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A theorem on compatibility of systems of sets with applications

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Abstract: A general theorem on compatibility of two systems of subsets of a separable metric space is proved. This theorem is used to deduce results about points of continuity of functions, filtrations and operator semigroups among other things.

In this paper we prove the following result which, in spirit, is very much similar to a classical 1908 result of W. H. Young [2, page 304] on real functions and unifies several well-known results.

Theorem. *Let X be a separable metric space and I be any uncountable subset of the real line. Suppose that $\{A_t : t \in I\}$ and $\{B_t : t \in I\}$ are two families of subsets of X with each A_t closed and satisfying the following condition:*

(*) for each $t \in I$, there is a $\delta_t > 0$ such that $A_t \supset B_s$ whenever $s \in (t, t + \delta_t) \cap I$.

Then for all but countably many $t \in I$, $A_t \supset B_t$. The same conclusion holds if, in condition (*), $(t, t + \delta_t)$ is replaced by $(t - \delta_t, t)$.

Proof. Set $I_\delta = \{t \in I : A_t \supset B_s$ whenever $s \in (t, t + \delta) \cap I\}$ and let ρ denote the metric on the space X. If the conclusion were false, then there is some $\delta > 0$ and an uncountable set $S \subset I_\delta$ such that for all $t \in S$ the assertion fails. Since each A_t is closed, we can get $\epsilon > 0$ such that for uncountably many $t \in S$ there exists $x_t \in B_t$ such that $\rho(x_t, A_t) > \epsilon$. Cutting down S, if necessary, we can and shall assume that this holds for all points t in S. Again no loss to assume that S is contained in an interval of length smaller than δ. Now , if $t < t'$ are two distinct points of S, then noticing that $t' \in (t, t + \delta)$ we see that $\rho(x_t, x_{t'}) > \epsilon$. Thus $\{x_t : t \in S\}$ is an uncountable set of elements of S with any two of them separated by distance larger than ϵ, contradicting separability of X. The other part is similarly proved. $\square$

The following propositions illustrate some applications of the theorem – perhaps there are others. In what follows, the closure of a set A is denoted by $\overline{A}$.

Proposition 1. *Let X and I be as above. Let $\{B_t : t \in I\}$ be any family of subsets of X. Then for all but countably many $t \in I$*

$$B_t \subset \bigcap_{\delta>0} \overline{\bigcup\{B_s : s \in (t, t + \delta) \cap I\}}$$

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and

$$B_t \subset \bigcap_{\delta>0} \overline{\bigcup\{B_s : s \in (t-\delta, t) \cap I\}}$$

Proof. Fix $\delta > 0$ and put $A_t = \overline{\bigcup\{B_s : s \in (t, t+\delta) \cap I\}}$ for each $t \in I$. Then the Theorem implies that $B_t \subset A_t$ for all but countably many $t \in I$. The proof is now completed by running δ through a sequence decreasing to zero. The second part follows similarly. □

Proposition 2. *Let f be any function defined on an open interval I into a separable metric space X. For each $t \in I$, define*

$$L_t^d = \bigcap_{\delta>0} \overline{f[(t-\delta, t) \cap I]} \qquad L_t = \bigcap_{\delta>0} \overline{f[(t-\delta, t] \cap I]}$$

and

$$R_t^d = \bigcap_{\delta>0} \overline{f[(t, t+\delta) \cap I]} \qquad R_t = \bigcap_{\delta>0} \overline{f[[t, t+\delta) \cap I]}$$

Then for all but countably many $t \in I$, $L_t^d = L_t = R_t^d = R_t$.

Proof. Since by definition, $L_t^d \subset L_t$ and $R_t^d \subset R_t$ for all $t \in I$, it suffices to show that $L_t \subset R_t^d$ and $R_t \subset L_t^d$ for all but countably many $t \in I$. Fixing $\delta > 0$ and putting for each $t \in I$, $A_t = \overline{f[(t, t+\delta) \cap I]}$, $B_t = L_t$, it follows from the Theorem that $B_t \subset A_t$ for all but countably many $t \in I$. Running δ through a sequence decreasing to zero, one obtains $L_t \subset R_t^d$ for all but countably many $t \in I$. The other inclusion $R_t \subset L_t^d$ is proved similarly. □

