is the generalized Lorenz curve used for determining second degree stochastic dominance. The generalized Lorenz curve is the g function derived from the quantile (inverse of F) income distribution. A different context where the convexity is useful is in verifying that a particular linear function of regression quantiles defines an empirical cdf. It is not at all obvious, for example, that a combination of regression quantiles defines an empirical cdf until the combination is recognized as the derivative of a convex function; see Bassett and Koenker (1982, Theorem 2.1, p.409) and Koenker and Bassett (1978).

when F is not lognormal, and even if investors do not have explicit probability assessments about future values. In situations where risk matters, there will be an implied cdf, though it need not agree with the process that governs prices at expiration.

Section 2 briefly describes the connection between convex, arbitrage-free, call values and the implied risk-neutral probability distribution. Arbitrage-free call values were first described in Merton (1973) (also, see Cox and Rubinstein (1985)), and Breeden and Litzenberger (1979)). By proceeding from prices to inferred probabilities, we reverse the standard approach in which prices arise out of causally prior probabilities. Bounds for the underlying cdf, given a discrete set of option prices, are presented in Section 3 along with the modifications needed when convexity is invalidated by non-zero transaction costs, bid-ask spreads, and nonsynchronous prices.

2 Option values and probabilities

Let $c(K)$ denote the time t value of a European call option with expiration $T > t$. The underlying asset's current value is S_t and the unknown value at expiration is S_T. Suppose initially that no dividends are paid between t and T, and that there is a continuum of strike prices in the interval $[0, K_{\max}]$, where $K_{\max}$ is large enough that $c(K_{\max}) = 0$.

2.1 Risk-neutral call values

Let the risk free rate of return be denoted by r. Suppose investors are risk-neutral; that is, indifferent between a riskless return and a random return with the same expected value; see Harrison and Pliska (1981). Prices in equilibrium are then determined by expected values. Let $F(s) = \Pr[S_T \leq s]$ represent a cdf for S_T. Since the value S_t invested today in risk free bonds at rate r yields $e^{r(T-t)}S_t$ at time T, the expected value for a risk-neutral investor must grow at the risk free rate. Hence, the expectation of F is required to satisfy, $E(S_T) = e^{r(T-t)}S_t$, but otherwise F is arbitrary.

The value of $c(K)$ at expiration is the random variable, $\max\{0, S_T - K\}$, whose expected value is,

$$E[\max\{0, S_T - K\}] = \int_K^\infty (s - K)dF(s).$$

Integration by parts yields the convenient expression,

$$E[\max\{0, S_T - K\}] = E(S_T) + \int_0^K (F(s) - 1)ds.$$

Finally, let $g(K : F)$ denote the discounted present value of the expectation,

$$g(K : F) = e^{-r(T-t)}E[\max\{0, S_T - K\}] = S_t + e^{-r(T-t)} \int_0^K (F(s) - 1)ds \tag{1}$$

This provides the basic expression for determining prices from probabilities, or probabilities from prices. In a risk neutral world with distribution F, call prices are $c(K) = g(K : F)$. Alternatively, given call prices $c(K)$ there exists the implied distribution F such that $g(K : F) = c(K)$.

From expression (1) we see that the first derivative of g with respect to K recovers the underlying cdf, and the second derivative produces the probability density; the respective derivatives are $F(K) - 1$ and $f(K)$, each scaled by the discount factor. Since the functions are related by integration/differentiation, the call price curve will be smoother than the cdf, which will be in turn smoother than the density. Finally, expression (1) shows that $g(0 : F) = S_t$, reflecting the equivalence between the underlying stock and a zero-strike call option. Since the shares, but not the options, may receive dividends, the identity has to be modified when dividends are nonzero.

In view of (1), risk neutral call prices satisfy certain basic properties: there has to be an F such that $c(K) = g(K : F)$. What does this imply about the form of $c(K)$? The following are the features of risk-neutral call values.

1. $c(K)$ is nonnegative with $c(0) = S_t$.
2. $c(K)$ is decreasing with $-e^{r(t-T)} < dc/dK < 0$.
3. $c(K)$ is convex.

The first property follows from $\max\{0, S_T - K\} \geq 0$; the second says the derivative is nonpositive,

$$dc/dK = dg/dK = e^{-r(T-t)}(F(K) - 1) \leq 0;$$

and the third follows from the fact that the first derivative increases with K, or, when there is a density the second derivative is $e^{-r(T-t)}f(K) \geq 0$.

