U-STATISTICS AND DOUBLE STABLE INTEGRALS

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Summary

We derive the tail behaviour of the double stable integral

$$I(h) = \int_0^1\int_0^1 h(x,y)\,X(dx)\,X(dy),$$

where X is a completely asymmetric stable process.

1. Introduction. First we shall show the relation between the double stable integral and a simple U-statistic. Let $\{X(t): 0 \leq t < \infty\}$ be a completely asymmetric stable process with characteristic exponent $\alpha \in (0,1)$ and $\beta = 1$. For the theory of stable distributions we refer to Gnedenko-Kolmogorov [GK 54], Breiman [Bre 68] or Feller[Fel 71]. A summary can be found in Mijnheer [Mij 75]. We use the notation as used in [Mij 75]. See [Mij 75] section 3.2 for a review of properties of stable processes. The random variables X_i, $i = 1, 2, \ldots$ are i.i.d. and have the same distribution as $X(1)$. $X \overset{d}{=} Y$ means that X and Y have the same distribution. $X \in D(\alpha, \beta)$ (resp. $D_N(\alpha,\beta)$) means that X belongs to the domain of (resp. normal) attraction of the stable distribution with parameters α and β. Then we have

$$n^{-2/\alpha}\sum_{\substack{i=1 \\ i\neq j}}^{n}\sum_{j=1}^{n} X_iX_j \overset{d}{=} n^{-2/\alpha}\sum_{i\neq j}\sum \{X(i) - X(i-1)\}\{X(j) - X(j-1)\}$$

$$\overset{d}{=} \sum_{i\neq j}\sum \{X(in^{-1}) - X((i-1)n^{-1})\}\{X(jn^{-1}) - X((j-1)n^{-1})\}.$$

This quadratic form is in a natural way related to the double stable integral

$$I(h) = \int_0^1\int_0^1 h(x,y)\,X(dx)\,X(dy) \tag{1.1}$$

where the function h is given by

$$h(x,y) = \begin{cases} 1 & 0 \leq x,y \leq 1 \ \ and \ \ x \neq y \\ 0 & \text{otherwise}. \end{cases} \tag{1.2}$$

The existence of double stable integrals is proved in Szulga and Woyczynski [SW 83]. Easy to investigate are integrals where the function h is of the type

$$h(x,y) = \phi(x)\phi(y). \tag{1.3}$$

For Wiener--Ito integrals we can restrict ourselves to functions h of this particular structure. See Denker [Den 85], lemma 2.2.3. In Section 3 we derive the tail behaviour of the integral (1.1) in the case h is defined by (1.2). In Section 4 we give the behaviour when h satisfies (1.3).

$\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n} X_iX_j$ is a (simple) example of a U-statistic. These U-statistics have been introduced by Hoeffding [Hoe 48]. For the general theory of U-statistics see Serfling [Ser 80], Shorack and Wellner [SW 86] and a review of Dehling [Deh 85].

Next we summarize the limit behaviour of the mentioned U-statistic. We distinguish several cases.

Case I. X_i has a finite second moment. $EX_1 = \mu$ and $\sigma^2(X_1) = \sigma^2$.

Case Ia. The **non-degenerate** case: $\mu \neq 0$. We write $X_i = \mu + \sigma U_i$, $i = 1,2, \ldots$ where U_i, $i = 1,2, \ldots$ are i.i.d. with $EU_i = 0$ and $\sigma^2(U_i) = 1$. Then we have

$$\begin{aligned} Y_n &= n^{-1}(n-1)^{-1}\sum_{i\neq j}\sum X_iX_j \\ &= \mu^2 + 2n^{-1}\mu\sigma\sum U_i + \sigma^2 n^{-1}(n-1)^{-1}\left\{(\sum U_i)^2 - \sum U_i^2\right\}. \end{aligned} \tag{1.4}$$

This implies

$$n^{1/2}(Y_n - \mu^2) = 2\mu\sigma n^{-1/2}(\sum U_i) + \sigma^2(n-1)^{-1}n^{-1/2}\left\{(\sum U_i)^2 - \sum U_i^2\right\}. \tag{1.5}$$

It follows from the central limit theorem that the first term on the right hand side of (1.5) has a normal limit distribution with expectation 0 and variance $(2\mu\sigma)^2$. The second term on the right converges in probability to 0 for $n \to \infty$, by the central limit theorem and the law of large numbers.