Corollary (W.H.Young [2]). *Let f be any real-valued function defined on an open interval I. For every $t \in I$, let*

$$\overline{f}(t-) = \limsup_{\substack{s \longrightarrow t \\ s<t}} f(s) \qquad \overline{f}(t+) = \limsup_{\substack{s \longrightarrow t \\ s>t}} f(s)$$

and

$$\underline{f}(t-) = \liminf_{\substack{s \longrightarrow t \\ s<t}} f(s) \qquad \underline{f}(t+) = \liminf_{\substack{s \longrightarrow t \\ s>t}} f(s)$$

Then for all but countably many $t \in I$, $\overline{f}(t-) = \overline{f}(t+)$ and $\underline{f}(t-) = \underline{f}(t+)$. In particular there is a countable set $D \subset I$ such that for $t \in I - D$, if one of the limits $\lim_{\substack{s \longrightarrow t \\ s<t}} f(s)$ or $\lim_{\substack{s \longrightarrow t \\ s>t}} f(s)$ exists, then so does the other and the two are equal.

Proof. In view of the order preserving homeomorphism $x \to \arctan x$, it suffices to consider bounded f only. To complete the proof now, one has to simply observe that $\overline{f}(t-) = \sup L_t^d$, $\overline{f}(t+) = \sup R_t^d$, $\underline{f}(t-) = \inf L_t^d$ and $\underline{f}(t+) = \inf R_t^d$ in the notation of Proposition 2. □

Remark 1. It is possible to improve the above corollary as follows:

"For any function f on an open interval I into a separable metric space X, there is a countable set $D \subset I$, such that, for $t \in I - D$, if either $\lim_{s \to t, s<t} f(s)$ or $\lim_{s \to t, s>t} f(s)$ exists in X, then f is continuous at t."

To see this, we first note that, by Proposition 2, there is a countable set $D \subset I$, such that, $L_t^d = L_t = R_t^d = R_t$, for $t \in I - D$. For such a t, existence of either of the limits stated in the proposition clearly implies that all these four sets L_t^d, L_t, R_t^d, R_t are equal to one singleton set, i.e., that f is continuous at t.

As an immediate consequence of this, we get

Corollary. *Any function $f : I \to X$ which has , at every point $t \in I$, either a left limit or a right limit, can have at most countably many points of discontinuity.*

Similar technique yields some results on differentiability properties of a real function on an interval. Let f be a real-valued function defined on an open interval I. For $x \in I$, let

$$D_{t+}^d = \bigcap_{\delta>0} \overline{\left\{ \frac{f(u)-f(v)}{u-v} : t < u < v < t+\delta \right\}}$$

and

$$D_{t-}^d = \bigcap_{\delta>0} \overline{\left\{ \frac{f(u)-f(v)}{u-v} : t-\delta < u < v < t \right\}}$$

D_{t+} and D_{t-} are defined analogously with the only exception that $t < u < v < t+\delta$ and $t-\delta < u < v < t$ are replaced by $t \leq u < v < t+\delta$ and $t-\delta < u < v \leq t$ respectively. It should be pointed out that the closures in the above definitions are closures in the extended real line. Using arguments similar to that of Proposition 2 we get

Proposition 3. *For all but countably many $t \in I$*

$$D_{t-}^d = D_{t-} = D_{t+} = D_{t+}^d.$$

From the definition of D_{t-}, it is clear that in case D_{t-} is a singleton then f must have a left derivative at t. Similar argument applies for D_{t+} as well. This easily yields the following

Corollary. *If $f : I \to \mathbb{R}$ is such that for all but countably many t in I, either D_{t-}^d or D_{t+}^d is a singleton then f is differentiable at all but countably many points.*

A more satisfactory result would have been to replace the hypothesis in the above corollary by the apparently weaker condition that f has a left derivative or a right derivative at all but countably many t. The main problem appears to be that f may have a left (right) derivative at a point t without D_{t-} (D_{t+}) being a singleton. But can this happen at uncountably many points t ? We do not know.

