Asymptotic Design of General Triangular Stopping Boundaries for Brownian Motion *

By Peng Huang

Medical University of South Carolina at Charleston

We consider triangular stopping boundaries for a Brownian motion with drift, with specified error probabilities at two given values for the drift. We consider the Kiefer-Weiss problem of finding boundaries which minimize the maximum expected stopping time asymptotically as the error probabilities tend to zero. A construction is given which minimizes the objective function through fourth order optimality. This extends earlier work for the simpler symmetric (equal error probabilities) case, where fifth order minimization was achieved.

1. Introduction. Consider testing the hypotheses $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ for the drift θ of a Brownian motion Y. Kiefer and Weiss [12] suggest searching for the test such that the maximum (over θ) of the average stopping time (AST) is minimized under some prespecified error probabilities (α, β) at θ_0 and θ_1. Lorden [14] combined two SPRTs, of θ_0 versus θ_m and of θ_0 versus θ_m for some intermediate θ_m, to form a particular class of tests called 2-SPRTs. He showed, for any fixed θ^*, a 2-SPRT can be chosen such that its stopping time T^* satisfies

$$E_{\theta^*} T^* = \inf_{T \in D(\alpha,\beta)} E_{\theta^*} T + o(1)$$

as $\min(\alpha, \beta) \to 0$, where $D(\alpha, \beta)$ is the class of all tests with error probability bounds (α, β). For Brownian motion, 2-SPRTs have triangular stopping boundaries.

In the symmetric case when $\beta = \alpha$, it is known that $\sup_\theta E_\theta T = E_{\theta_m} T$ for all $T \in D(\alpha, \alpha)$, where $\theta_m = (\theta_0 + \theta_1)/2$. Hence the 2-SPRT stopping time T_m with respect to this θ_m satisfies

$$\sup_\theta E_\theta T_m = \inf_{T \in D(\alpha,\alpha)} \sup_\theta E_\theta T + o(1).$$

Lai [13] also showed that, in the symmetric case, the asymptotic shape of the minimax (Kiefer-Weiss) stopping boundaries are triangular. In the asymmetric case, Huffman [11] extended Lorden's results to show that by solving $\tilde{\theta}$ from some equation numerically, the stopping time $\tilde{T}$ of 2-SPRT with respect to this $\tilde{\theta}$ satisfies

$$\sup_\theta E_\theta \tilde{T} = \inf_{T \in D(\alpha,\beta)} \sup_\theta E_\theta T + o(|\log \alpha|^{1/2})$$

as $\alpha \to 0, \beta \to 0$ and $0 < C_1 < \log \alpha / \log \beta < C_2 < +\infty$, where C_1 and C_2 are constants. Note that $|\log \alpha|^{1/2} \to \infty$ as $\alpha \to 0$. Such results were extended further by Dragalin and Novikov [3]. They showed that

$$\sup_\theta E_\theta \tilde{T} = \inf_{T \in D(\alpha,\beta)} \sup_\theta E_\theta T + O(1)$$

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for this same 2-SPRT. Asymptotic expansions of the two error probabilities and the value of the maximum AST for 2-SPRT were given by Dragalin and Novikov [4].

Various formulas associated with general triangular tests were first given by Anderson [1]. Equivalent formulas were given by Hall [8]. Whitehead [15], and his PEST software [2], provide theory and method for symmetric 2-SPRTs. For asymmetric cases ($\beta \neq \alpha$), they find θ_1' for which the symmetric 2-SPRT of θ_0 versus θ_1' with error probabilities (α, α) has error probability β at θ_1. This choice is not minimax. However, various results and software for symmetric designs can be easily adapted for this asymmetric case. Hall [7] found, numerically, the minimax triangular tests (MTT) for several choices of (α, β), and noted that the resulting average stopping time (AST) functions are uniformly smaller than those of designs given by PEST. Huang, Dragalin and Hall [10] and Huang [9] utilized Hall's [8] formulas to study mathematically how the error probabilities affect the AST functions asymptotically among symmetric triangular designs and found asymptotic expansions for the parameters of minimax triangular stopping boundaries. The asymptotic minimax triangular tests (AMTT) achieving fifth order optimality are found and simple constructions are given. The AMTT stopping time T_a satisfies

$$\sup_{\theta} E_\theta T_a = \inf_{T \in D(\alpha)} \sup_{\theta} E_\theta T + O(|\log \alpha|^{-3/2}),$$

where $D(\alpha)$ is the class of triangular tests with equal error probabilities approximated to the order $O(\alpha/|\log \alpha|^2)$. Note that $|\log \alpha|^{-3/2} \to 0$ as $\alpha \to 0$. (If the error term is $O(|\log \alpha|^{1-d/2})$, we say the order of optimality is d.) Analytic and numerical comparison showed that family of AMTT achieves uniform reduction in AST function compared to the family of 2-SPRT.

In this paper, results in [10] are extended to asymmetric triangular tests ($\beta > \alpha$). The performance of the resulting AMTT are compared to designs from PEST which adapt symmetric 2-SPRTs to asymmetric triangular designs. A family of tests satisfying

$$\sup_{\theta} E_\theta T_a = \inf_{T \in D_3} \sup_{\theta} E_\theta T + O(|\log \alpha|^{-1})$$

is found (achieving fourth order optimality) and a construction is given, where D_3 is defined in Theorem 3.

By a suitable rescaling

$$X(t) = \frac{\theta_1 - \theta_0}{2} Y\left(\frac{4}{(\theta_1 - \theta_0)^2} t\right) - \frac{\theta_1 + \theta_0}{\theta_1 - \theta_0} t,$$

the original hypotheses about θ become $H_0 : \delta = -1$ versus $H_1 : \delta = 1$ for the drift δ of X, where $\delta = 2\theta/(\theta_1 - \theta_0) - (\theta_1 + \theta_0)/(\theta_1 - \theta_0)$. Hence, without loss of generality, we will confine our attention to hypotheses $H_0 : \delta = -1$ versus $H_1 : \delta = 1$ for the drift δ of Brownian motion X.

Section 2 studies the asymptotic behaviors of the operating characteristic function (OC) and AST using Hall's [8] formulas for OC and AST functions. The neighborhood of δ where the maximum of AST occurs is found. Section 3 shows how stopping boundary parameters are affected by its error probability functions asymptotically. Based on such relation, we can choose the design parameters in order to achieve the desired error probabilities asymptotically. We then search for asymptotic minimax triangular stopping boundaries. Families of first, second, third and fourth order asymptotic minimax triangular tests (AMTT) are found, and simple constructions are given. Numerical comparisons show that design parameters of AMTT come very close to those of the exact minimax triangular tests (MTT) obtained numerically by Hall [7]. Section 4 compares the performances of AMTT with Whitehead's designs given by PEST. Figure 4 shows that AMTT achieves uniform reduction in AST function compared to that of PEST design. Throughout the paper, *Mathematica*^{TM} [16] is used for some of the calculations. Huang [9] provides more details. Methods for analyzing results from triangular tests — p-values, median unbiased estimates and confidence intervals for the drift are given in [9].

Brownian motion provides a good approximation whenever a sequential stopping rule is based on a cumulative sum of independent identically distributed terms — with a moderately large number of terms. In statistical quality control, it can be used to make decisions on acceptance or rejection of manufactured or purchased product, to test if there are any assignable causes (special causes) in production procedures. Lack of control is often indicated by points falling outside the control limits (stopping boundaries). See Grant and Leavenworth [5].

2. Asymptotic behavior of OC and AST. We consider a Brownian motion $X = \{X(t), t \geq 0\}$ with drift δ and general triangular stopping boundaries

$$x = a - bt, \quad x = -a' + b't, \quad (a > 0,\ a' > 0,\ b + b' > 0). \tag{1}$$

Let T be the boundary hitting time. The hypothesis H_0 is rejected on the event $U = \{X(T) = a - bT\}$ and accepted on the event $L = \{X(T) = -a' + b'T\}$. The *operating characteristic function* and *average stopping time function* are defined by $OC(\delta) = P_\delta\{L\}$ and $AST(\delta) = E_\delta T$, respectively.

