LECTURE 9

Curve Estimation and Long-Range Dependence

It is of some interest to see what one can say about probability density estimates in some domain of long-range dependence. Let us consider a stationary process Y_k, with a density function, given by

$$Y_k = G(X_k)$$

with X_k Gaussian and stationary. One can show [see Ibragimov and Rozanov (1978)] that if the correlation [of (X_k)]

$$r(s) \simeq q|s|^{-\alpha}, \qquad \alpha > 1, \tag{9.1}$$

then X_k is a process with asymptotic correlation zero and with corresponding mixing coefficient $\rho(s) = o(|s|^{-\beta})$ for $\beta < \alpha - 1$. This is also true of the process Y_k since it is obtained by an instantaneous function applied to the process X_k. Suppose we wish to consider a kernel estimate of the density function of Y_k,

$$f_n(y) = \frac{1}{nb(n)} \sum_{k=1}^{n} \omega\left(\frac{y - Y_k}{b(n)}\right).$$

A theorem of Bradley (1983) as applied here would give us the usual result on asymptotic distribution of the estimates if ω is nonnegative, bounded and band limited with integral one.

Let us now consider the case in which α in (9.1) is such that $0 < \alpha < 1$. The process X_k is then long-range dependent by the remarks in the previous section. If we expand in terms of Hermite polynomials as in the last section,

$$\omega\left(\frac{y - G(x)}{b(n)}\right) = \sum c_{j,n} H_j(x)$$

with

$$c_{j,n} = \frac{1}{j!} \int \omega\left(\frac{y - G(x)}{b(n)}\right) H_j(x) \frac{\exp(-x^2/2)}{\sqrt{2\pi}}\, dx.$$

The second moment of a single term is

$$\int \omega\left(\frac{y - G(x)}{b(n)}\right)^2 \frac{\exp(-x^2/2)}{\sqrt{2\pi}}\, dx = \sum_{k=0}^{\infty} c_{k,n}^2 k!$$

and the variance is $\sum_{k=1}^{\infty} c_{k,n}^2 k!$. The covariance of two terms in terms of jointly Gaussian variables X, X' with covariance r is

$$\operatorname{cov}\left(\omega\left(\frac{y - G(X)}{b(n)}\right), \omega\left(\frac{y - G(X')}{b(n)}\right)\right) = \sum_{k=1}^{\infty} c_{k,n}^2 k! r^k.$$

Let us assume that ω is bounded, of finite support and such that

$$\int \omega(u)\, du = 1.$$

Further let G be continuously differentiable with $G'(G^{-1})(y)) \neq 0$. We can then get a first order asymptotic intermediate estimate for $c_{j,n}$. Formally setting $u = (y - G(x)/b(n)$, we have

$$c_{j,n} = \frac{1}{j!} \int \omega(u) H_j(G^{-1}(y - b(n)u)) \exp\left\{-\frac{(G^{-1}(y - b(n)u))^2}{2}\right\}$$
$$\times b(n) \frac{1}{\sqrt{2\pi}} \frac{du}{G'(G^{-1}(y - b(n)u))}.$$

As $b(n) \downarrow 0$, we obtain to the first order for fixed j

$$(9.2)\quad c_{j,n} \simeq \frac{1}{j!} H_j(G^{-1}(y)) \exp\left\{-\frac{(G^{-1}(y))^2}{2}\right\} \frac{1}{\sqrt{2\pi}} |G'(G^{-1}(y))|^{-1} b(n).$$

From Szegö's orthogonal polynomials [(1975), page 199 ff.], we obtain the following asymptotic expression for $H_j(x)$, namely that

$$e^{j/2} 2^{-1/2} j^{-j/2} H_j(x) = \exp\left(\frac{x^2}{4}\right) \cos\left((2j+1)^{1/2} \frac{x}{\sqrt{2}} - \frac{n\pi}{2}\right) + O(j^{-1/2})$$

holds uniformly over any finite x interval as $j \to \infty$. This implies that the asymptotic expression (9.2) is still valid as long as $j^{1/2} = o(b(n)^{-1})$ and $H_j(G^{-1}(y)) \neq 0$. If we make the same change of variable $u = (y - G(x))/b(n)$ and use the same estimation procedure as that employed on (9.2), we find

$$\operatorname{var}\left(\omega\left(\frac{y - G(X)}{b(n)}\right)\right)$$
$$\simeq b(n) \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(G^{-1}(y))^2}{2}\right\} |G'(G^{-1}(y))|^{-1} \int \omega(u)^2\, du.$$