For the next few propositions, which are of interest in the context of stochastic processes, we fix the following set-up and notations. $(\Omega, \mathcal{F}, P)$ denotes a probability space where $\mathcal{F}$ is the P-completion of a countably generated σ-field. It is well known that $\mathcal{F}$ is then a polish space with the metric $\rho(A, B) = P(A \Delta B)$, provided one identifies sets A and B in $\mathcal{F}$ whenever $P(A \Delta B) = 0$. For two sub-σ-fields $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{F}$, we say that $\mathcal{A}$ and $\mathcal{B}$ are equal upto P-null sets, and write $\mathcal{A} \sim \mathcal{B}$ to mean that they generate the same σ-field on augmentation by P-null sets of $\mathcal{F}$. Note that any sub-σ-field of $\mathcal{F}$, on augmentation by P-null sets, becomes a closed subset (modulo the above identification) of the separable metric space $\mathcal{F}$. We will use the same notation for a sub-σ-field of $\mathcal{F}$ as well as the closed subset it gives rise to. In this language, $\mathcal{A} \sim \mathcal{B}$ simply means that $\mathcal{A}$ and $\mathcal{B}$ are equal as closed sets. Also for any family $\{\mathcal{F}_\alpha, \alpha \in \Lambda\}$ of σ-fields the smallest σ-field containing $\mathcal{F}_\alpha$ for all $\alpha \in \Lambda$ will be denoted by $\vee_{\alpha\in\Lambda}\mathcal{F}_\alpha$.

Proposition 4. *Let $\{\mathcal{F}_t, t \in I\}$ be a monotone non-decreasing family of sub-σ-fields of $\mathcal{F}$ where I is an open interval. For each $t \in I$, let $\mathcal{F}_{t-} = \vee_{s<t}\mathcal{F}_s$ and $\mathcal{F}_{t+} = \cap_{s>t}\mathcal{F}_s$. Then for all but countably many $t \in I$,*

$$\mathcal{F}_{t-} \sim \mathcal{F}_t \sim \mathcal{F}_{t+}$$

Proof. Take $A_t = \mathcal{F}_{t-}$ and $B_t = \mathcal{F}_{t+}$, and note that $B_s \subset A_t$ whenever $s < t$. From the Theorem one gets $B_t \subset A_t$ for all but countably many $t \in I$. The proof is now completed in view of the fact that $\mathcal{F}_{t-} \subset \mathcal{F}_t \subset \mathcal{F}_{t+}$ for all $t \in I$. □

As a consequence we have

Proposition 5. *If $\{X_t, t \geq 0\}$ is a stochastic process on $(\Omega, \mathcal{F}, P)$ and if, for each $t > 0$, $\mathcal{F}_t = \sigma\langle X_u, 0 \leq u \leq t\rangle$, then for all but countably many $t > 0$, $\mathcal{F}_{t-} \sim \mathcal{F}_t \sim \mathcal{F}_{t+}$.*

It is interesting to note that the exceptional set of t's in the above proposition, to be denoted by $D(X)$, depends only on the law of the process $\{X_t\}$; that is, for two processes $\{X_t\}$ and $\{Y_t\}$ on $(\Omega, \mathcal{F}, P)$, having the same finite dimensional distributions, $D(X) = D(Y)$. In particular, if $\{X_t, t \geq 0\}$ is a process with stationary increments, then, for any $s \geq 0$, denoting the process $\{X_{s+t} - X_s, t \geq 0\}$ by $\{Y_t\}$, one has $D(X) = D(Y)$. If moreover, the increments of X are independent, then one can show, using the above, that the complement of $D(X)$ is a right interval and, hence, has to contain $(0, \infty)$. The same argument can be used to show that $D(X)$ is actually empty. Thus we have

Proposition 6. *If $\{X_t, t \geq 0\}$ is a process on $(\Omega, \mathcal{F}, P)$ with stationary independent increments, then for all t, $\mathcal{F}_{t-} \sim \mathcal{F}_t \sim \mathcal{F}_{t+}$.*

This is what is usually known as Blumenthal's $0-1$ law (see for example [3]), for which the usual proof is via a right continuous modification of the process $\{X_t\}$ and the strong Markov property.

