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DESCRIPTIVE CHARACTERIZATIONS OF PETTIS AND STRONGLY MCSHANE INTEGRALS

Abstract

Using different types of absolute continuity, we characterize additive interval functions which are the primitives of Pettis or strongly McShane integrable functions.

1 Introduction and Preliminaries

There are known characterizations of additive interval functions which are the primitives of Pettis or strongly McShane integrable functions in terms of their scalar derivatives or derivatives, respectively, see Theorem 5.1 in [14] and Theorem 7.4.14 in [15]. Here, we characterize these additive interval functions in terms of their average ranges, Theorem 6 and Theorem 8.

Throughout this paper X denotes a real Banach space with its norm $||.||$. We denote by $B(x,\varepsilon)$ the open ball with center x and radius $\varepsilon > 0$ and by X^* the topological dual to X.

We denote by $\mathcal{I}$ the family of all non-degenerate closed subintervals of $[0,1]$, by λ the Lebesgue measure on $[0,1]$ and by $\mathcal{L}$ the family of all Lebesgue measurable subsets of $[0,1]$. We will identify an interval function $\widetilde{F} : \mathcal{I} \to X$ with the point function $F(t) = \widetilde{F}([0,t]), t \in [0,1]$; and conversely, we will identify a point function $F : [0,1] \to X$ with the interval function $\widetilde{F}([u,v]) = F(v) - F(u), [u,v] \in \mathcal{I}$. An interval function $\widetilde{F} : \mathcal{I} \to X$ is said to be *additive* if for each nonoverlapping interval $I, J \in \mathcal{I}$ with $I \cup J \in \mathcal{I}$, we have

$$\widetilde{F}(I \cup J) = \widetilde{F}(I) + \widetilde{F}(J).$$

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The intervals I and J are said to be *nonoverlapping* if $\operatorname{int}(I) \cap \operatorname{int}(J) = \emptyset$, where $\operatorname{int}(I)$ denotes the interior of I.

Let $F : [0,1] \to X$ be a function. The function is said to be *strongly absolutely continuous (sAC)* if for every $\varepsilon > 0$ there exists $\eta > 0$ such that for every finite collection $\{I_i : i = 1, 2, ..., p\}$ of nonoverlapping subintervals in $\mathcal{I}$, we have

$$\sum_{i=1}^{p} \lambda(I_i) < \eta \Rightarrow \sum_{i=1}^{p} ||\widetilde{F}(I_i)|| < \varepsilon. \tag{1.1}$$

Replacing (1.1) by

$$\sum_{i=1}^{p} \lambda(I_i) < \eta \Rightarrow ||\sum_{i=1}^{p} \widetilde{F}(I_i)|| < \varepsilon,$$

we obtain the definition of *absolute continuity (AC)*.

Let us consider $t \in [0,1]$. We put

$$\Delta F(t,h) = \frac{F(t+h) - F(t)}{h}, \quad A_F(t,\delta) = \{\Delta F(t,h) : 0 < |h| < \delta\}$$

and

$$A_F(t) = \bigcap_{\delta > 0} \overline{A_F}(t,\delta),$$

where $\overline{A_F}(t,\delta)$ is the closure of $A_F(t,\delta)$. The set $A_F(t)$ is said to be *the average range of F at t*.

The function F is said to be *differentiable at the point t* if there is a vector $x \in X$ such that

$$\lim_{h \to 0} ||\Delta F(t,h) - x|| = 0.$$

We denote $x = F'(t)$ the derivative of F at t.

We say that F has a *scalar derivative* on $[0,1]$ if there exists a function $f : [0,1] \to X$ such that for each $x^* \in X^*$, $(x^* \circ F)'(t)$ exists and $(x^* \circ F)'(t) = (x^* \circ f)(t)$ a.e. on $[0,1]$ (the exceptional set may vary with x^*). The function f is said to be a scalar derivative of F on $[0,1]$.