2.2 Arbitrage-free call prices

Suppose now that risk neutrality is relaxed and investors have arbitrary risk preferences and perhaps even know nothing of probability. Suppose, however, that all arbitrage opportunities are exploited; that is, call prices $c(K)$ are such that there are no riskless profit opportunities from buying or selling calls, or investing at the risk free interest rate (assuming zero transaction costs). What does this imply about $c(K)$?

It is now well known that arbitrage-free call prices are identical to risk-neutral call prices; call values are arbitrage-free if and only if there is an F such that $c(K) = g(K : F)$. In a risk neutral world the F representing the beliefs of investors is the same as the F implied by call prices, whereas in a non risk-neutral world the F implicit in call prices is the equivalent martingale measure. This may seem surprising since the arbitrage-free requirement says nothing about probabilities. The intuition behind the equivalence is similar to Dutch book explanations for coherent beliefs regarding probability assessments. When your beliefs are not consistent with the probability axioms you can make book against yourself and win (lose). To see why arbitrage-free call values must be nonnegative, decreasing, and convex, as well as the risk free arbitrage opportunities that would occur if one of the conditions is violated, see Cox and Rubinstein (1985, p.237); also, see Cox and Ross (1976) for option valuation with stochastic processes other than geometric Brownian motion.

2.3 Call price curves

Given expression (1) we can identify an F from a $c(K)$, or a $c(K)$ from an F. The types of curves that arise in simple special cases are illustrated in the following examples. For simplicity r is assumed to be zero.

2.3.1 Discrete probabilities

Suppose S_T is a discrete random variable that takes values s_j with probabilities p_j, $j = 1, ..., J$. Then the cdf $F(s) = \Pr[S_T \leq s]$ is a discontinuous jump function, and call prices are a linear spline,

$$c(K) = c(s_j) - (1 - F(s_j))(K - s_j), \ s_j \leq K \leq s_{j+1}.$$

This situation is illustrated in Figure 1.

2.3.2 Histogram probabilities

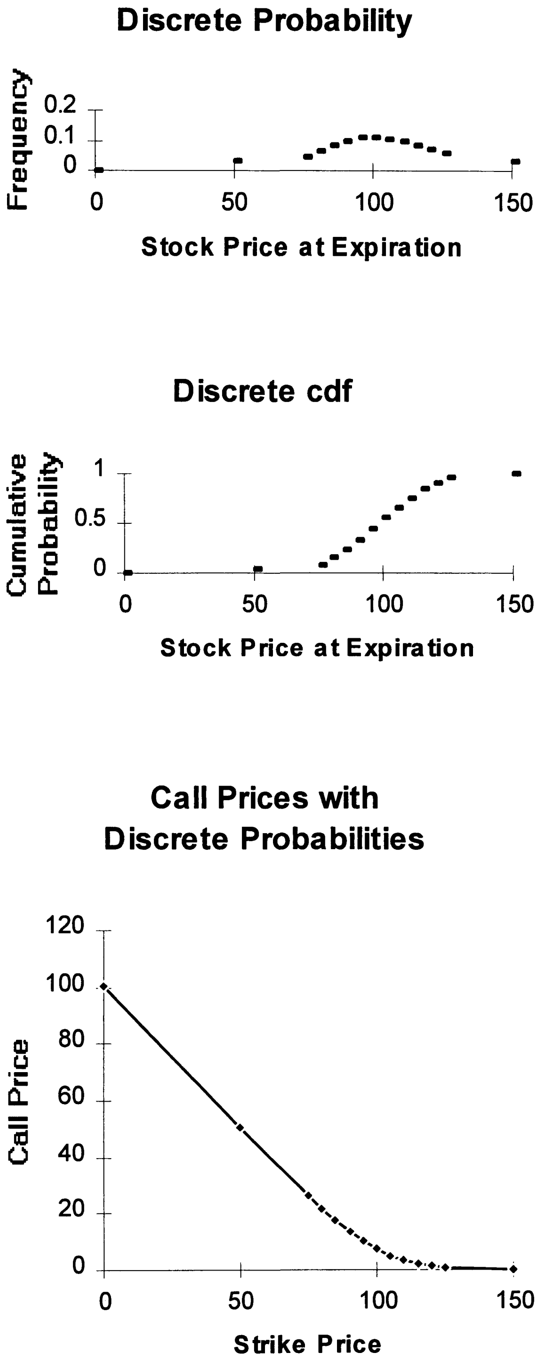
Let the probability density for prices at expiration be a histogram: in the interval, $[s_j, s_{j+1}]$, S_T is uniformly distributed. Integrating a histogram gives a piecewise linear cdf, and integrating again gives a quadratic spline for call prices. To see this, write S_T as a mixture of uniform distributions,

$$F(s) = \sum_{j=1}^{J} p_j U_j(s)$$

where,

$$U_j(s) = \begin{cases} 0 & s < s_j \\ \frac{s - s_j}{s_{j+1} - s_j} & s_j \leq s \leq s_{j+1} \\ 1 & s > s_j \end{cases}$$