Let $\phi(x), \Phi(x)$ and $\overline{\Phi}(x)$ be the density, distribution and survival functions of the $\mathcal{N}(0,1)$ distribution. Let $M(x) = \overline{\Phi}(x)/\phi(x)$ be Mill's ratio, $\tau_L = -\delta + b'$, $\tau_U = \delta + b$, $t_v = (a + a')/(b + b')$, $B = (a'\tau_U - a\tau_L)^2/[2(a + a')(b + b')]$, $c = a + a'$, $s_j = jc + a'1_{(j=even)} + a1_{(j=odd)}$, and $r_j = (2j + 1)c - s_j$. Formulas for $OC(\delta)$ and $AST(\delta)$ are ([8]);

$$\begin{aligned} OC(\delta) &= \frac{e^{-B}}{\sqrt{2\pi}} \sum_{j=0}^{+\infty} (-1)^j \left[M\Big(\frac{s_j - \tau_L t_v}{\sqrt{t_v}}\Big) + M\Big(\frac{s_j + \tau_L t_v}{\sqrt{t_v}}\Big) \right], \\ &= 1 - \frac{e^{-B}}{\sqrt{2\pi}} \sum_{j=0}^{+\infty} (-1)^j \left[M\Big(\frac{r_j - \tau_U t_v}{\sqrt{t_v}}\Big) + M\Big(\frac{r_j + \tau_U t_v}{\sqrt{t_v}}\Big) \right] \end{aligned} \tag{2}$$

$$AST(\delta) = E_\delta T = E_\delta(T\ 1_U) + E_\delta(T\ 1_L), \tag{3}$$

where

$$E_\delta(T\ 1_U) = \frac{e^{-B}}{\tau_U \sqrt{2\pi}} \sum_{j=0}^{+\infty} (-1)^j r_j \left[M\Big(\frac{r_j - \tau_U t_v}{\sqrt{t_v}}\Big) - M\Big(\frac{r_j + \tau_U t_v}{\sqrt{t_v}}\Big) \right],$$

and $E_\delta(T\ 1_L)$ is similar but with r_j and τ_U replaced by s_j and τ_L.

Recursive formulas for $OC(\delta)$ and $AST(\delta)$ are also derived by Hall [6].

PROPOSITION 1 HALL. *(i) Let $d = 2(b+b')$. Then*

$$OC(\delta) = e^{-2a'(\delta-b')} - e^{2a(d-\delta-b)-2a'(\delta-b')}(1 - OC(\delta - d)).$$

(ii) For $\delta \neq -b, b'$

$$AST(\delta) = \frac{a(b'-\delta) + [a'b - ab' + (a+a')\delta]OC(\delta) + (b+b')OC'(\delta)}{(b+\delta)(b'-\delta)},$$

$$AST(-b) = a^2 - \frac{a}{b+b'} + (\frac{a+a'}{b+b'} - a^2)OC(-b) + OC''(-b),$$

$$AST(b') = \frac{a}{b+b'} + (a'^2 - \frac{a+a'}{b+b'})OC(b') - OC''(b').$$

Based on Proposition 1 we only need to study $OC(\delta)$ and $AST(\delta)$ in any interval of width $4\bar{b} = 2(b+b')$. We hence consider the interval $(w_{-1}, w_1]$, where $w_{-1} = w_0 - 2a\bar{b}/\bar{a}$, $w_1 = w_0 + 2a'\bar{b}/\bar{a}$, $w_0 = (ab' - a'b)/(a+a')$, $\bar{a} = (a+a')/2$, $\bar{b} = (b+b')/2$.

The asymptotic minimax problem in the class of triangular tests is to find stopping boundaries (1) such that, when the resulting error probabilities

$$(4) \qquad \alpha = P_{-1}(\text{ reject } H_0) \to 0, \quad \beta = P_1(\text{ reject } H_1) \to 0,$$

the AST function $E_\delta T$ satisfies $\sup_\delta E_\delta T \to \inf_{T'} \sup_\delta E_\delta T'$ for all T' defined on triangular boundaries for which $1 - OC(-1) = \alpha$ and $OC(1) = \beta$. Throughout, we assume that $\beta = \alpha^\rho$ for some positive constant ρ.

The following lemma, obtained directly from (2), (3) and the expansion for Mill's ratio $M(x) = 1/x - 1/x^3 + 3/x^5 + O(x^{-7})$ $(x > 0)$, will be used to construct the asymptotic minimax design. The interval $(w_{-1}, w_1]$ has width d. A proof is given in Appendix A.

LEMMA 1. *Suppose boundaries (1) are used. Let b and b' be fixed, $(a+a') \to +\infty$. Define $\bar{a} = (a+a')/2$, $\bar{b} = (b+b')/2$, $B = \bar{a}(\delta - w_0)^2/(2\bar{b})$, $w_0 = (ab' - a'b)/(a+a')$, $w_{-1} = w_0 - 2a\bar{b}/\bar{a}$, $w_1 = w_0 + 2a'\bar{b}/\bar{a}$, $q = a/(2\bar{a})$, $\xi = (\delta+b)/(2\bar{b})$.*
(i) For $w_{-1} < \delta < w_0$,

$$OC(\delta) = 1 - \frac{\sqrt{\pi}e^{-B}}{4\sqrt{2\bar{a}\bar{b}}}\left[\cot\left(\frac{(q+\xi)\pi}{2}\right) + \cot\left(\frac{(q-\xi)\pi}{2}\right) + O\Big(\frac{1}{\bar{a}}\Big)\right],$$

$$AST(\delta) = \frac{a'}{b'-\delta} + O(\sqrt{\bar{a}}\,e^{-B}).$$

(ii) For $\delta = w_0$,

$$OC(\delta) = \frac{1}{2} + O\Big(\frac{1}{\sqrt{\bar{a}}}\Big),$$

$$AST(\delta) = \frac{\bar{a}}{\bar{b}} - \frac{\bar{a}^{3/2}}{\sqrt{2\pi}\,a'q\,\bar{b}^{3/2}}$$
$$+\frac{\sqrt{2}\pi^{3/2}\,\bar{a}^{3/2}}{96\,aa'\bar{b}^{5/2}}\left[5 + \cos(2q\pi)\right]\csc^2(q\pi) + O\Big(\frac{1}{\bar{a}^{3/2}}\Big).$$

(iii) For $w_0 < \delta < w_1$, $OC(\delta)$ is given by $1 - OC(\delta)$ in (i) with "$-\tan$" replacing "cot"; the $AST(\delta)$ is similar but with leading term $a/(b+\delta)$.
(iv) For $\delta = w_1$,

$$OC(\delta) = \left(\frac{1}{2} + O(\frac{1}{\sqrt{\bar{a}}})\right) \exp\left[\frac{-2a'^2\,\bar{b}}{\bar{a}}\right],$$

$$AST(\delta) = \frac{a\bar{a}}{(a+2a')\bar{b}} + O(\bar{a}e^{-B}).$$

Note that these reduce correctly in the symmetric case to formulas given in Lemma 1 in [10]. In the symmetric case (when $\rho = 1$), it is known that the maximum of $AST(\delta)$ occurs at $\delta_m = 0$. However, for a general triangular test, the δ_m where the supremum of $AST(\delta)$ occurs does not have a closed form. It is not difficult to see that δ_m falls in some finite interval for tests satisfying (4). The following lemma, whose proof is given in [9], states that δ_m is not far from w_0.

LEMMA 2. *Let b and b' be fixed in (1), $a/a' + a'/a = O(1)$, and $w_0 = (ab' - a'b)/(a+a')$. Then for any given $c > 0$, $0 < r < 1/2$, b and b' $(b + b' > 0)$, we can choose a, depending only on c, r, b, and b', large enough such that*

$$\sup_{|\delta - w_0| \geq c/a^r} AST(\delta) < AST(w_0).$$

3. Asymptotic minimax designs. In the remainder of the paper, we will assume that $a/a'+a'/a = O(1)$. Since $0 < r < 1/2$ implies $0 < r+(1/2-r)/2 < 1/2$, an immediate consequence of Lemma 2 and Lemma 1(ii) is

$$\sup_{\delta} AST(\delta) = AST\left(w_0 + o(\frac{1}{a^{1/2+\epsilon}})\right) \sim \frac{a+a'}{b+b'} \tag{5}$$

for any $\epsilon > 0$. This enables determination of the first order term in the asymptotic expansion of the minimax AST and the design parameters which assure it. The minimax AST is of order $O(m)$ with $m = -\log\alpha$. We refer to this result as the "first order asymptotic minimax construction".