A finite collection of interval-point pairs $\{(I_i, t_i) : i = 1, 2, ..., m\}$ is said to be an $\mathcal{M}$-partition of $[0,1]$ if $t_i \in [0,1]$, for all $i = 1, 2, ..., m$, and $\{I_i : i = 1, 2, ..., m\}$ is a finite collection of pairwise nonoverlapping intervals of $\mathcal{I}$ such that

$$\bigcup_{i=1}^{m} I_i = [0,1].$$

A function $\delta : [0,1] \to (0,+\infty)$ is said to be a gauge on $[0,1]$. An $\mathcal{M}$-partition π is said to be δ-fine if for each interval-point pair $(I,t) \in \pi$, we have

$$I \subset (t - \delta(t), t + \delta(t)).$$

Definition 1. A function $f : [0,1] \to X$ is said to be *McShane integrable* on $[0,1]$ if there is a vector $w \in X$ such that for every $\varepsilon > 0$, there a gauge δ on $[0,1]$, such that for every δ-fine $\mathcal{M}$-partition π of $[0,1]$, we have

$$|| \sum_{(I,t)\in\pi} f(t)\lambda(I) - w|| < \varepsilon.$$

We denote $w = (M)\int_{[0,1]} f$. A function $f : S \to X$ is said to be *McShane integrable on* $E \subset [0,1]$ if the function $f.\chi_E : [0,1] \to X$ is McShane integrable on $[0,1]$, where χ_E is the characteristic function of the set E. The McShane integral of f over E will be denoted by $(M)\int_E f$. If f is McShane integrable on $[0,1]$ then we obtain by Theorem 4.1.6 in [15] that for every $E \in \mathcal{L}$ the function f is McShane integrable on E.

Definition 2. A function $f : [0,1] \to X$ is said to be strongly McShane integrable on $[0,1]$ if there exists a function $F : [0,1] \to X$ such that for every $\varepsilon > 0$, there a gauge δ on $[0,1]$, such that for every δ-fine $\mathcal{M}$-partition π of $[0,1]$, we have

$$\sum_{(I,t)\in\pi} ||f(t)\lambda(I) - \widetilde{F}(I)|| < \varepsilon.$$

The function F is said to be the primitive of f. Clearly, if f is strongly McShane integrable on $[0,1]$ with the primitive F, then f is McShane integrable on $[0,1]$ and we obtain by Proposition 3.6.16 in [15] that

$$\widetilde{F}(I) = (M)\int_I f \quad \text{for every} \quad I \in \mathcal{I}.$$

For more information about the McShane integral we refer to [2], [4], [5], [6]-[8], [17], [18] and [19].

Definition 3. A function $f : [0,1] \to X$ is said to be Pettis integrable, if $x^* \circ f \in L_1([0,1])$ for all $x^* \in X^*$ and for any $E \in \mathcal{L}$ there exists an $x_E \in X$ such that

$$x^*(x_E) = \int_E (x^* \circ f)(t)d\lambda$$

whenever $x^* \in X^*$. The vector x_E is then called the Pettis integral of f on E and we set $x_E = (P)\int_E f(t)d\lambda$.

We refer to [11], [12], [13], [16] and [14] for Pettis integrability.

2 Main Results

The main results are Theorem 6 and Theorem 8. Lemma 4 and Example 5 highlight the local relation between the differential and the average range.

Lemma 4. *Let $F : [0,1] \to X$ be a function and let $t_0 \in [0,1]$. If F is differentiable at t_0, then*

$$A_F(t_0) = \{F'(t_0)\}. \tag{2.1}$$

Proof. First, we will show that $F'(t_0) \in A_F(t_0)$. To see this, we choose a sequence (h_k) of real numbers such that

$$0 < |h_k| < \frac{1}{k} \quad \text{for all} \quad k \in \mathbb{N}.$$

Then

$$\lim_{k\to\infty} ||\Delta F(t_0, h_k) - F'(t_0)|| = 0$$

and since

$$\Delta F(t_0, h_k) \in A_F(t_0, \frac{1}{n})$$

for all $k \in \mathbb{N}$ such that $k \geq n$, it follows that

$$F'(t_0) \in \bigcap_{n=1}^{\infty} \overline{A_F}(t_0, \frac{1}{n}) = \bigcap_{\delta>0} \overline{A_F}(t_0, \delta) = A_F(t_0).$$