Figure 1


Discrete Probability
Frequency
0.2
0.1
0
0
50
100
150
Stock Price at Expiration
Discrete cdf
Cumulative Probability
1
0.5
0
0
50
100
150
Stock Price at Expiration
Call Prices with Discrete Probabilities
Call Price
120
100
80
60
40
20
0
0
50
100
150
Strike Price

Substituting into (1) gives the quadratic spline,

$$c(K) = c(s_j) + \sum_{j=1}^{j-1} p_j (K - s_j) + \frac{1}{2} p_j \left[\frac{(K - s_j)^2}{(s_{j+1} - s_j)} \right] - (K - s_j), \;\; s_j \leq K \leq s_{j+1}.$$

Conversely, if $c(K)$ were a quadratic spline then the implied F would be a histogram density. This situation is illustrated in Figure 2.

2.3.3 Mixture models

The representation of the histogram as a mixture of uniform distributions can be extended directly to cases where the mixing distributions are not uniform. That is, now let

$$F(s) = \sum_{j=1}^{J} w_j F_j(s).$$

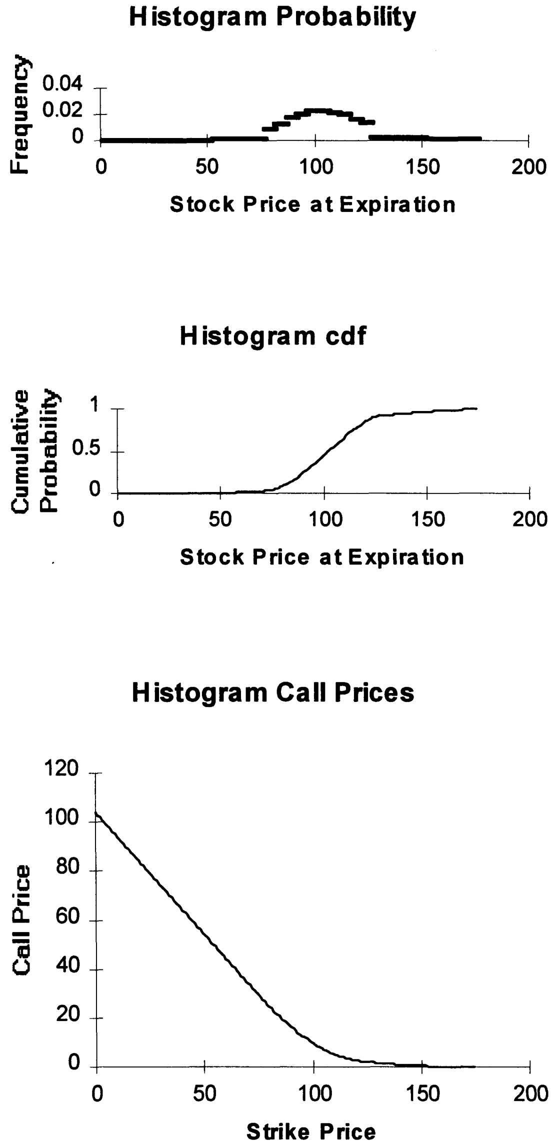
This says that S_T is, with probability w_j, the random variable S_j with cdf F_j. In the case where all the F_js are lognormal with different variances, this corresponds to a lognormal stochastic volatility model in which Hull and White (1987) showed that call prices are the average of the call prices over the mixture. This extends to risk-neutral valuation with any mixing distributions, because

$$g\left(K : \sum_{j=1}^{J} w_j F_j(s) \right) = \sum_{j=1}^{J} w_j g\left(K : F_j(s) \right).$$

3 Estimating and bounding implied probability distributions

We first consider the case where call prices are arbitrage-free and hence convex, but where there are only a discrete number of strikes. Let $c_i = c(K_i)$, $i = 0, ..., n$, denote European call option prices on the same underlying asset with the same expiration date T, but different strike prices K_i, where $K_i < K_{i+1}$ and $K_0 = 0$. The price of the zero price call is set equal to the current price of the stock, S_t. (If there are dividends then $c(0) = e^{-\delta(T-t)} S_t$ where δ is the payout rate through the expiration date). Assuming arbitrage-free valuation, the remaining call prices must satisfy, $c_i = g(K_i : F)$, for some unknown cdf F.