THEOREM 1. *Suppose X is a Brownian motion with drift δ. Let D_1 be the class of all triangular tests with stopping boundaries of the form (1) and $1 - OC(-1) \sim \alpha$, $OC(1) \sim \beta = \alpha^\rho$ as $\alpha \to 0$. Let $m = -\log\alpha \to \infty$, $R = 1/(1+\sqrt{\rho})$. Let $D_1' \subset D_1$ be those tests for which*

$$a \sim \frac{m}{2R},\ a' \sim \frac{(1-R)m}{2R^2},\ \ b \sim (1-R),\ b' \sim R. \tag{6}$$

(i) Then $\inf_{T\in D_1} \sup_\delta E_\delta T \sim m/(2R^2)$. (ii) For any $T' \in D_1'$, $\sup_\delta E_\delta T' \sim m/(2R^2)$. (iii) For any $T'' \in D_1 - D_1'$, $\sup_\delta E_\delta T'' - \sup_\delta E_\delta T' \to +\infty$.

This theorem states that asymptotic minimax designs in class D_1 can be found from its subset D_1'.

PROOF. Consider tests in D_1. Let $Y(s) = c[X(s/c^2) - ds/c^2]$ with $c = b+b'$, $d = (b'-b)/2$. Then Y is a Brownian motion with drift $\theta = (\delta - d)/c$. The corresponding stopping boundaries for Y are $y = A - s/2$ and $y = -A' + s/2$, where $A = a(b+b')$, $A' = a'(b+b')$. The hypotheses are $H_0 : \theta = -(1+d)/c = \theta_{-1}$ versus $H_1 : \theta = (1-d)/c = \theta_1$. Let T and S be the stopping times for X and Y respectively, and $\theta_0 = (A - A')/(2(A+A'))$. Then $OC_Y(\theta_{-1}) = P_{\theta_{-1}}(Y(S) = -A' + S/2) = P_{-1}(X(T) = -a' + b'T) = OC_X(-1)$, $OC_Y(\theta_1) = P_{\theta_1}(Y(S) = -A' + S/2) = P_1(X(T) = -a' + b'T) = OC_X(1)$. Applying (5),

$$\sup_\delta E_\delta T = \frac{1}{4}(\theta_1 - \theta_{-1})^2 \sup_\theta E_\theta S \sim \frac{1}{4}(\theta_1 - \theta_{-1})^2 (A + A'),$$

$$\inf_{T \in D_1} \sup_\delta E_\delta T = \inf_{a,a',b,b'} \sup_\delta E_\delta T \sim \inf_{A,A',\theta_{-1},\theta_1} \frac{1}{4}(\theta_1 - \theta_{-1})^2 (A + A').$$

Let θ'_{-1} and θ'_1 be the leading terms of θ_{-1} and θ_1 respectively. Then we must have $\theta'_{-1} < \theta_0 < \theta'_1$ based on Lemma 1 (since $\alpha \to 0$). Define $\lambda = A'/A$, $u_{-1} = -1/2 - 1/(1+\lambda)$, $u_1 = 3/2 - 1/(1+\lambda)$, $c_1 = (2\theta'_1 - 1)(1+\lambda)/(2\lambda)$, $c_0 = (2\theta'_{-1} + 1)(1+\lambda)/2$, and the reflection mapping $\eta : (A, A', \theta, m, \rho) \longmapsto (A', A, -\theta, \rho m, 1/\rho)$. There are in total 9 cases for the pair $(\theta'_{-1}, \theta'_1) : \theta'_{-1} <,=,> u_{-1};\ \ \theta'_1 <,=,> u_1$. The following discussions are all using methods similar to the derivation of Lemma 1, and the fact that $1 - OC_Y(\theta_{-1}) \sim \alpha$ and $OC_Y(\theta_1) \sim \beta = \alpha^\rho$. Hence they will not be stated in full detail.

1^o If $\theta'_{-1} = u_{-1}$ and $\theta'_1 = u_1$, then $\lambda \sim \sqrt{\rho}$, $A \sim m/(2R)$. Hence $(\theta_1 - \theta_{-1})^2 (A + A')/4 \sim m/(2R^2)$. Similar to Lemma 1(*ii*), it can be shown that the second order term in $(1/4)(\theta_1 - \theta_{-1})^2 \sup_\theta E_\theta S$ is of order $O(\sqrt{m})$ with negative coefficient for $\sqrt{m}$.

2^o If $\theta'_{-1} = u_{-1}$ and $\theta'_1 > u_1$, then $c_1 > 1$, $\lambda \sim \sqrt{\rho/c_1}$, $A \sim (1+\lambda)m/2$. Hence $(\theta_1 - \theta_{-1})^2 (A + A')/4 \sim [2 + \sqrt{\rho}(1/\sqrt{c_1} + \sqrt{c_1})]^2 m/8$. Note that $[2 + \sqrt{\rho}(1/\sqrt{c_1} + \sqrt{c_1})]^2 m/8 > m/(2R^2)$, with equality iff $c_1 = 1$.

3^o If $\theta'_{-1} < u_{-1}$ and $\theta'_1 = u_1$, then $c_0 < -1$, $(\theta_1 - \theta_{-1})^2 (A + A')/4 \sim ([2\sqrt{\rho} + (1/\sqrt{-c_0} + \sqrt{-c_0})]^2 m/8$ by applying η to case 2^o. Note that $([2\sqrt{\rho} + (1/\sqrt{-c_0} + \sqrt{-c_0})]^2 m/8 > m/(2R^2)$, with equality iff $c_0 = -1$.

4^o If $u_{-1} < \theta'_{-1}$ $(< \theta_0)$ and $\theta'_1 > u_1$, then $|c_0| < 1$, $c_1 > 1$, $\lambda \sim (1 - c_0)\sqrt{\rho}/(2\sqrt{c_1})$, $A \sim \rho(1+\lambda)m/(2c_1\lambda^2)$. Hence $(\theta_1 - \theta_{-1})^2 (A + A')/4 \sim \rho(2/\sqrt{\rho} + \sqrt{c_1} + 1/\sqrt{c_1})^2 m/8$. Note that $\rho(2/\sqrt{\rho} + \sqrt{c_1} + 1/\sqrt{c_1})^2 m/8 > m/(2R^2)$, with equality iff $c_1 = 1$.

5^o If $\theta'_{-1} < u_{-1}$ and $(\theta_0 <)$ $\theta'_1 < u_1$, then similar results can be obtained by applying η to 4^o.

6^o If $\theta'_{-1} < u_{-1}$ and $\theta'_1 > u_1$, then $c_0 < -1$, $c_1 > 1$, $\lambda \sim \sqrt{-c_0\rho/c_1}$, $A \sim (1+\lambda)m/(-2c_0)$. Hence $(\theta_1 - \theta_{-1})^2 (A + A')/4 \sim [\sqrt{\rho}(\sqrt{c_1} + 1/\sqrt{c_1}) + (\sqrt{-c_0} + 1/\sqrt{-c_0})]^2 m/8$. Note that $[\sqrt{\rho}(\sqrt{c_1} + 1/\sqrt{c_1}) + (\sqrt{-c_0} + 1/\sqrt{-c_0})]^2 m/8 > m/(2R^2)$, with equality iff $c_1 = -c_0 = 1$.

7^o If $u_{-1} < \theta'_{-1} (< \theta_0)$ and $(\theta_0 <)$ $\theta'_1 < u_1$, then $|c_0| < 1$, $|c_1| < 1$, $\lambda \sim \rho(1 - c_0)/(1+c_1)$, $A \sim 2\rho(1+\lambda)m/(\lambda^2(1+c_1)^2)$. Hence $(\theta_1 - \theta_{-1})^2 (A + A')/4 \sim m/(2R^2)$. But the second order term in $(1/4)(\theta_1 - \theta_{-1})^2 \sup_\theta E_\theta S$ is of order $O(\sqrt{m})$ with coefficient larger than that in case 1^o.

8^o If $u_{-1} < \theta'_{-1} (< \theta_0)$ and $\theta'_1 = u_1$, then results similar to those in 7^o follow, except $|c_0| < 1$, $c_1 = 1$, $\lambda \sim \sqrt{\rho}(1-c_0)/2$, $A \sim \rho(1+\lambda)m/(2\lambda^2)$.