Secondly, we will show that $A_F(t_0) \subset \{F'(t_0)\}$. Assume that $x \in A_F(t_0)$ is given. Then, for each $n \in \mathbb{N}$, we have

$$B(x, \frac{1}{n}) \bigcap A_F(t_0, \frac{1}{n}) \neq \emptyset.$$

Therefore, there is a sequence (h'_n) of real numbers such that

$$0 < |h'_n| < \frac{1}{n} \quad \text{and} \quad \Delta F(t_0, h'_n) \in B(x, \frac{1}{n}) \quad \text{for all} \quad n \in \mathbb{N}.$$

Hence

$$\lim_{n\to\infty} ||\Delta F(t_0, h'_n) - x|| = 0,$$

and we infer that $x = F'(t_0)$. □

The following example shows that there is a function $F : [-1,1] \to l_p$, $p > 1$, such that $A_F(0) = \{(0)\}$ but F is not differentiable at $t = 0$.

Example 5. *Let $F : [-1,1] \to l_p$ be a function given as follows*

$$F(t) = \begin{cases} (0, ..., 0, ...) & \text{if} \quad t \neq \frac{1}{n} \\ (0, ..., 0, \frac{1}{n}, 0, ...) & \text{if} \quad t = \frac{1}{n} \end{cases} \qquad t \in [-1,1] \quad n = 1, 2, 3, ...$$

Since

$$\Delta F(0,h) = \begin{cases} (0, ..., 0, ...) & \text{if} \quad h \neq \frac{1}{n} \\ (0, ..., 0, 1, 0, ...) & \text{if} \quad h = \frac{1}{n} \end{cases}$$

we have

$$diam(\overline{A_F}(0,\delta)) = 2^{\frac{1}{p}} \quad \text{for all} \quad \delta > 0. \tag{2.2}$$

We claim that

$$A_F(0) = \{(0, ..., 0, ...)\}. \tag{2.3}$$

Let us consider an arbitrary element $x_0 \in A_F(0)$. Since

$$A_F(0) = \bigcap_{k=1}^{\infty} \overline{A_F}(0, \frac{1}{k})$$

then there is a sequence $(h_k) \subset \mathbb{R}$ such that for each $k \in \mathbb{N}$, we have

$$0 < |h_k| < \frac{1}{k} \quad \text{and} \quad ||\Delta F(0,h_k) - x_0||_{l_p} < \frac{1}{k}.$$

Therefore

$$\lim_{k\to\infty} ||\Delta F(0,h_k) - x_0||_{l_p} = 0.$$

Hence

$$\lim_{k\to\infty} x^*(\Delta F(0,h_k)) = x^*(x_0) \quad \text{for all} \quad x^* \in (l_p)^*. \tag{2.4}$$

Fix an arbitrary $x^* \in (l_p)^*$. Since $(l_p)^* = l_q$, there is a sequence $(a_n) \in l_q$ such that

$$x^*(x) = \sum_{n=1}^{+\infty} a_n x_n \quad \text{for all} \quad x = (x_n) \in l_p,$$

and since

$$x^*(\Delta F(0,h_k)) = \begin{cases} 0 & \text{if} \quad h_k \neq \frac{1}{n} \\ a_n & \text{if} \quad h_k = \frac{1}{n} \end{cases} \qquad (n > k),$$

we obtain

$$\lim_{k\to\infty} x^*(\Delta F(0,h_k)) = 0.$$

Hence, by (2.4), it follows that

$$x^*(x_0) = 0 \quad \text{for all} \quad x^* \in (l_p)^*,$$

because x^* was arbitrary. Therefore, we obtain by Hahn-Banach Theorem that

$$x_0 = (0, ..., 0, ...)$$

and consequently (2.3) *holds true.*

Assume by contradiction that F is differentiable at $t = 0$. Then, we obtain by Lemma 4 that $F'(0) = (0, ..., 0, ...)$. Hence

$$\lim_{n\to\infty} ||\Delta F(0, \frac{1}{n}) - (0, ..., 0, ...)||_{l_p} = 0.$$

On the other hand

$$\lim_{n\to\infty} ||\Delta F(0, \frac{1}{n}) - (0, ..., 0, ...)||_{l_p} = 1,$$

because for each $n \in \mathbb{N}$, we have $||\Delta F(0, \frac{1}{n}) - (0, ..., 0, ...)||_{l_p} = 1$. This contradiction shows that F is not differentiable at $t = 0$.