Case Ib. In the **degenerate** case, i.e. $\mu = 0$, we have

$$Y_n = \sigma^2 n^{-1}(n-1)^{-1}\left\{(\sum U_i)^2 - \sum U_i^2\right\}. \tag{1.6}$$

Thus

$$nY_n\sigma^{-2} = (n-1)^{-1}n\left\{(n^{-1/2}\sum U_i)^2 - n^{-1}\sum U_i^2\right\} \tag{1.7}$$

which converges by the central limit theorem and the law of large numbers to a random variable $\chi_1^2 - 1$, where χ_1^2 has a chi--square distribution with one degree of freedom.

Case II. Let $X_1 \in D(\alpha, \beta)$ with $1 < \alpha < 2$. Then $\mu = EX_1$ exists and we can write $X_i = \mu + \sigma U_i$, where $EU_i = 0$ and σ is some scale parameter. (Remark that the variance of X_1 is infinite.)

Case IIa. In the non--degenerate case we have (1.4). Now there exists a slowly varying function h such that $n^{-1/\alpha}h^{-1}(n)\sum_{i=1}^{n} U_i$ converges in distribution to a stable random variable with distribution function $F(\cdot;\, \alpha,\beta)$. We write

$$h^{-1}(n)\,n^{1-1/\alpha}(Y_n - \mu^2) = 2\mu\sigma h^{-1}(n)\,n^{-1/\alpha}\sum_{i=1}^{n} U_i$$

$$+ n^{-1/\alpha}(n-1)^{-1}h^{-1}(n)\,\sigma^2\left\{(\sum U_i)^2 - \sum U_i^2\right\} \tag{1.8}$$

Thus the first term on the right hand side has a stable limit distribution. Since $1 < \alpha < 2$ it follows that $n^{-1/\alpha}(n-1)^{-1}h^{-1}(n)\,(\sum U_i)^2$ converges to 0 in probability. The random variable $U_i^2 \in D(\alpha/2\,,\, 1)$; thus there exists a slowly varying function h_1 such that $n^{-2/\alpha}h_1^{-1}(n)\sum U_i^2$ converges in distribution to a stable random variable with distribution function $F(\cdot;\ \alpha/2,1)$. Hence it follows that $n^{-1/\alpha}(n-1)^{-1}h^{-1}(n)\sum U_i^2$ converges in probability to 0.

General U-statistics in this case are considered in Malevich and Abdalimov [MA 77]. For many random variables $X_i \in D(\alpha, \beta)$ we can choose $h_1 = h^2$. But this is not in general true. From now on we restrict ourselves to those random variables in the domain of attraction where we have $h_1 = h^2$.

Case IIb. In the degenerate case we have

$$\sum_{i \neq j}\sum X_i X_j = \sigma^2\left\{(\sum U_i)^2 - \sum U_i^2\right\}.$$

As in Case IIa and with the assumption made above, we have that the distribution of $n^{-1/\alpha}h^{-1}(n)\sum U_i$ converges weakly to $F(\cdot;\ \alpha,\beta)$ and the distribution of $n^{-2/\alpha}h^{-2}(n)\sum U_i^2$ to $F(\cdot;\ \alpha/2,1)$. Thus the statistics $\sigma^{-2}n^{-2/\alpha}h^{-2}(n)\sum_{i \neq j}\sum X_i X_j$ converge in distribution to the random variable $S_\alpha^2 - S_{\alpha/2}$, where S_α and $S_{\alpha/2}$ are <u>dependent</u> stable random variables with distribution functions $F(\cdot;\ \alpha,\beta)$ and $F(\cdot;\ \alpha/2,1)$.

This is a special case of a theorem on products of stable random variables. See Avram and Taqqu [AT 86].

We delete the case $\alpha = 1$ for well-known normalizing difficulties.

Case III. $X_1 \in D(\alpha,\beta)$ with $0 < \alpha < 1$. The random variable X_1 has no finite expectation. This case is the same as Case IIb.