9^o If $\theta'_{-1} = u_{-1}$ and $(\theta_0 <)\ \theta'_1 < u_1$, then similar results can be obtained by applying η to 8^o.

Summarizing all of the above cases, we see that case 1^o results in the smallest $\sup_\delta E_\delta T$ value when $\alpha \to 0$. Since conditions in 1^o are equivalent to (6), (i), (ii) and (iii) follow.

Based on Theorem 1 and Lemma 2, we will confine attention to tests with

$$
\begin{aligned}
a &= \frac{m}{2R} + \sum_{i\ge -1} \frac{a_i}{m^{i/2}}, \quad a' = \frac{(1-R)m}{2R^2} + \sum_{i\ge -1} \frac{a'_i}{m^{i/2}}, \\
b &= 1 - R + \sum_{i\ge 1} \frac{b_i}{m^{i/2}}, \quad b' = R + \sum_{i \ge 1} \frac{b'_i}{m^{i/2}}.
\end{aligned} \tag{7}
$$

Using $Mathematica^{TM}$, the corresponding $OC(1)$ and $1 - OC(-1)$, based on (2) and Mill's ratio expansion, have expansions of the form

$$
\begin{aligned}
1 - OC(-1) &= e^{-m + (b_1/R - 2a_{-1}R)\sqrt{m}} \Big(c_{10} + \frac{c_{11}}{m^{1/2}} + \frac{c_{12}}{m} + O(m^{-3/2}) \Big), \\
OC(1) &= e^{-\rho m - (1-R)(2a'_{-1} - b'_1/R^2)\sqrt{m}} \Big(c_{20} + \frac{c_{21}}{m^{1/2}} + \frac{c_{22}}{m} + O(m^{-3/2}) \Big).
\end{aligned} \tag{8}
$$

Expressions for c_{ij} $(i = 1,\ 2,\ j = 0, 1, 2)$ are given in Appendix B.

First we set $c_{10} \exp[(b_1/R - 2a_{-1}R)\sqrt{m}\ ] = 1$ and $c_{20} \exp[-(1-R)(2a'_{-1} - b'_1/R^2)\sqrt{m}\ ] = 1$ so that the error probabilities are correct to relative order $O(m^{-1/2})$. We then solve for the slope coefficients in terms of the intercept coefficients, obtaining

$$
\begin{aligned}
b_1 &= 2a_{-1}R^2, \quad b_2 = 2a_0 R^2 - 4a_{-1}^2 R^3 - R \log \overline{\Phi}(2\sqrt{2}a_{-1}R), \\
b'_1 &= 2a'_{-1}R^2, \quad b'_2 = 2a'_0 R^2 - \frac{4{a'_{-1}}^2 R^4}{(1-R)} - \frac{R^2 \log \overline{\Phi}(2\sqrt{2}a'_{-1}R)}{1-R}.
\end{aligned} \tag{9}
$$

By substituting them into the AST function, we are able to locate where the maximum AST occurs within order $O(m^{-1})$. And this is sufficient to find the second term in the minimax AST through $O(1)$. This leads to a second and third order asymptotic minimax construction.

THEOREM 2. *Let D_2 be the class of tests of form (7) satisfying* $1 - OC(-1) = \alpha(1 + O(m^{-1/2}))$, $OC(1) = \beta(1 + O(m^{-1/2}))$, $m = -\log\alpha \to \infty$. *Let $T_0 \in D_2$ and $\delta_m = \sum_{i\ge 0} \delta_i / m^{i/2}$ be defined by*

$$
\inf_{T\in D_2} \sup_\delta E_\delta T \sim \sup_\delta E_\delta T_0 \sim E_{\delta_m} T_0.
$$

Then

(i) $\delta_0 = 2R - 1$, $\delta_1 = \sqrt{2}R\Phi^{-1}(R)$.

(*ii*) $\inf_{T\in D_2}\sup_\delta E_\delta T = \mathcal{E}_1 m + \mathcal{E}_2\sqrt{m} + \mathcal{E}_3 + O(m^{-1/2})$ *where*

$$
\begin{aligned}
\mathcal{E}_1 &= 1/(2R^2),\\
\mathcal{E}_2 &= -1/\big[2\sqrt{\pi}(1-R)R^2 e^{(\delta_1/(2R))^2}\big],\\
\mathcal{E}_3 &= \frac{g_0+g_1+g_2}{2(1-R)^2R^3},
\end{aligned}
$$

$$
(10)\qquad
\begin{aligned}
g_0 &= \delta_1^2(1-R) - \delta_1(2R-1)e^{-(\delta_1/(2R))^2}/\sqrt{\pi},\\
g_1 &= \inf_x G_1(x)\ (> -\infty),\\
g_2 &= \inf_x G_2(x)\ (> -\infty),\\
G_1(x) &= 4(1-R)^2R^4x^2 + 2(1-R)^2R^2(\delta_1 + e^{-(\delta_1/(2R))^2}/\sqrt{\pi})x\\
&\quad +(1-R)^2R^2\log\overline{\Phi}(2^{3/2}R\,x),\\
G_2(x) &= 4(1-R)R^5x^2 + R^3\big[-2\delta_1(1-R) + 2R\,e^{-(\delta_1/(2R))^2}/\sqrt{\pi}\big]x\\
&\quad +(1-R)R^3\log\overline{\Phi}(2^{3/2}R\,x).
\end{aligned}
$$

(*iii*) *If a test T_1 of the form (7) satisfies (9), then $T_1\in D_2$ and*

$$\sup_\delta E_\delta T_1 = \mathcal{E}_1 m + \mathcal{E}_2\sqrt{m} + O(1).$$

That is, T_1 achieves second order minimax in class D_2.

(*iv*) *If a test T_2 of the form (7) satisfies (9), and if a_{-1} and a'_{-1} solve equations*

$$
(11)\qquad
\begin{aligned}
&\delta_1 + \frac{e^{-(\delta_1/(2R))^2}}{\sqrt{\pi}} + 4a_{-1}R^2 - \frac{R\,e^{-(2a_{-1}R)^2}}{\sqrt{\pi}\overline{\Phi}(2^{3/2}a_{-1}R)} = 0,\\
&\delta_1 - \frac{R\,e^{-(\delta_1/(2R))^2}}{\sqrt{\pi}(1-R)} - 4a'_{-1}R^2 + \frac{R\,e^{-(2a'_{-1}R)^2}}{\sqrt{\pi}\overline{\Phi}(2^{3/2}a'_{-1}R)} = 0,
\end{aligned}
$$

where $\delta_1 = \sqrt{2}R\Phi^{-1}(R)$, then $T_2\in D_2$ and

$$\sup_\delta E_\delta T_2 = \mathcal{E}_1 m + \mathcal{E}_2\sqrt{m} + \mathcal{E}_3 + O(m^{-1/2}).$$

That is, T_2 achieves third order minimax in class D_2.

PROOF. Suppose T is any test in D_2. Then it must satisfy (9). Expanding its AST function at $\delta = \sum_{i\geq 0}\delta_i/m^{i/2}$, we find $E_\delta T = e_1 m + e_2\sqrt{m} + e_3 + O(m^{-1/2})$ where

$$e_1 = \frac{1-R}{2(R-\delta_0)R^2}\,1_{(-1<\delta_0<2R-1)} + \frac{1}{2R^2}\,1_{(\delta_0=2R-1)} + \frac{1}{2R(1+\delta_0-R)}\,1_{(2R-1<\delta_0<1)}.$$

When $|\delta_0|\geq 1$, values of e_1 can be obtained by Proposition 1. It is seen that e_1 reaches its maximum $\mathcal{E}_1$ when $\delta_0 = 2R-1$. Substituting it into e_2, we have

$$e_2 = -\left[R\,e^{-(\delta_1/(2R))^2}/\sqrt{\pi} - \delta_1 R + \delta_1\Phi(\frac{\delta_1}{\sqrt{2}R})\right]/\left[2(1-R)R^3\right].$$

Since $\lim_{\delta_1 \to \pm\infty} e_2 = -\infty$ and $\partial^2 e_2/\partial\delta_1^2 = -e^{-(\delta_1/(2R))^2}/[4\sqrt{\pi}(1-R)R^4] < 0$, e_2 reaches its maximum $\mathcal{E}_2$ when δ_1 satisfies $\partial e_2/\partial\delta_1 = 0$, yielding (i).