Theorem 6. *Let $F : [0, 1] \to X$ be a function. Then the following statements are equivalent.*

(i) F is the primitive of a Pettis integrable function f, i.e.,

$$\widetilde{F}(I) = (P)\int_I f(t)d\lambda \quad \textit{for all} \quad I \in \mathcal{I},$$

(ii) F is AC and f is a scalar derivative of F on $[0, 1]$,

(iii) F is AC and there exists a function $f : [0, 1] \to X$ such that for each $x^* \in X^*$, we have $(x^* \circ f)(t) \in A_{x^*\circ F}(t)$ a.e. on $[0, 1]$ (the exceptional set may vary with x^*).

Proof. $(i) \Leftrightarrow (ii)$ By Theorem 5.1 in [14], this equivalence holds true.

$(ii) \Rightarrow (iii)$ Assume that F is AC and f is a scalar derivative of F on $[0, 1]$. Fix an arbitrary $x^* \in X^*$. Then, there exists $Z^{(x^*)} \subset [0, 1]$ with $\lambda(Z^{(x^*)}) = 0$ such that $(x^* \circ F)'(t)$ exists and

$$(x^* \circ F)'(t) = (x^* \circ f)(t) \quad \text{for all} \quad t \in [0, 1] \setminus Z^{(x^*)}.$$

Hence, we obtain by Lemma 4 that

$$(x^* \circ f)(t) \in A_{x^*\circ F}(t) \quad \text{for all} \quad t \in [0, 1] \setminus Z^{(x^*)}$$

and since x^* was arbitrary it follows that (iii) holds.

$(iii) \Rightarrow (ii)$ Assume that (iii) holds and let x^* be an arbitrary element of X^*. Then, there is a subset $Z^{(x^*)} \subset [0,1]$ with $\lambda(Z^{(x^*)}) = 0$ such that

$$(x^* \circ f)(t) \in A_{x^* \circ F}(t) \quad \text{for all} \quad t \in [0,1] \setminus Z^{(x^*)}$$

Since $x^* \circ F$ is AC, there exists $Z_1^{(x^*)} \in [0,1]$ with $\lambda(Z_1^{(x^*)}) = 0$, such that $(x^* \circ F)'(t)$ exists for all $t \in [0,1] \setminus Z_1^{(x^*)}$. Therefore, we obtain by Lemma 4 that

$$(x^* \circ F)'(t) = (x^* \circ f)(t) \quad \text{for all} \quad t \in [0,1] \setminus (Z^{(x^*)} \cup Z_1^{(x^*)}).$$

Since x^* was arbitrary, the last result yields that f is a scalar derivative of F on $[0,1]$. □

The following lemma makes it possible to present clearly Theorem 8. We refer to [1] for the notions used in this lemma.

Lemma 7. *Let $F : [0,1] \to X$ be a function. If F is sAC, then there exists an unique countable additive vector measure $F_{\mathcal{L}} : \mathcal{L} \to X$ of bounded variation, λ-continuous and such that*

$$\widetilde{F}(I) = F_{\mathcal{L}}(I) \quad \textit{for all} \quad I \in \mathcal{I}. \tag{2.5}$$

PROOF. Let $\mathcal{I}_0$ be the set of all subintervals $I \subset [0,1]$ having one of two forms $[0,b]$ or $(a,b]$ where $0 < a < b \leq 1$. For such intervals, place

$$\widetilde{F}_0([0,b]) = \widetilde{F}([0,b]) \quad \text{and} \quad \widetilde{F}_0((a,b]) = \widetilde{F}([a,b]).$$