2. The Characteristic Function. In this section we consider i.i.d. positive random variables $X_1, X_2, \ldots$ with common distribution function G given by

$$1 - G(x) = x^{-\alpha} \quad x > 1 \quad 0 < \alpha < 1. \tag{2.1}$$

Thus we have $X_1 \in D_N(\alpha, 1)$. Let U_1, U_2, ... be i.i.d. uniformly distributed on (0,1). Thus $G^{-1}(U_1) \stackrel{d}{=} X_1$. Then we have

$$\begin{aligned} T_n = \sum_{i \neq j}\sum X_i X_j &\stackrel{d}{=} \{\sum_{i=1}^{n} G^{-1}(U_i)\}^2 - \sum_{i=1}^{n}\{G^{-1}(U_i)\}^2 \\ &\stackrel{d}{=} \{\sum_{i=1}^{n} U_i^{-1/\alpha}\}^2 - \sum_{i=1}^{n} U_i^{-2/\alpha} \\ &= \{\sum_{i=1}^{n} U_{(i)}^{-1/\alpha}\}^2 - \sum_{i=1}^{n} U_{(i)}^{-2/\alpha}, \end{aligned}$$

where $U_{(1)} \ < , \ \cdot \ \cdot \ \cdot \ , \ < \ U_{(n)}$ a.s.

$$= 2U_{(1)}^{-1/\alpha}\{\sum_{i=2}^{n} U_{(i)}^{-1/\alpha}\} + \{\sum_{i=2}^{n} U_{(i)}^{-1/\alpha}\}^2 - \sum_{i=2}^{n} U_{(i)}^{-2/\alpha}.$$

Given $U_{(1)} = u$ we have

$$T_n | U_{(1)} = u \stackrel{d}{=} 2u^{-1/\alpha}\{\sum_{i=2}^{n} \bar{U}_i^{\ -1/\alpha}\} + \{\sum_{i=2}^{n} \bar{U}_i^{\ -1/\alpha}\}^2 - \sum_{i=2}^{n} \bar{U}_i^{\ -2/\alpha} \tag{2.2}$$

where $\bar{U}_2, \ldots, \bar{U}_n$ are i.i.d. and uniformly distributed on (u,1). $\bar{U}_2$ has a finite expectation and finite variance. We have for $u \downarrow 0$

$$\mu := \quad E\bar{U}_2^{\ -1/\alpha} = (1-u)^{-1}\alpha(1-\alpha)^{-1}(u^{-(1-\alpha)/\alpha} - 1) \tag{2.3}$$

$$\sim (1-u)^{-1}\alpha(1-\alpha)^{-1}u^{-1/\alpha+1}$$

and

$$\sigma^2 := \quad E\bar{U}_2^{\ -2/\alpha} = (1-u)^{-1}\alpha(2-\alpha)^{-1}(u^{-(2-\alpha)/\alpha} - 1) \tag{2.4}$$

$$\sim (1-u)^{-1}\alpha(2-\alpha)^{-1}\upsilon^{-2/\alpha+1}.$$

This implies that

$$\sigma^2(\bar{U}_2^{\ -1/\alpha}) \sim \sigma^2 \quad \text{for} \quad u \downarrow 0.$$

Given $U_{(1)} = u$, the random variable

$$V_{n-1} = \left(\sum_{i=2}^{n} \bar{U}_i^{\ -1/\alpha}\right)^2 - \sum_{i=2}^{n} \bar{U}_i^{\ -2/\alpha} \tag{2.5}$$

is again a U-statistic. Define, for $i = 1, \ldots, n$ $Y_i = \bar{U}_i^{-1/\alpha} - \mu$. Then we have

$$V_{n-1} = EV_{n-1} + \mu(n-2)\sum_{i=2}^{n} Y_i + \sum_{i=2}^{n}\sum_{\substack{j=2 \\ i \neq j}}^{n} Y_i Y_j. \tag{2.6}$$