Continuing to substitute (i) into e_3, we have $e_3 = [g_0 + G_1(a_{-1}) + G_2(a'_{-1})]/[2(1-R)^2R^3]$. Note that functions $G_1(x)$ and $G_2(x)$ have finite lower bounds. By setting $G'_1(a_{-1}) = 0$ and $G'_2(a'_{-1}) = 0$, we obtain (11). Hence (ii), (iii) and (iv) follow.

Now set $c_{11} = c_{21} = 0$ in (8) so that the error probabilities are correct to relative order $O(m^{-1})$. Solving, we find

$$\begin{aligned} b_3 = {} & \Big(b_2 + (b_2 + b'_2)R + 2a_0R^2 - 2(a_0 + a'_0)R^3 - \pi R^2\cot(R\pi)\Big)\cdot \\ & e^{-(2a_{-1}R)^2}/[2\sqrt{\pi}\overline{\Phi}(2^{3/2}a_{-1}R)] - 2a_{-1}b_2R + 2a_1R^2 - 4a_0a_{-1}R^3, \\ b'_3 = {} & \Big((b_2 + 2b'_2)R - (b_2 + b'_2)R^2 + 2(a_0 + a'_0)R^4 + R^3(-2a_0 \\ & +\pi\cot(R\pi)]\Big)e^{-(2a'_{-1}R)^2}/[2\sqrt{\pi}(1-R)\overline{\Phi}(2^{3/2}a'_{-1}R)] \\ & +2R^2[a'_1 - a'_{-1}(b'_2 + 2a'_0R^2)/(1-R)]. \end{aligned} \tag{12}$$

If we substitute (12) into $AST(\sum_{i\geq 0}\delta_i/m^{i/2})$ for any test T of form (7) satisfying (9) and (11), where δ_0 and δ_1 are given by Theorem 2(i), then $AST(\delta_m) = \mathcal{E}_1 m + \mathcal{E}_2\sqrt{m} + \mathcal{E}_3 + e_4 m^{-1/2} + O(m^{-1})$. Here

$$\begin{aligned} e_4 = {} & -e^{(\delta_1/(2R))^2}/[8\sqrt{\pi}(1-R)R^4]\delta_2^2 + e_{41}\delta_2 + e_{42}, \\ e_{41} = {} & [2 + 4\delta_1\sqrt{\pi}e^{\delta^2/(4R^2)} + \log\overline{\Phi}(2^{3/2}a_{-1}R) + R\{-2(2 + 2\delta_1\sqrt{\pi}e^{\delta^2/(4R^2)} + \\ & \log\overline{\Phi}(2^{3/2}a_{-1}R)) + R[\log\overline{\Phi}(2^{3/2}a_{-1}R) - \log\overline{\Phi}(2^{3/2}a'_{-1}R) \\ & -4(a'_{-1}R - a_{-1}(1-R))(a'_{-1}R \\ & +(a_{-1} + \sqrt{\pi}e^{\delta^2/(4R^2)})(1-R))]\}]/(4\sqrt{\pi}R^3e^{\delta^2/(4R^2)}(1-R)^2), \end{aligned}$$

and e_{42} is a constant. Thus e_4 reaches its maximum $\mathcal{E}_4$ when

$$\begin{aligned} \delta_2 = {} & 4\sqrt{\pi}R\left[\delta_1 + a_{-1}R^2 - (a_{-1} + a'_{-1})R^3\right]e^{(\delta_1/(2R))^2} \\ & + \Big[(1-R)^2\log\overline{\Phi}(2^{3/2}a_{-1}R) - R^2\log\overline{\Phi}(2^{3/2}a'_{-1}R) + 2 - 4R \\ & +4a_{-1}^2R^2(1-R)^2 - 4{a'_{-1}}^2R^4\Big]R/(1-R). \end{aligned} \tag{13}$$

Now we have obtained the following fourth order asymptotic minimax construction:

THEOREM 3. *Let D_3 be the class of tests of form (7) satisfying* $1 - OC(-1) = \alpha(1 + O(m^{-1}))$, $OC(1) = \beta(1 + O(m^{-1}))$, $m = -\log\alpha \to \infty$. *Let $T_0 \in D_3$ and $\delta_m = \sum_{i\geq 0}\delta_i/m^{i/2}$ be defined by*

$$\inf_{T\in D_3}\sup_\delta E_\delta T \sim \sup_\delta E_\delta T_0 \sim E_{\delta_m}T_0.$$

Then

(i) δ_0 and δ_1 are given by Theorem 2 (i), δ_2 is given by (13).

(ii) $\inf_{T\in D_3}\sup_\delta E_\delta T = \mathcal{E}_1 m + \mathcal{E}_2\sqrt{m} + \mathcal{E}_3 + \mathcal{E}_4 m^{-1/2} + O(m^{-1})$ *where* $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ *are given in (10),* $\mathcal{E}_4$ *is some constant.*

(iii) If a test T_1 *of the form (7) satisfies (9), (11) and (12), then* $T_1 \in D_3$, *and*

$$\sup_\delta E_\delta T_1 = \mathcal{E}_1 m + \mathcal{E}_2\sqrt{m} + \mathcal{E}_3 + \mathcal{E}_4 m^{-1/2} + O(m^{-1}).$$

That is, T_1 *achieves fourth order minimax in class* D_3.

In summary, by properly choosing the leading terms of a, a', b and b', we can minimize the leading term of $\sup_\delta E_\delta T$ in class D_1. Then the second term of $\inf_{T\in D_1}\sup_\delta E_\delta T$ is free of the design parameters if the tests are restricted to those in class D_2. Hence there is no need for minimization for the design parameters: all of the a_i's and a_i''s are arbitrary, but tests in D_2 require (b_1, b_1', b_2, b_2') to be functions of $(a_{-1}, a'_{-1}, a_0, a_0')$. If we choose the second terms of a and a' properly, we can minimize the third term of $\sup_\delta E_\delta T$ in class D_2. The fourth term of $\inf_{T\in D_2}\sup_\delta E_\delta T$ is free of the design parameters if the tests are restricted to those in class D_3. Hence no need for minimization for the remaining design parameters, but tests in D_3 require (b_3, b_3') to be functions of (a_1, a_1'). This rule continues to be followed for higher order terms, as was seen in the symmetric case [10].

Theorem 3 *(iii)* gives a class of tests — with $\{a_i, a_i' \mid i \geq 0\}$ and $\{b_j, b_j' \mid i \geq 4\}$ arbitrary — achieving fourth order optimality in D_3. We could continue to obtain higher order minimax construction, but formulas for (b_i, b_i') $(i \geq 4)$ become much longer. For symmetric case, however, (b_j, b_j') $(j = 4,\ 5)$ are given by [10]. In order to be consistent with the symmetric case, we propose to use regression to fit all free parameters (a_i, a_i') and (b_j, b_j') $(i = 0,\ 1,\ 2,\ 3,\ j = 4, 5)$ to the MTT parameters using 20 different combinations of (α, β) for which Hall [7] determined the MTT designs numerically. This yields the following *asymptotic minimax triangular test* (AMTT) construction, achieving properties of T_1 in Theorem 3.