Let $\mathscr{A}$ consist of all finite unions of such intervals. It is clear that $\mathscr{A}$ is an algebra and that if a set $E \in \mathscr{A}$ has the form

$$E = I_1 \cup I_2 \cup ... \cup I_n,$$

where I_i are disjoint intervals of type described, then

$$\widetilde{F}_0(I_1) + \widetilde{F}_0(I_2) + ... + \widetilde{F}(I_n),$$

is independent of the particular family of disjoint intervals $I_1, I_2, ..., I_n$, whose union is E. Thus, we may define the vector $F_{\mathscr{A}}(E)$ by the equation

$$F_{\mathscr{A}}(E) = \widetilde{F}_0(I_1) + \widetilde{F}_0(I_2) + ... + \widetilde{F}_0(I_n).$$

Clearly, $F_{\mathscr{A}} : \mathscr{A} \to X$ is a unique vector measure such that

$$F_{\mathscr{A}}(I) = \widetilde{F}_0(I) = \widetilde{F}(\overline{I}) \quad \text{for all} \quad I \in \mathcal{I}_0, \tag{2.6}$$

where $\overline{I}$ is the closure of I. Since F is sAC, it easy to see that $F_{\mathscr{A}}$ is of bounded variation and

$$\lim_{\lambda(A)\to 0: A\in\mathscr{A}} F_{\mathscr{A}}(A) = 0. \tag{2.7}$$

Let us consider a sequence (A_n) of pairwise disjoint members of $\mathscr{A}$ such that $\cup_{n=1}^{\infty} A_n \in \mathscr{A}$. Denote $A = \cup_{n=1}^{\infty} A_n$. Since

$$\lim_{n\to\infty} \lambda(A \setminus \cup_{i=1}^{n} A_i) = 0$$

we obtain by (2.7) that

$$\lim_{n\to\infty} ||F_{\mathscr{A}}(A) - \cup_{i=1}^{n} F_{\mathscr{A}}(A_i)|| = \lim_{n\to\infty} ||F_{\mathscr{A}}(A \setminus \cup_{i=1}^{n} A_i)|| = 0.$$

Thus, $F_{\mathscr{A}}$ is a countable additive measure on $\mathcal{A}$, and since it is of bounded variation we obtain by Proposition I.1.15 in [1] that $F_{\mathscr{A}}$ is also strongly additive. Consequently, by Caratheodory-Hahn-Kluvanek extension theorem in [10] or by Theorem I.5.2 in [1], $F_{\mathscr{A}}$ has a unique countable additive extension $F_{\mathscr{B}} : \mathscr{B} \to X$.

Claim 1. The vector measure $F_{\mathscr{B}}$ is λ-continuous. To see this, let us consider the semimetric space $\mathscr{B}(\lambda)$ consisting of members of $\mathscr{B}$ equipped with the semimetric

$$\rho(B_1, B_2) = \lambda(B_1 \Delta B_2),$$

where $B_1 \Delta B_2 = (B_1 \setminus B_2) \cup (B_2 \setminus B_1)$. By Lemma III.7.1 in [3], $\mathscr{A}(\lambda)$ consisting of elements of $\mathscr{A}$ is dense in $\mathscr{B}(\lambda)$. By (2.7) and

$$F_{\mathscr{A}}(B_1) - \varphi_{\mathscr{A}}(B_2) = F_{\mathscr{A}}(B_1 \setminus B_1 \cap B_2) - F_{\mathscr{A}}(B_2 \setminus B_1 \cap B_2),$$

the function $F_{\mathscr{A}} : \mathscr{A} \to X$ is uniformly continuous. Consequently, $F_{\mathscr{B}}$ is the unique uniformly continuous extension of $F_{\mathscr{A}}$. Hence

$$\lim_{\lambda(B\Delta\emptyset)\to 0: B\in\mathscr{B}} ||F_{\mathscr{B}}(B)|| = 0.$$

Therefore, by Theorem I.2.1 in [1], it follows that $F_{\mathscr{B}}$ is λ-continuous.