One easily computes for small u

$$E2n^{-2/\alpha}u^{-1/\alpha}\left(\sum_{i=2}^{n} \bar{U}_i^{-1/\alpha}\right) \approx 2\alpha(1-\alpha)^{-1}(nu)^{-2/\alpha+1},$$

$$En^{-2/\alpha}V_{n-1} \approx \alpha^2(1-\alpha)^{-2}(nu)^{-2/\alpha+2},$$

$$\sigma^2\left(2u^{-1/\alpha}n^{-2/\alpha}\left(\sum_{i=2}^{n}\overline{U_i}^{-1/\alpha}\right)\right) \approx 4\alpha(2-\alpha)^{-1}(nu)^{-4/\alpha+1},$$

$$\sigma^2\left(n^{-2/\alpha}\mu(n-2)\sum_{i=2}^{n}Y_i\right) \approx c(nu)^{-4/\alpha+3}, \text{ for some constant c}$$

and

$$\sigma^2\left(n^{-2/\alpha}\sum_{\substack{i=2\\ i\neq j}}^{n}\sum_{j=2}^{n} Y_iY_j\right) \approx (nu)^{-4/\alpha+2}.$$

We shall derive the characteristic function (cf.) f^* of the limit distribution of

$$n^{-2/\alpha}T_n^* = 2n^{-2/\alpha}U_{(1)}^{-1/\alpha}\sum_{i=2}^{n}\overline{U_i}^{-1/\alpha} + n^{-2/\alpha}EV_{n-1}. \tag{2.7}$$

In section 3 we shall give an estimate for the difference of this cf. f^* and the cf. f of the double stable integral for small values of the argument. For technical reasons we restrict ourselves to the case $\frac{1}{2} \le \alpha < 1$. See section 3.

One easily obtains that $n\,U_{(1)}$ converges for $n \to \infty$, in distribution to the exponential distribution. For $k_n \to \infty$ and $k_n = o(n)$ for $n \to \infty$ we have

$$P((nk_n)^{-1} < U_{(1)} < n^{-1}k_n) \to 1 \tag{2.8}$$

for $n \to \infty$.

Take $t > 0$

$$Ee^{itn^{-2/\alpha}T_n^*} = E_{U_{(1)}}\left\{Ee^{itn^{-2/\alpha}T_n^*} \mid U_{(1)}\right\}$$

$$= n\int_0^1 (1-u)^{n-1}\left\{Ee^{itn^{-2/\alpha}T_n^*} \mid U_{(1)} = u\right\}du.$$

From (2.7) we have

$$E\left(e^{itn^{-2/\alpha}T_n^*} \mid U_{(1)} = u\right) = Ee^{2itn^{-2/\alpha}u^{-1/\alpha}\sum_{j=2}^{n}\overline{U_j}^{-1/\alpha}}\, e^{itn^{-2/\alpha}n(n-1)\mu^2}$$

$$= e^{itn^{-2/\alpha}n(n-1)\mu^2}\left\{Ee^{2itn^{-2/\alpha}\overline{U}_2^{-1/\alpha}}\right\}^{n-1},$$

where

$$Ee^{2itn^{-2/\alpha}u^{-1/\alpha}\overline{U}_2^{-1/\alpha}} = (1-u)\int_u^1 e^{2itn^{-2/\alpha}u^{-1/\alpha}y^{-1/\alpha}}dy.$$

Consider the integral

$$I = (1-u)^{-1}\int_u^1 (e^{2itn^{-2/\alpha}u^{-1/\alpha}y^{-1/\alpha}} - 1)\,dy$$

$$= (1-u)^{-1}\alpha 2^{\alpha}t^{\alpha}n^{-2}u^{-1}\int_{2n^{-2/\alpha}u^{-1\alpha}t}^{2n^{-2/\alpha}u^{-2/\alpha}t} (e^{iv}-1)\,v^{-1-\alpha}dv.$$

This integral is well-known in the derivation of the characteristic function of a stable random variable. We define

$$\phi(x) = \int_0^x (e^{iv}-1)\,v^{-1-\alpha}dv. \tag{2.9}$$

We have

$$\phi(\infty) = -\alpha^{-1}\Gamma(1-\alpha)\,e^{-\pi i(\alpha/2)}. \tag{2.10}$$