Figure2



The estimation problem is to identify the F that generates the call prices $(c_0, ..., c_n)$. Additional structure can be imposed by introducing restrictions on the set of allowed cdfs. When F is restricted to be lognormal with unknown variance the problem reduces to estimating the volatility smile. Less structure is imposed by Shimko (1993), who essentially estimates F and its associated density from a smoothed quadratic fit to the implied volatilities. The method in Rubinstein (1994) is still more general as it essentially finds the risk-neutral density that is (least squares) closest to the lognormal density that could have generated call prices.

Consider the set of all cdf's that are consistent with the observed call prices, namely,

$$\{F \quad | \quad g(K_i : F) = c(K_i), i = 0, ..., n\}.$$

If there was a continuum of strike prices, this set would consist of a single F and the implied cdf would be exactly

$$F(K) = e^{r(T-t)}[dc(K)/dK] + 1.$$

For the discrete strike case, let $\widehat{F}(K_i)$ be the analogous difference quotient,

$$\widehat{F}(K_i) = e^{r(T-t)}\left[\frac{c(K_{i+1}) - c(K_i)}{K_{i+1} - K_i}\right] + 1 \qquad i = 0, ..., n-1,$$

and set $\widehat{F}(K_n) = 1$.

These $\widehat{F}(K_i)$ values can be used to bound the set of cdfs that could have generated call prices. Substitute $g(K_i : F)$ for $c(K_i)$ in the definition of $\widehat{F}(K_i)$ and use

$$F(K_i) \leq \frac{1}{(K_{i+1} - K_i)} \int_{K_i}^{K_{i+1}} F(s)ds \leq F(K_{i+1})$$

to show,

$$F(K_{i-1}) \leq \widehat{F}(K_i) \leq F(K_i).$$

This yields upper and lower bounds on F at each strike price; namely,

$$\widehat{F}(K_{i-1}) \leq F(K_i) \leq \widehat{F}(K_i).$$

When the K_is are close together this interpolation provides tight bounds for the allowed probabilities. When strike prices are far apart the bounds will be correspondingly large as it is then necessary to interpolate F over a large range of unobserved strike values.

Since F is nondecreasing, the $\widehat{F}(K_i)$ values can be used to bound the entire F function. Upper and lower cdfs are given by the discrete cdfs with jumps at the K_i values. Tighter bounds are given by assuming the underlying F is continuous. In this case the upper and lower cdfs are given by linearly interpolating between the $\widehat{F}(K_i)$ values.

Define an upper cdf by,

$$\widehat{F}_U(K) = \widehat{F}(K_i) + (K - K_i)\left[\frac{\widehat{F}(K_{i+1}) - \widehat{F}(K_i)}{K_{i+1} - K_i}\right],$$

$$K_i \leq K \leq K_{i+1}, \quad i = 0, ..., n-1,$$

and a lower cdf by,

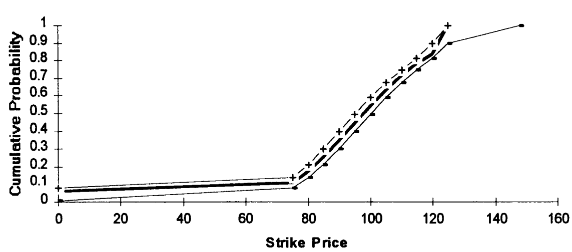
$$\widehat{F}_L(K) = \widehat{F}(K_{i-1}) + (K - K_{i-1}) \left[\frac{\widehat{F}(K_i) - \widehat{F}(K_{i-1})}{K_i - K_{i-1}} \right],$$