Claim 2. The vector measure $F_{\mathscr{B}}$ is of bounded variation. Let $\{B_i : i = 1, 2, ..., m\}$ be a finite collection of pairwise disjoint members of $\mathscr{B}$ and let $\varepsilon > 0$ be given. Since $\mathscr{A}$ is dense in $\mathscr{B}(\lambda)$ and $F_{\mathscr{B}}$ is uniformly continuous, for each B_i there is an A_i such that for each $i = 1, 2, ..., m$, we have

$$||F_{\mathscr{B}}(B_i) - F_{\mathscr{A}}(A_i)|| < \frac{\varepsilon}{2 \cdot m^2}$$

and since $F_{\mathscr{A}}$ is of bounded variation there exists $M > 0$ such that

$$\sum_{i=1}^{m} ||F_{\mathscr{B}}(B_i)|| < \sum_{i=1}^{m} ||F_{\mathscr{A}}(A_i)|| + \frac{\varepsilon}{2} < M + \varepsilon$$

This means that $F_{\mathscr{B}}$ is of bounded variation.

It is known that

$$\mathcal{L} = \{B \cup Z' : B \in \mathscr{B} \text{ and } Z' \subset Z \text{ for some } Z \in \mathscr{Z}\},$$

where $\mathscr{Z} = \{Z \in \mathscr{B} : \lambda(Z) = 0\}$.

Claim 3. The vector measure $F_{\mathscr{B}}$ has a countable additive extension $F_{\mathcal{L}} : \mathcal{L} \to X$ that is of bounded variation and λ-continuous. Indeed, let us define $F_{\mathcal{L}} : \mathcal{L} \to X$ as follows

$$F_{\mathcal{L}}(B \cup Z') = F_{\mathscr{B}}(B) \quad \text{for all} \quad B \cup Z' \in \mathcal{L}.$$

This is well defined, since if $B_1 \cup Z_1' = B_2 \cup Z_2'$, then $\lambda(B_1 \setminus B_2) = \lambda(B_2 \setminus B_1) = 0$ and from this it follows that

$$\begin{aligned} F_{\mathscr{B}}(B_1) = F_{\mathscr{B}}(B_1 \setminus B_2) + F_{\mathscr{B}}(B_1 \cap B_2) = \\ F_{\mathscr{B}}(B_1 \cap B_2) = \\ F_{\mathscr{B}}(B_2 \setminus B_1) + F_{\mathscr{B}}(B_1 \cap B_2) = F_{\mathscr{B}}(B_2). \end{aligned}$$

Clearly, $F_{\mathcal{L}}$ is of bounded variation, λ-continuous and a countable additive vector measure such that $F_{\mathcal{L}}(B) = F_{\mathscr{B}}(B)$ for all $B \in \mathscr{B}$.

Claim 4. The vector measure $F_{\mathcal{L}}$ is unique. Suppose that there is another vector measure $G_{\mathcal{L}} : \mathcal{L} \to X$ that is of bounded variation, λ-continuous and a countable additive extension of $F_{\mathscr{B}}$ to $\mathcal{L}$. Let $B \cup Z'$ be an arbitrary element of $\mathcal{L}$. We can assume that $B \cap Z' = \emptyset$ (otherwise, replace Z' by $Z' \setminus B$). Since the vector measure $G_{\mathcal{L}}$ is λ-continuous and $\lambda(Z') = 0$, we obtain

$$G_{\mathcal{L}}(B \cup Z') = G_{\mathcal{L}}(B) + G_{\mathcal{L}}(Z') = G_{\mathcal{L}}(B) = F_{\mathscr{B}}(B) = F_{\mathcal{L}}(B \cup Z').$$

Hence, we infer that $F_{\mathcal{L}}$ is unique.