> *For given* α *and* $\beta = \alpha^\rho$, *choose the parameters in (7), where* $m = -\log\alpha$, $R = 1/(1+\sqrt{\rho})$, $R_1 = R - 1/2$, $\delta_1 = \sqrt{2}R\Phi^{-1}(R)$, a_{-1} *and* a'_{-1} *satisfy*
>
> $$\delta_1 + \frac{e^{-(\delta_1/(2R))^2}}{\sqrt{\pi}} + 4a_{-1}R^2 - \frac{R\, e^{-(2a_{-1}R)^2}}{\sqrt{\pi}\overline{\Phi}(2^{3/2}a_{-1}R)} = 0,$$
>
> $$\delta_1 - \frac{R\, e^{-(\delta_1/(2R))^2}}{\sqrt{\pi}(1-R)} - 4a'_{-1}R^2 + \frac{R\, e^{-(2a'_{-1}R)^2}}{\sqrt{\pi}\overline{\Phi}(2^{3/2}a'_{-1}R)} = 0,$$
>
> $$a_0 = -1.569116 - 3.561621R_1,$$
>
> $$a_0' = -1.569116 - 10.80427R_1,$$
>
> $$a_1 = 0.85205 + 4.319348R_1 - 21.95502R_1^2,$$
>
> $$a_1' = 0.85205 + 26.18879R_1 - 2.98425R_1^2,$$
>
> $$a_2 = 0.95506 - 7.44207R_1,\quad a_2' = 0.95506 - 16.88119R_1,$$

$$a_3 = a_3' = -1.06270,$$
$$b_1 = 2a_{-1}R^2, \;\; b_2 = 2a_0R^2 - 4a_{-1}^2R^3 - R\log\overline{\Phi}(2\sqrt{2}a_{-1}R),$$
$$b_1' = 2a_{-1}'R^2, \;\; b_2' = 2a_0'R^2 - \frac{4{a'_{-1}}^2R^4}{(1-R)} - \frac{R^2\;\log\overline{\Phi}(2\sqrt{2}a_{-1}'R)}{1-R},$$
$$b_3 = \Big(b_2 + (b_2 + b_2')R + 2a_0R^2 - 2(a_0 + a_0')R^3 - \pi R^2\cot(R\pi)\Big)\cdot$$
$$e^{-(2a_{-1}R)^2}/[2\sqrt{\pi}\overline{\Phi}(2^{3/2}a_{-1}R)] - 2a_{-1}b_2R + 2a_1R^2 - 4a_0a_{-1}R^3,$$
$$b_3' = \Big((b_2 + 2b_2')R - (b_2 + b_2')R^2 + 2(a_0 + a_0')R^4 + R^3(-2a_0$$
$$+\pi\cot(R\pi)]\Big)e^{-(2a_{-1}'R)^2}/[2\sqrt{\pi}(1-R)\overline{\Phi}(2^{3/2}a_{-1}'R)]$$
$$+2R^2[a_1' - a_{-1}'(b_2' + 2a_0'R^2)/(1-R)],$$
$$b_4 = -0.04241 - 30.6437R_1^2, \;\; b_4' = -0.04241 - 165.3697R_1^2,$$
$$b_5 = 0.30625, \;\; b_5' = 0.30625 + 4.93333R_1.$$

Design parameters for AMTT for commonly used combinations of (α, β) are given in Table 1. Numerical comparisons show that, when $\alpha \leq 20\%$ and $\beta \leq 20\%$, the relative differences between AMTT and Hall's MTT in design parameters are within 0.2% for a and a', within 6% for b and b', and the δ values where the maximum of AST occur for both designs are within 10% of each other. The OC functions of the two designs are within 4.6% of each other with maximum difference occurring when both OC values are above 0.65. The AST functions of AMTT are within 2.5% of those of MTT when $-2 \leq \delta \leq 2$.

Triangular designs are constructed for testing against one-sided alternatives — in our notation, $\delta = -1$ versus $\delta > 1$. However, they may be adapted for two-sided alternatives by rejecting in favor of $\delta < -1$ if the early part, say $T \leq t_0$ of the lower boundary is reached, and choosing t_0 so that $P_{-1}(T \leq t_0, X(T) = -a' + b'T) = \alpha$. The test then has significance level 2α, with negligible effect on the true β.

4. Numerical comparisons with PEST designs. The only available commercial software to provide triangular designs for tests concerning the drift of a continuous time Brownian motion is PEST [2], which uses only symmetric 2-SPRTs to provide the designs. The hypotheses considered in PEST are $H_0 : \theta = 0$ versus $H_1 : \theta = \theta_R$. To obtain stopping boundaries with error probabilities α and β at 0 and θ_R respectively, PEST finds a $\theta_R'(> \theta_R)$ for which a symmetric 2-SPRT of 0 versus θ_R' with both error probabilities α has error probability β at θ_R. The Brownian motion, the drift and the boundaries may then be transformed to yield a test with drift δ and error probabilities α and β at $\delta = \mp 1$. This adaption of a sym-

Table 1: AMTT with stopping boundaries $x = a - bt$ and $x = -a' + b't$.

α	β	a	a'	b	b'	δ_m
0.005	0.005	4.19249	4.19249	0.41497	0.41497	0.00000
0.005	0.010	4.01179	3.68946	0.38927	0.42844	0.06000
0.005	0.025	3.74717	2.99777	0.34737	0.44118	0.15996
0.005	0.050	3.51873	2.44691	0.30678	0.44403	0.25957
0.005	0.100	3.25186	1.86139	0.25383	0.43670	0.39410
0.005	0.200	2.91899	1.22212	0.17899	0.41360	0.59405
0.010	0.010	3.53296	3.53296	0.40272	0.40272	0.00000
0.010	0.025	3.29308	2.86912	0.36111	0.42086	0.10042
0.010	0.050	3.08561	2.34198	0.32039	0.42559	0.20102
0.010	0.100	2.84213	1.78330	0.26665	0.41589	0.33764
0.010	0.200	2.53554	1.17537	0.18965	0.38062	0.54205
0.025	0.025	2.67230	2.67230	0.38025	0.38025	0.00000
0.025	0.050	2.49670	2.18120	0.34046	0.39605	0.10149
0.025	0.100	2.28888	1.66358	0.28666	0.39034	0.24058
0.025	0.200	2.02230	1.10397	0.20726	0.34119	0.45107
0.050	0.050	2.03275	2.03275	0.35564	0.35564	0.00000
0.050	0.100	1.85709	1.55133	0.30410	0.36970	0.14039
0.050	0.200	1.62667	1.03467	0.22464	0.32175	0.35527
0.100	0.100	1.40561	1.40561	0.32034	0.32034	0.00000
0.100	0.200	1.21934	0.93767	0.24725	0.31818	0.21828

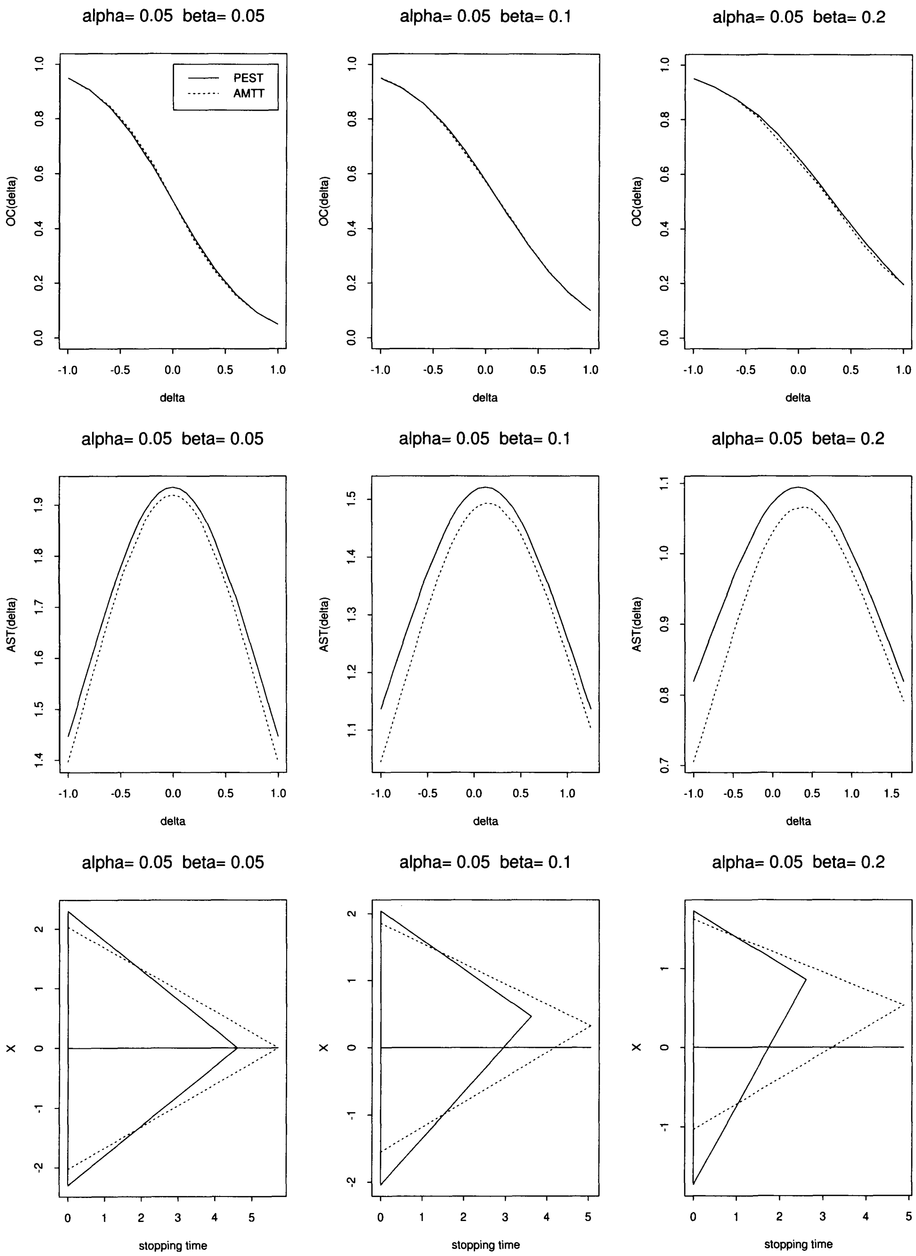
metric 2-SPRT provides an approach to deal with asymmetric hypothesis testing problems, but no optimality criterion is used.