See Laha and Rohatgi [LR79] p. 333. Thus

$$f^*(t) = \lim_{n\to\infty} Ee^{itn^{-2/\alpha}T_n^*}$$

$$= \lim_{n\to\infty} n\int_{(nk_n)^{-1}}^{k_n n^{-1}} (1-u)^{n-1}(1+I)^{n-1}e^{it\alpha^2(1-\alpha)^{-2}(un)^{-2/\alpha+2}}du$$

$$= \lim_{n\to\infty}\int_{k_n^{-1}}^{k_n} e^{-nu}e^{nI}e^{it\alpha^2(1-\alpha)^{-2}(un)-2/\alpha+2}dnu \tag{2.11}$$

$$= \int_0^\infty e^{-y}e^{it\alpha^2(1-\alpha)^{-2}y^{-2/\alpha+2}}e^{\alpha 2^{\alpha}t^{\alpha}y^{-1}\phi(2ty^{-2/\alpha})}dy.$$

3. Tail Behaviour of the Stable Integral. In order to obtain the behaviour of the tail of the double stable integral we need the expansion of its cf. f near the origin. We rewrite the cf. f^* given in (2.11). Let

$$\phi_1(x) = \int_0^x (\cos\ v-1)\, v^{-1-\alpha} dv \tag{3.1}$$

$$\phi_2(x) = \int_0^x \sin v\ v^{-1-\alpha} dv \tag{3.2}$$

and let the function g_t be defined by

$$g_t(y) = \begin{cases} e^{-y} e^{\alpha 2^\alpha t^\alpha y^{-1}\phi_1(2ty^{-2/\alpha})} & for\ 0<y<\infty \\ 0 & else \end{cases} \tag{3.3}$$

Then $\phi_1(\infty)$ and $\phi_2(\infty)$ follow from (2.10). We have

$$f^*(t) = \int_0^\infty e^{it\alpha^2(1-\alpha)^{-2}y^{-2/(\alpha+2)}} e^{i\alpha 2^\alpha t^\alpha y^{-1}\phi_2(2ty^{-2/\alpha})} g_t(y)\,dy. \tag{3.4}$$

Since for $y > (2t)^{\alpha/2}$ both $2^\alpha t^\alpha y^{-1}$ and $2ty^{-2/\alpha}$ are small, we easily obtain

$$\int_{(2t)^{\alpha/2}}^\infty g_t(y)\,dy = \int_{(2t)^{\alpha/2}}^\infty e^{-y} dy + O(t^\alpha) \tag{3.5}$$

for $t{\downarrow}0$. Obviously we have

$$\int_0^{\alpha 2^\alpha t^\alpha} g_t(y)\,dy = O(t^\alpha) \tag{3.6}$$

for t 0. Then

$$\int_{\alpha 2^\alpha t^\alpha}^{(2t)^{\alpha/2}} g_t(y)\,dy = \int_{\alpha 2^\alpha t^\alpha}^{(2t)^{\alpha/2}} \left\{1-y+\alpha 2^\alpha t^\alpha y^{-1}\phi_1(2ty^{12/\alpha})\right\} dy + \text{error.} \tag{3.7}$$

Using the definition of ϕ_1 and by partial integration we obtain

$$\alpha \int_{\alpha 2^\alpha t^\alpha}^{(2t)^{\alpha/2}} y^{-1}\phi_1(2ty^{-2/\alpha})\,dy = 2\int_1^{\alpha^{-2/\alpha}2^{-1}t^{-1}} z^{-1}\phi_1(z)\,dz$$

$$= 2\phi_1(\alpha^{-2/\alpha}2^{-1}t^{-1})\log(\alpha^{-2/\alpha}2^{-1}t^{-1}) + O(t^\alpha) \quad \text{for } t{\downarrow}0.$$

With similar calculations as above we show that the error in the right hand side of (3.7) is $O(t^{\alpha})$ for $t\downarrow 0$. Combining the results in (3.5), (3.6) and the result above, we obtain

$$\int_0^\infty g_t(y)\,dy = 1+2^{1+\alpha}t^{\alpha}\phi_1(\alpha^{-2/\alpha}2^{-1}t^{-1})\log(t^{-1}) + O(t^{\alpha})$$

for $t\downarrow 0$.