Claim 5. The vector measure $F_{\mathcal{L}}$ satisfies (2.5). Indeed, since $\mathcal{I} \subset \mathscr{B}$, we obtain by (2.6) that

$$F_{\mathscr{B}}(I) = \widetilde{F}(I) \quad \text{for all} \quad I \in \mathcal{I},$$

and since $F_{\mathcal{L}}$ is an extension of $F_{\mathscr{B}}$ to $\mathcal{L}$, it follows that $F_{\mathcal{L}}$ satisfies (2.5) and the proof is finished. $\square$

Theorem 8. *Let $F : [0,1] \to X$ be a function. Then the following statements are equivalent.*

(i) F is the primitive of a strongly McShane integrable function f, i.e.,

$$\widetilde{F}(I) = (M)\int_I f \quad \textit{for all} \quad I \in \mathcal{I}, \tag{2.8}$$

(ii) F is sAC, $F'(t)$ exists and $F'(t) = f(t)$ a.e. on $[0,1]$,

(iii) F is sAC and there exists a function $f : [0,1] \to X$ such that $f(t) \in A_F(t)$ a.e. on $[0,1]$.

Proof. $(i) \Leftrightarrow (ii)$ By Theorem 7.4.14 in [15], this equivalence holds true.

$(ii) \Rightarrow (iii)$ Assume that (ii) holds. Then, there is a subset $Z \subset [0,1]$ with $\lambda(Z) = 0$ such that $F'(t)$ exists and $F'(t) = f(t)$ for all $t \in [0,1] \setminus Z$. Hence, by Lemma 4, $f(t) \in A_F(t)$ for all $t \in [0,1] \setminus Z$.

$(iii) \Rightarrow (ii)$ Assume that F is sAC and $f(t) \in A_F(t)$ for all $t \in [0,1] \setminus Z$, where $Z \subset [0,1]$ with $\lambda(Z) = 0$. Since

$$\overline{x^*(A_F(t))} \subset A_{x^* \circ F}(t) \quad \text{for all} \quad x^* \in X^* \quad \text{and} \quad t \in [0,1] \setminus Z,$$

we obtain by Theorem 6 that f is Pettis integrable on $[0,1]$ and

$$\widetilde{F}(I) = (P)\int_I f(t)dt \quad \text{for all} \quad I \in \mathcal{I}.$$

Claim 1. The function f is strongly measurable. Since F is sAC the function F is continuous on $[0,1]$, and because this the set $\{F(t) : t \in [0,1]\} \subset X$ is compact and therefore separable. If $Y \subset X$ is the closed linear subspace spanned by the set $\{F(t) : t \in [0,1]\}$, then Y is separable. Since $\Delta F(t,h) \in Y$ for all $t \in [0,1]$ and $h \neq 0$, we obtain that $A_F(t) \subset Y$ for all $t \in [0,1] \setminus Z$. Thus, we have $f(t) \in Y$ for all $t \in [0,1] \setminus Z$. This means that f is almost everywhere separable valued, and since f is Pettis integrable on $[0,1]$, we obtain by the Pettis measurability theorem, Theorem II.1.2 in [1], that f is strongly measurable.

Claim 2. The function f is Bochner integrable on $[0,1]$. We set

$$\nu(E) = (P)\int_E f(t)dt \quad \text{for all} \quad E \in \mathcal{L}.$$

Since ν is a countable additive vector measure such that

$$\nu(I) = \widetilde{F}(I) \quad \text{for all} \quad I \in \mathcal{I},$$

and since F is sAC, we obtain by Lemma 7 that ν is of bounded variation. Hence, if (E_k) is a sequence of pairwise disjoint members of $\mathcal{L}$ such that $\cup_{k=1}^{+\infty} E_k = [0,1]$, then

$$\int_{\cup_{k=1}^{n} E_k} ||f(s)||dt \leq |\nu|(S) < +\infty \quad \text{for all} \quad n \in \mathbb{N}.$$

By the Monotone Convergence Theorem, the last result yields that the function $||f(.)||$ is Lebesgue integrable on $[0,1]$. Therefore, we obtain by Theorem II.2.2 in [1] that f is Bochner integrable. Since the Bochner and Pettis integrals coincide whenever they coexist, we have

$$\nu(E) = (B)\int_E f(t)d\lambda \quad \text{for every } E \in \mathcal{L},$$

and consequently

$$\widetilde{F}(I) = (B)\int_I f(t)d\lambda \quad \text{for every } I \in \mathcal{I}.$$

Hence, by Theorem 7.4.15 in [15], $F'(t)$ exists and $F'(t) = f(t)$ a.e. on $[0,1]$. □

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