$$K_{i-1} \leq K \leq K_i, \quad i = 1, ..., n.$$

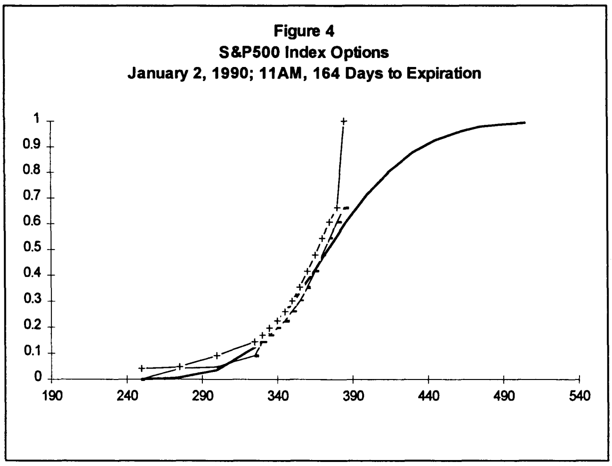
These upper and lower cdfs bound the F that could have generated call prices, the only restriction being that the underlying cdf is continuous.

The bounds are illustrated in Figure 3. The data is from an example in Rubinstein (1994, p.781) in which call prices are generated by the Black Scholes model. For comparison the figure also shows a lognormal cdf. Since the call prices are generated from the Black Scholes model the lognormal cdf falls nicely between the bounding cdfs.

Figure 3
Implied Probability Bounds



Another illustration, also from Rubinstein (1994, p. 784), is shown in Figure 4. Call prices are for the S&P500 Index at 11AM on January 2, 1990. The reference lognormal cdf is now seen to not fall within the bounding cdfs. The bounding cdfs are consistent with the estimated density function shown in Rubinstein in that the upper tail is much shorter and the left tail much longer than lognormal. Market prices imply a cdf with much greater chance of downward price movements than would be suggested by a lognormal cdf.



Remark 1 *The distributions that fall between the upper and lower cdfs do not all have the same expectation. The allowed risk-neutral cdfs are those that fall within the bounds and which have their discounted expectation equal to the current stock price.*

3.1 Estimation

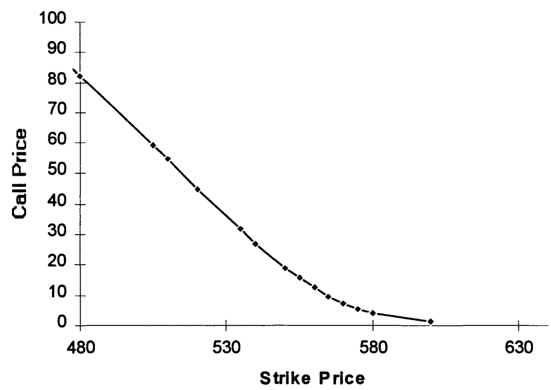
Now let $g(K_i : F)$ be a model for actual prices that are observed with error. The error term stands for all the non arbitrage reasons for differences between the call price and its g value. These reasons include: the bid-ask spread, nonsynchronous prices, and positive transaction costs.

(It will be assumed that the difference between g and observed option prices does not depend on K. A more general analysis would permit the error variance to depend on trading volume, or, what is practically the same, the moneyness of the option, $|S_t - K|$.)

The presence of the error term means actual prices need not be convex. Hence, linear interpolation between adjacent strike prices need not yield a convex function, the resulting implied "cdf" based on the difference quotient need not be decreasing, and the implied density and probabilities could be negative.

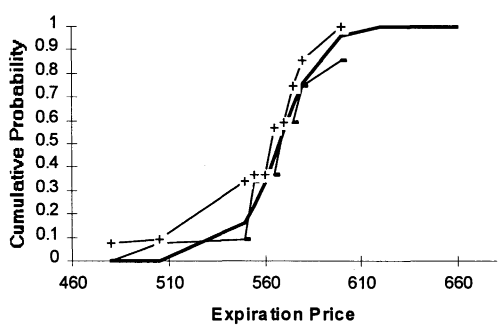
Figure 5 shows call prices for the closing S&P500 call on July 13, 1995. Prices are almost convex; there is slight concavity in the deep in the money calls. (Note that, unlike the above example with 11AM prices, these closing prices are likely susceptible to slight departures from convexity on account of nonsynchronous prices near the close.) Since prices are nearly convex, the convex hull of observed prices is used for the values of g at the given strike prices.

Figure 5
S&P500 September Call
July 13, 1995



The interest rate and dividend discount factors were estimated using put-call parity as in Shimko (1993). Bounds based on the convex hull of call prices are shown in Figure 6. The figure depicts bounding cdfs that are jumpier than shown in the previous figures, perhaps due to nonsynchronous closing prices. Similar to the other figures, however, there is a fat left-hand tail and a large difference from lognormality.

Figure 6
S&P September Call
July 13, 1995, Close



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