For the expansion of f^* we have to distinguish two cases: $\frac{1}{2}\le\alpha<1$ and $0<\alpha<\frac{1}{2}$. As mentioned before, we only consider the case $\frac{1}{2}\le\alpha<1$. See after (3.11). Now we have $t^{\alpha/(2(1-\alpha))}<t^{\alpha}<t^{\alpha/2}$.

$$f^*(t) = \int_0^\infty e^{it\alpha^2(1-\alpha)^{-2}y^{-2/\alpha+2}}e^{i\alpha 2^{\alpha}t^{\alpha}y^{-1}\phi_2(2ty^{-2/\alpha})}g_t(y)\,dy$$

$$= \int_0^{\alpha 2^{\alpha}t^{\alpha}} + \int_{\alpha 2^{\alpha}t^{\alpha}}^{(2t)^{\alpha/2}} + \int_{(2t)^{\alpha/2}}^{\infty} = I_1+I_2+I_3.$$

Obviously we have

$$I_1 = O(t^{\alpha}) \quad \text{for } t\downarrow 0,$$

For $y>(2t)^{\alpha/2}$ we have that $ty^{-2/\alpha+2}$, $2^{\alpha}t^{\alpha}y^{-1}$ and $2ty^{-2/\alpha}$ are small. Expansion of the integrand gives

$$I_3 = \int_{(2t)^{\alpha/2}}^{\infty} e^{-y}dy + O(t^{\alpha}) \quad \text{for } t\downarrow 0.$$

I_2 is the most interesting part. We have

$$I_2 = \int_{\alpha 2^{\alpha}t^{\alpha}}^{(2t)^{\alpha/2}} \left\{1+it\alpha^2(1-\alpha)^{-2}y^{-2/\alpha+2}+i\alpha 2^{\alpha}t^{\alpha}y^{-1}\phi_2(2ty^{-2/\alpha})\right\}g_t(y)\,dy \tag{3.9}$$

$$+ \text{error}.$$

For $\alpha\ge\frac{1}{2}$ we have

$$t\int_{\alpha 2^{\alpha}t^{\alpha}}^{(2t)^{\alpha/2}} y^{-2/\alpha+2}dy = O(t^{\alpha}).$$

And, as in computation of the integral of g_t, we have

$$\alpha t^{\alpha} \int_{\alpha 2^{\alpha} t^{\alpha}}^{(2t)^{\alpha/2}} y^{-1}\phi_2(2ty^{-2/\alpha})\,dy = 2t^{\alpha}\phi_2(\alpha^{-2/\alpha}2^{-1}t^{-1})\log(\alpha^{-2/\alpha}2^{-1}t^{-1}) + O(t^{2\alpha}).$$

One easily shows that the error on the right hand side of (3.9) is also $O(t^{\alpha})$ for $t\downarrow 0$. Combining the foregoing results, one obtains in the case $\frac{1}{2} \le \alpha < 1$

$$f^*(t) - 1 = ct^{\alpha}\log(t^{-1}) + O(t^{\alpha}) \tag{3.10}$$

for $t\downarrow 0$. We have

$$f(t) - f^*(t) = \lim_{n\to\infty} E_{U_{(1)}}\left[E\left\{e^{itn^{-2/\alpha}T_n} \middle| U_{(1)}\right\} - E\left\{e^{itn^{-2/\alpha}T_n^*} \middle| U_{(1)}\right\}\right]$$

$$= \lim_{n\to\infty} E_{U_{(1)}}\left[Ee^{2itn^{-2/\alpha}U_{(1)}^{-1/\alpha}\sum_{j=2}^{n}\bar{U}_j^{-1/\alpha}}\left\{e^{itn^{-2/\alpha}V_{n-1}} - e^{itn^{-2/\alpha}EV_{n-1}}\right\}\middle| U_{(1)}\right].$$

Applying the inequality

$$|e^{ix} - e^{iy}| \le |x-y|$$

and similar calculations as above, we obtain, for $\frac{1}{2} \le \alpha < 1$,

$$|f(t) - f^*(t)| = O(t^{\alpha}) \quad \text{for } t\downarrow 0.$$

This implies the following expansion for f

$$f(t) - 1 \sim ct^{\alpha}\log(1/t) \tag{3.11}$$

for $t\downarrow 0$.