Figure 4 shows that the OC functions for the AMTT and PEST designs are almost identical, but the AST of AMTT is uniformly smaller than that of the PEST design for different combinations of (α, β). The PEST designs have smaller maximum stopping times and, in asymmetric case, steeper slopes for the lower boundaries. It is quite possible that AMTT has a stopping time distribution with a larger median or $90th$ percentile. Hence, it will be more interesting to study the stopping boundaries such that the q-th percentile of the stopping time $Q(\delta, q)$, solving $P_\delta\{T < Q(\delta, q)\} = q$, is small under certain criteria. For example, we can consider finding the design which has minimax in $Q(\delta, q)$, or the one which has minimum of some weighted average of $Q(\delta, q)$. The value of q can be chosen as 50%, 90% or 95%, for example.

5. Acknowledgments. The author wants to express her sincere thanks to her thesis advisor W. J. Hall for guidance in the preparation of this work, to Vladimir Dragalin for helpful suggestions, and to John E. Kolassa for help in using *Mathematica*^{TM}.

Appendix A: Proof of Lemma 1. Define $\lambda = a'/a$.

Figure 1: OC functions (first row), AST functions (second row) and stopping boundaries (third row) for AMTT and PEST designs



(i) For $w_{-1} < \delta < w_0$: Applying (2) and (3) and Mill's ratio expansion,

$$
\begin{aligned}
1 - OC(\delta) &= \frac{\sqrt{t_v}e^{-B}}{\sqrt{2\pi}} \left[\sum_{j=0}^{+\infty} (-1)^j \frac{2r_j}{r_j^2 - \tau_U^2 t_v^2} + O\left(\frac{1}{\bar{a}^2}\right) \right] \\
&= \frac{\sqrt{\pi}e^{-B}}{4\sqrt{2\bar{a}\bar{b}}} \left[\cot\left(\frac{a'(b+\delta) + a(2b+b'+\delta)}{2(a+a')(b+b')} \right) \right. \\
&\quad \left. - \cot\left(\frac{a(2b+b'+\delta) + a'(3b+2b'+\delta)}{2(a+a')(b+b')} \right) + O(\frac{1}{\bar{a}}) \right]
\end{aligned}
$$

$$E_\delta(T\ 1_U) = O(\sqrt{\bar{a}}e^{-B}),$$

$$
\begin{aligned}
E_\delta(T\ 1_L) &= \frac{s_0}{\tau_L} \left[\overline{\Phi}\left(\frac{s_0 - \tau_L t_v}{\sqrt{t_v}} \right) - \frac{e^{-B}}{\sqrt{2\pi}} M\Big(\frac{s_0 + \tau_L t_v}{\sqrt{t_v}} \Big) \right] \\
&\quad + \frac{e^{-B}}{\tau_L\sqrt{2\pi}} \sum_{j=1}^{+\infty} (-1)^j s_j \left[M\Big(\frac{s_j - \tau_L t_v}{\sqrt{t_v}} \Big) - M\Big(\frac{s_j + \tau_L t_v}{\sqrt{t_v}} \Big) \right] \\
&= \frac{s_0}{\tau_L} - \frac{s_0 e^{-B}}{\tau_L\sqrt{2\pi}} \left[M\Big(\frac{\tau_L t_v - s_0}{\sqrt{t_v}} \Big) + M\Big(\frac{s_0 + \tau_L t_v}{\sqrt{t_v}} \Big) \right] + O(\sqrt{\bar{a}}e^{-B}), \\
&= \frac{a'}{b' - \delta} + O(\sqrt{\bar{a}}e^{-B}),
\end{aligned}
$$

completing (i).

(ii) For $\delta = w_0$: From (2) we have

$$
\begin{aligned}
OC(\delta) &= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} M\Big(\frac{s_0 + \tau_L t_v}{\sqrt{t_v}} \Big) \\
&\quad + \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{+\infty} (-1)^j \left[M\Big(\frac{s_j - \tau_L t_v}{\sqrt{t_v}} \Big) + M\Big(\frac{s_j + \tau_L t_v}{\sqrt{t_v}} \Big) \right] \\
&= \frac{1}{2} + O(\frac{1}{\sqrt{\bar{a}}}).
\end{aligned}
$$

To obtain $E_\delta T$, we first calculate it when $b = b'$. Applying (3),

$$
\begin{aligned}
E_\delta(T\ 1_U) &= \frac{r_0}{2\tau_U} + \frac{\sqrt{t_v}}{\tau_U\sqrt{2\pi}} \left[- \frac{r_0}{r_0 + \tau_U t_v} - \frac{r_1}{r_1 - \tau_U t_v} + \frac{r_1}{r_1 + \tau_U t_v} \right. \\
&\quad \left. + \frac{r_0 t_v}{(r_0 + \tau_U t_v)^3} + \frac{r_1 t_v}{(r_1 - \tau_U t_v)^3} - \frac{r_1 t_v}{(r_1 + \tau_U t_v)^3} \right] \\
&\quad + \frac{1}{\sqrt{2\pi}}\ H + O(\frac{1}{\bar{a}^{3/2}}),
\end{aligned}
$$

where H equals

$$2a\lambda t_v^{3/2}\sum_{n=1}^{+\infty}\left[\frac{1}{(r_{2n}-\tau_U t_v)(r_{2n+1}-\tau_U t_v)}+\frac{1}{(r_{2n}+\tau_U t_v)(r_{2n+1}+\tau_U t_v)}\right]+$$

$$\frac{t_v^{3/2}}{\tau_U}\sum_{n=1}^{+\infty}\left[\frac{-1}{(r_{2n}-\tau_U t_v)^2}+\frac{1}{(r_{2n}+\tau_U t_v)^2}+\frac{1}{(r_{2n+1}-\tau_U t_v)^2}-\frac{1}{(r_{2n+1}+\tau_U t_v)^2}\right]+$$

$$t_v^{5/2}\sum_{n=1}^{+\infty}\left[\frac{-1}{(r_{2n}-\tau_U t_v)^3}-\frac{1}{(r_{2n}+\tau_U t_v)^3}+\frac{1}{(r_{2n+1}-\tau_U t_v)^3}+\frac{1}{(r_{2n+1}+\tau_U t_v)^3}\right].$$

Summing the series, we obtain

$$\begin{aligned}E_\delta(T\,1_U) = {}& \frac{a+a'}{4b}-\frac{\sqrt{a+a'}\left[2+2\lambda-\pi\cot\left(\frac{\pi}{1+\lambda}\right)\right]}{8\sqrt{\pi}b^{3/2}}+\\ & \frac{\pi^{3/2}}{192b^{5/2}\sqrt{a+a'}}\left[\left(6+6\lambda-3\pi\cot\left(\frac{\pi}{1+\lambda}\right)\right)\csc^2\left(\frac{\pi}{1+\lambda}\right)\right.\\ & \left.-2(1+\lambda)\right]+O\left(\frac{1}{\bar{a}^{3/2}}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}E_\delta(T\,1_L) = {}& \frac{a+a'}{4b}-\frac{\sqrt{a+a'}\left[2+2\lambda-\pi\lambda\cot\left(\frac{\pi\lambda}{1+\lambda}\right)\right]}{8\sqrt{\pi}\lambda b^{3/2}}+\\ & \frac{\pi^{3/2}}{192b^{5/2}\lambda\sqrt{a+a'}}\left[\left(6+6\lambda-3\pi\lambda\cot\left(\frac{\pi\lambda}{1+\lambda}\right)\right)\csc^2\left(\frac{\pi\lambda}{1+\lambda}\right)\right.\\ & \left.-2(1+\lambda)\right]+O\left(\frac{1}{\bar{a}^{3/2}}\right).\end{aligned}$$