From (9.2), it is clear that if $j^{1/2} = o(b(n)^{-1})$, we would expect

$$c_{j,n} \simeq e^{j/2} e^{-j/2-1} 2^{-1/2} \pi^{-1/2} \times \cos\left((2j+1)^{1/2} \frac{G^{-1}(y)}{\sqrt{2}} - \frac{n\pi}{2} \right) |G'(G^{-1}(y))|^{-1/2} b(n)$$

and for $j^{1/2} \asymp b(n)^{-1}$, one would still expect

$$c_{j,n} = O(e^{j/2} j^{-j/2-1/2}) b(n).$$

Here $A \asymp B$ means that the ratios A/B, B/A are bounded. However if $b(n)^{-1} = o(j^{1/2})$ with ω piecewise smooth,

$$c_{j,n} = O\left(e^{j/2} j^{-j/2-1/2} \{ j^{1/2} b(n) \}^{-1} \right) b(n).$$

Thus if $j^{1/2} = o(b(n)^{-1})$,

$$c_{j,n}^2 j! \simeq j^{-1/2} (2\pi)^{-1} \cos^2\left((j+1)^{1/2} G^{-1}(y)/\sqrt{2} - (n\pi)/2 \right) b(n)^2$$

and

$$\sum_{j=1}^{Ab(n)^{-2}} c_{j,n}^2 j! = O(b(n)).$$

But

$$\sum_{j-Ab(n)^{-2}}^{\infty} c_{j,n}^2 j! = O\left(\sum_{j-Ab(n)^{-2}}^{\infty} j^{-1/2} j^{-1} b(n)^{-2} \right) b(n)^2 = O(A^{-1/2} b(n)).$$

Now

$$\sum_{j=1}^{n} \omega\left(\frac{y - G(X_j)}{b(n)} \right) = \sum_{k=1}^{\infty} c_{k,n} \sum_{j=1}^{n} H_k(X_j).$$

If $k\alpha < 1$, the variance of $\sum_{j=1}^{n} H_k(X_j)$ is

$$\sum_{s=-n}^{n} (n - |s|) r(s)^k \asymp \sum_{s=-n}^{n} (n - |s|)(1 + |s|)^{-k\alpha} \asymp n^{2-k\alpha},$$

while if $k\alpha > 1$, the variance grows at a rate proportional to n. Let m be the largest positive integer such that $m\alpha < 1$. The variance contribution of the first term ($k = 1$) is

$$\sigma^2\left(c_{1,n} \sum_{j=1}^{n} H_1(X_j) \right) \asymp n^{2-\alpha} b(n)^2.$$

The variance contributions of the second ($k = 2$) up to the mth terms ($k = m$) will be $\asymp n^{2-2\alpha} b(n)^2, \dots, n^{2-m\alpha} b(n)^2$, respectively. The sum of the remaining terms will be $\asymp nb(n)$. The question as to whether the first term dominates is one of whether $nb(n) = o(n^{2-\alpha} b(n)^2)$ or $n^{1-\alpha} b(n) \to \infty$. Let us assume $n^{1-\alpha} b(n) \to \infty$ and that $G^{-1}(y) \neq 0$ [is not a zero of $H_1(\cdot)$]. Let us also consider another value y' such that $G^{-1}(y') \neq 0$. Then

$$n^{\alpha/2}[f_n(y) - Ef_n(y)], \qquad n^{\alpha/2}[f_n(y') - Ef_n(y')]$$

are jointly asymptotically normal with mean zero, variances $\alpha(y)^2$ and $\alpha(y')^2$, respectively, and covariance $\alpha(y)\alpha(y')$, where

$$\alpha(y) = G^{-1}(y)\exp\left(-\frac{G^{-1}(y)^2}{2}\right)\frac{1}{\sqrt{2\pi}}|G'(G^{-1}(y))|^{-1}.$$

The process $n^{\alpha/2}[f_n(y) - Ef_n(y)]$ if $G^{-1}(y) \neq 0$ just appears to be asymptotically degenerate in distribution and of the form

$$\alpha(y)Z$$

with Z $N(0,1)$.