In the case $0 < \alpha < 1/2$ it is more delicate to obtain an estimate for $f - f^*$. For that reason we delete this case.

From the theory of characteristic functions we obtain from the expansion of f

$$P(I(h) > x) \sim ax^{-\alpha}\log x \quad \text{as } x \to \infty.$$

See, for example, Feller [Fel. 71] section xvii.12, problem 14.

4. Extension. In the previous section we considered the integral

$$I(h) = \iint h(x,y)X(dx)X(dy)$$

where X is a completely asymmetric stable process and h is given by (1.2). We can extend the integral to functions given by (1.3).

THEOREM 4.1. Let $I(h)$ be given as above and

$$h(x,y) = \begin{cases} \phi(x)\phi(y) & for\ x \neq y \\ 0 & for\ x = y \end{cases}.$$

Suppose ϕ positive and $\int_0^1 \phi(x)^\alpha dx < \infty$. Then

$$I(h) \stackrel{d}{=} \left\{\int_0^1 \phi^\alpha(x)\,dx\right\}^{2/\alpha} \left\{S_\alpha^2 - S_{\alpha/2}\right\}$$

where S_α and $S_{\alpha/2}$ are dependent stable random variables.

PROOF.

$$I(h) = \lim\left[\{\sum_{i=1}^{n} \phi(i/n)n^{-1/\alpha}X_i\}^2 - \sum_{i=1}^{n} \phi^2(i/n)n^{-2/\alpha}X_i^2 \right]$$

$$\stackrel{d}{=} \left[\left\{\int_0^1 \phi^\alpha(x)\,dx\right\}^{1/\alpha} X\right]^2 - \left\{\int_0^1 \phi^\alpha(x)\,dx\right\}^{2/\alpha} S_{\alpha/2}$$

$$= \left\{\int_0^1\int_0^1 h^\alpha(x,y)\,dxdy\right\}^{1/\alpha} \left\{S_\alpha^2 - S_{\alpha/2}\right\}.$$

See also section 1, case IIb. The first limit follows from properties of stable random variables. One shows the second limit by using characteristic functions. □

Remark. We can obtain the tail behaviour as in the case h satisfies (1.2). Note that the dependency of S_a and $S_{\alpha/2}$ depends on ϕ.

In Samorodnitsky and Szulga [SS 88] the asymptotic behaviour of the tail of $I(h)$ is given in the case of a symmetric stable process. They obtain if h satisfies the conditions of Theorem 4.1

$$P(|I(h)| > x) \sim C_\alpha(h)x^{-\alpha}\log x \text{ as } x \to \infty.$$

5. The Behaviour of the Extremal Term. In this section we consider the term $U_{(1)}^{-1/\alpha}U_{(2)}^{-1/\alpha}$ where $U_{(1)}$ and $U_{(2)}$ are the order statistics of a uniform distribution. Let Γ_i, $i = 1,2, \ldots$ be the arrivals of a Poisson process. We have

$$
\begin{aligned}
P(U_{(1)}^{-1/\alpha}U_{(2)}^{-1/\alpha} > n^{2/\alpha}x) &= P(U_{(1)}U_{(2)} \le n^{-2}x^{-\alpha}) \\
&= P\left(\frac{\Gamma_1^{-1/\alpha}\Gamma_2^{-1/\alpha}}{\Gamma_n^{-2/\alpha}} > n^{2/\alpha}x\right) \\
&\sim P(\Gamma_1^{-1/\alpha}\Gamma_2^{-1/\alpha} > x) = P(\Gamma_1\Gamma_2 < x^{-\alpha}) \\
&= x^{-\alpha}\int_{x^{-\alpha/2}}^{\infty} y^{-1}e^{-y}dy + \int_0^{x^{-\alpha/2}} ye^{-y}dy \\
&= x^{-\alpha}\int_{x^{-\alpha/2}}^{\infty} y^{-1}e^{-y}dy + 1 - e^{-x^{-\alpha/2}} - x^{-\alpha/2}e^{-x^{-\alpha/2}} \\
&\sim \frac{1}{2}\alpha x^{-\alpha}\log x \quad \text{for large } x.
\end{aligned}
$$

We have seen a similar tail behaviour for the double α-stable integral.

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