Hence

$$\begin{aligned}E_\delta T = {}& \frac{\bar{a}}{b}-\frac{\bar{a}^{3/2}}{\sqrt{2\pi}\,a'q\,b^{3/2}}\\ & +\frac{\sqrt{2}\pi^{3/2}\,\bar{a}^{3/2}}{96\,aa'b^{5/2}}\left[5+\cos(2q\pi)\right]\csc^2(q\pi)+O\left(\frac{1}{\bar{a}^{3/2}}\right).\end{aligned}\tag{14}$$

The formula for $E_\delta T$ when $b\neq b'$ can be obtained by replacing b in (14) by $\bar{b}=(b+b')/2$.

(*iii*) For $w_0<\delta<w_1$: Applying (2) and (3),

$$\begin{aligned}OC(\delta) = {}& \frac{\sqrt{t_v}e^{-B}}{\sqrt{2\pi}}\left[\frac{1}{r_0+\tau_U t_v}+\frac{1}{r_0-\tau_U t_v}-\frac{1}{r_1+\tau_U t_v}-\frac{1}{r_1-\tau_U t_v}\right.\\ & +\sum_{n=1}^{+\infty}\left(\frac{1}{r_{2n}+\tau_U t_v}-\frac{1}{r_{2n+1}+\tau_U t_v}+\frac{1}{r_{2n}-\tau_U t_v}-\frac{1}{r_{2n+1}-\tau_U t_v}\right)\end{aligned}$$

$$+O\left(\frac{1}{\bar{a}^2}\right)\Bigg].$$

Summing the series yields the formula in (iii). The formula for $E_\delta T$ can be obtained as in (i).

(iv) For $\delta = w_1$: The formulas can be obtained as in (ii).

Appendix B: Constants c_{ij} in (8)

$$\begin{aligned}
c_{10} &= \overline{\Phi}(2\sqrt{2}a_{-1}R)\exp[b_2/R - 2a_0R + 4a_{-1}^2R^2],\\
c_{11} &= \frac{1}{2R}\,e^{b_2/R-2a_0R}\Big[\frac{1}{\sqrt{\pi}}\Big(2(a_0+a_0')R^3 + \pi R^2\cot[\pi R] - b_2 - (b_2+b_2')R\\
&\quad -2a_0R^2\Big) + 2e^{4a_{-1}^2R^2}(b_3 + 2R(-(a_1R) + a_{-1}(b_2 + 2a_0R^2)))\overline{\Phi}(2\sqrt{2}a_{-1}R)\Big],\\
c_{12} &= (\exp(b_2/R - 2a_0R)((6(b_3 + 2R(-(a_1R) + a_{-1}(b_2 + 2a_0R^2)))(-b_2\\
&\quad -b_2R - b_2'R - 2a_0R^2 + 2a_0R^3 + 2a_0'R^3 + \pi R^2Cot[\pi R]))/(\sqrt{\pi}R^2)\\
&\quad +6\exp(4a_{-1}^2R^2)((b_3/R - 2a_1R + 2a_{-1}(b_2 + 2a_0R^2))^2 + 2(2a_0b_2 + b_4/R\\
&\quad -2a_2R + 2a_{-1}(b_3 + 2a_1R^2)))\overline{\Phi}(2\sqrt{2}a_{-1}R) + (2(-3(b_3(1+R)\\
&\quad -a_{-1}(b_2(1+R) + R(b_2' - 2R(a_0(R-1) + a_0'R)))^2\\
&\quad +R(b_3' + 2R(a_1 - a_1R - R(a_1' + (a_{-1} + a_{-1}')(b_2 + b_2' - 2(a_0 + a_0')R^2)))))\\
&\quad +\pi R^2\csc(\pi R)^2(\pi R^2(6a_{-1}'R + a_{-1}(-1+6R)) + a_{-1}\pi R^2\cos(2\pi R)\\
&\quad -3(a_{-1}'R^2 + a_{-1}(b_2(1+R) + R(b_2' + R(1 - 2a_0(R-1)\\
&\quad -2a_0'R)))) \sin(2\pi R))))/(\sqrt{\pi}R)))/12,\\
c_{20} &= \exp(2a_0'(R-1) + (b_2' - b_2'R + 4{a_{-1}'}^2R^4)/R^2)\overline{\Phi}(2\sqrt{2}a_{-1}'R),\\
c_{21} &= ((\exp(((R-1)(-b_2' + 2a_0'R^2))/R^2)(b_2'(-2+R) + b_2(R-1)\\
&\quad -2R^2(-a_0 + a_0R + a_0'R)))/(\sqrt{\pi}R) - \exp(((R-1)(-b_2'\\
&\quad +2a_0'R^2))/R^2)\sqrt{\pi}R\cot(\pi R) - (\exp(2a_0'(R-1) + (b_2' - b_2'R\\
&\quad +4{a_{-1}'}^2R^4)/R^2)(b_3' - b_3'R + 2R^2(a_1'(R-1)\\
&\quad +a_{-1}'(b_2' + 2a_0'R^2)))(2\overline{\Phi}(2\sqrt{2}a_{-1}'R)))/R^2)/2,\\
c_{22} &= ((3\exp(((R-1)(-b_2' + 2a_0'R^2))/R^2)(b_2'(-2+R) + b_2(R-1)\\
&\quad -2R^2(-a_0 + a_0R + a_0'R))(b_3' - b_3'R + 2R^2(a_1'(R-1)\\
&\quad +a_{-1}'(b_2' + 2a_0'R^2))))/(\sqrt{\pi}R^3) - (3\exp(((R-1)(-b_2' + 2a_0'R^2))/R^2)(b_3\\
&\quad -b_3'(-2+R) - b_3R - a_{-1}'(b_2 - b_2'(-2+R) - b_2R + 2R^2(-a_0\\
&\quad +a_0R + a_0'R))^2 + 2R^2(-(a_{-1}'b_2) - a_{-1}'b_2' + a_1(R-1) + a_1'R + a_{-1}'b_2R\\
&\quad +a_{-1}'b_2'R + 2a_0a_{-1}'R^2 + 2a_0'a_{-1}'R^2 - 2a_0a_{-1}'R^3 - 2a_0'a_{-1}'R^3
\end{aligned}$$

$$+a_{-1}(R-1)(b_2+b_2'-2(a_0+a_0')R^2))))/(\sqrt{\pi}R)+6\exp(2a_0'(R-1)$$
$$+(b_2'-b_2'R+4{a'_{-1}}^2R^4)/R^2)(a_0'b_2'+a'_{-1}b_3'+a_2'(R-1)$$
$$-(b_4'(R-1))/(2R^2)+2a'_{-1}a_1'R^2+(a'_{-1}b_2'+a_1'(R-1)$$
$$-(b_3'(R-1))/(2R^2)+2a_0'a'_{-1}R^2)^2)(2\overline{\Phi}(2\sqrt{2}a'_{-1}R))$$
$$+(\exp(((R-1)(-b_2'+2a_0'R^2))/R^2)\sqrt{\pi}(3(R-1)(b_3'+2R^2(-a_1'$$
$$-a'_{-1}(b_2+b_2'-2(a_0+a_0')R^2)))\cot(\pi R)-R^4(2a'_{-1}\pi$$
$$-3(a_{-1}+a'_{-1})\csc(\pi R)^2(-2\pi(R-1)+\sin(2\pi R)))))/R)/6.$$

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Department of Biometry and Epidemiology
Medical University of South Carolina
135 Cannon Street, Suite 303
Charleston, SC 29425 USA
huangp@musc.edu