Proposition 7. *Let I be any open interval and $\{\mathcal{G}_t, t \in I\}$ any family of sub-σ-fields of $\mathcal{F}$. For each $t \in I$, define*

$$\mathcal{G}^d_{t+} = \cap_{\delta>0} \vee \{\mathcal{G}_s, t < s < t+\delta\}, \qquad \mathcal{G}_{t+} = \cap_{\delta>0} \vee \{\mathcal{G}_s, t \leq s < t+\delta\}$$
$$\mathcal{G}^d_{t-} = \cap_{\delta>0} \vee \{\mathcal{G}_s, t-\delta < s < t\}, \qquad \mathcal{G}_{t-} = \cap_{\delta>0} \vee \{\mathcal{G}_s, t-\delta < s \leq t\}$$

Then for all but countably many $t \in I$, $\mathcal{G}^d_{t-} \sim \mathcal{G}_{t-} \sim \mathcal{G}^d_{t+} \sim \mathcal{G}_{t+}$.

Proof. Fix $\delta > 0$ and take $A_t = \vee\{\mathcal{G}_s, t < s < t+\delta\}$, $B_t = \mathcal{G}_{t-}$. The Theorem implies that, for all but countably many $t \in I$, $A_t \supset B_t$. By arguments similar to those used in Proposition 2, one concludes that $\mathcal{G}_{t-} \subset \mathcal{G}^d_{t+}$ for all but countably many $t \in I$. Similarly one shows that $\mathcal{G}_{t+} \subset \mathcal{G}^d_{t-}$ for all but countably many $t \in I$. The proof is now complete in view of the inclusions $\mathcal{G}^d_{t-} \subset \mathcal{G}_{t-}$ and $\mathcal{G}^d_{t+} \subset \mathcal{G}_{t+}$. □

In particular, this gives

Proposition 8 (V. S. Borkar [1]). *For a stochastic process* $\{X_t, t \geq 0\}$ *on* $(\Omega, \mathcal{F}, P)$

$$\cap_{\delta>0}\sigma\langle X_s, t-\delta < s < t\rangle \sim \cap_{\delta>0}\sigma\langle X_s, t-\delta < s \leq t\rangle \sim$$
$$\cap_{\delta>0}\sigma\langle X_s, t < s < t+\delta\rangle \sim \cap_{\delta>0}\sigma\langle X_s, t \leq s < t+\delta\rangle$$

for all but countably many $t > 0$.

Remark 2. In an analogous manner, one gets
For a stochastic process $\{X_t, t \geq 0\}$ on $(\Omega, \mathcal{F}, P)$

$$\begin{aligned} &\cap_{\delta>0} \quad \sigma\langle X_u - X_s,\ t-\delta < s < u < t\rangle \\ &\quad \sim \cap_{\delta>0} \quad \sigma\langle X_u - X_s,\ t-\delta < s < u \leq t\rangle \\ &\quad \sim \cap_{\delta>0} \quad \sigma\langle X_u - X_s,\ t < s < u < t+\delta\rangle \\ &\quad \sim \cap_{\delta>0} \quad \sigma\langle X_u - X_s,\ t \leq s < u < t+\delta\rangle \end{aligned}$$

for all but countably many $t > 0$.

We end this note with one more application which may have interesting consequences for Markov processes.

Proposition 9. *Let* $\{T_t, t > 0\}$ *be a semigroup of bounded linear operators on a separable Banach space* B *such that, for every* $x \in B$, $\lim_{t\to 0+} T_t x$ *exists in the strong operator topology. Then* $(T_t, t > 0)$ *is strongly continuous. Moreover, the set* $\{x \in B : T_{0+}x = x\}$ *is precisely the closed span of* $\cup_{t>0} T_t B$.

Proof. By the uniform boundedness principle, T_t are uniformly bounded for t in any bounded interval. For any $x \in B$, the map $t \to T_t x$ has, by the corollary following Remark 1, only countably many discontinuities. Separability of B and the boundedness property noted above permit us to choose one countable set of t's, outside of which the map $t \mapsto T_t x$ is continuous for all $x \in B$. The semigroup property, on the other hand, would assert that the continuity points form a right interval. The proof is complete. □

Remark 3. Without separability of X the Theorem fails. For example, put $I = \mathbb{R}$, $A_t = (t, \infty)$, and $B_t = [t, \infty)$ and let X be the real line with discrete topology. It is clear that the Theorem does not hold. However in the non-separable case the Theorem will remain true if we replace *countably many* by *at most* $\aleph$ *many* where $\aleph$ is the weight of X, that is, the least cardinality of a dense set in X. Interestingly, for finite X, the exceptional set of t's cannot have a right accumulation point of order equalling $card(X)$.

References

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