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A FUNCTIONAL ANALYTIC PROOF OF THE LEBESGUE–DARST DECOMPOSITION THEOREM

Abstract

The aim of this paper is to give a functional analytic proof of the Lebesgue–Darst decomposition theorem [1]. We show that the decomposition of a nonnegative valued additive set function into absolutely continuous and singular parts with respect to another derives from the Riesz orthogonal decomposition theorem employed in a corresponding Hilbert space.

1 Introduction

Throughout this paper we fix a ring $\mathscr{R}$ over a nonempty set T, that is $\mathscr{R}$ is defined to be a nonempty family $\mathscr{R} \subseteq \mathscr{P}(T)$ which is closed under the operations of union, intersection, and difference. For a subset E of T we define the characteristic function χ_{E} by letting

$$\chi_{E}(t) = \begin{cases} 1, & \text{if } t \in E, \\ 0, & \text{else.} \end{cases}$$

The function lattice of the $\mathbb{R}$-valued $\mathscr{R}$-simple functions (i.e., the $\mathbb{R}$-linear span of the characteristic functions of $\mathscr{R}$-measurable sets) is denoted by $\mathscr{S}$. If a

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nonnegative valued additive set function ν on $\mathscr{R}$ is given, then we set

$$(\varphi\,|\,\psi)_\nu := \int_T \varphi\cdot\psi\; d\nu, \qquad (\varphi,\psi\in\mathscr{S}),$$

which defines a semi inner product on $\mathscr{S}$. By factorizing with the kernel of $(\cdot\,|\,\cdot)_\nu$, as usual, $\mathscr{S}$ becomes a (real) pre-Hilbert space. The corresponding equivalence class of a function $\varphi\in\mathscr{S}$ will be denoted also by the symbol φ. Let $\mathscr{L}^2(\nu)$ stand for completion of $\mathscr{S}$ with respect to the corresponding Hilbert norm $\|\cdot\|_\nu$, so that $\mathscr{L}^2(\nu)$ becomes a (real) Hilbert space in which $\mathscr{S}$ forms a dense linear manifold by definition. Note that $\mathscr{L}^2(\nu)$ in fact does not consist of proper $T\to\mathbb{R}$ functions. Nevertheless, each element of $\mathscr{L}^2(\nu)$ can be approximated with $\mathscr{R}$-simple functions, i.e., for each $h\in\mathscr{L}^2(\nu)$ there is a sequence $(\varphi_n)_{n\in\mathbb{N}}$ from $\mathscr{S}$ such that

$$\|h-\varphi_n\|_\nu^2 := (h-\varphi_n\,|\,h-\varphi_n)_\nu \to 0.$$

We notice here that Darst [1] treated only the case when $\mathscr{R}$ is an *algebra*, that is when $T\in\mathscr{R}$, or equivalently, when the function 1 belongs to $\mathscr{S}$. Nevertheless, if we assume ν to be *bounded*, that is

$$C(\nu) := \sup_{E\in\mathscr{R}} \nu(E) < \infty,$$

then the linear functional

$$\varphi\mapsto \int_T \varphi\; d\nu, \qquad (\varphi\in\mathscr{S})$$

turns out to be continuous with respect to the norm $\|\cdot\|_\nu$ (by norm bound $\sqrt{C(\nu)}$), so that the Riesz representation theorem yields a (unique) vector $\widehat{\nu}\in\mathscr{L}^2(\nu)$ such that

$$(\varphi\,|\,\widehat{\nu})_\nu = \int_T \varphi\; d\nu, \qquad (\varphi\in\mathscr{S}).$$

Of course, if $\mathscr{S}$ is an algebra, then $\widehat{\nu}=1$. But in the general case $\widehat{\nu}$ must not belong to $\mathscr{S}$.

Henceforth, we fix another bounded nonnegative additive set function μ on $\mathscr{R}$, and we define the objects $(\cdot\,|\,\cdot)_\mu$ and $\mathscr{L}^2(\mu)$ just as above. We say that ν is *absolutely continuous* with respect to μ if for any sequence $(E_n)_{n\in\mathbb{N}}$ from $\mathscr{R}$ $\mu(E_n)\to 0$ implies $\nu(E_n)\to 0$. On the other hand, ν is said to be *singular* with respect to μ if for any nonnegative additive set function ϑ inequalities $\vartheta \leq \mu$ and $\vartheta \leq \nu$ imply $\vartheta = 0$, see [2].

Let us consider first the following linear submanifold of $\mathscr{L}^2(\mu) \times \mathscr{L}^2(\nu)$:

$$J := \{(\varphi, \varphi) \mid \varphi \in \mathscr{S}\}, \tag{1}$$

that is the identical "mapping" from $\mathscr{S} \subseteq \mathscr{L}^2(\mu)$ into $\mathscr{L}^2(\nu)$. Note that the μ- and ν-equivalence classes of a function $\varphi \in \mathscr{S}$ can completely differ from each other, therefore, one concludes that J is only a so called "multivalued operator", i.e., a linear relation, unless ν is μ-absolutely continuous. In particular, the following linear manifold

$$\mathfrak{M} := \big\{ f \in \mathscr{L}^2(\nu) \;\big|\; (0, f) \in \overline{J} \big\},$$

the so called *multivalued part* of $\overline{J}$, can be nontrivial (see e.g. [3]). On the other hand, one easily verifies that $\mathfrak{M}$ is closed, and that

$$\mathfrak{M} = \big\{ f \in \mathscr{L}^2(\nu) \;\big|\; \exists (\varphi_n)_{n\in\mathbb{N}} \subset \mathscr{S} \text{ such that } \|\varphi_n\|_\mu \to 0, \|f - \varphi_n\|_\nu \to 0 \big\}.$$

Therefore we have the following orthogonal decomposition of the Hilbert space $\mathscr{L}^2(\nu)$ along $\mathfrak{M}$: $\mathscr{L}^2(\nu) = \mathfrak{M} \oplus \mathfrak{M}^\perp$, thanks to the classical Riesz orthogonal decomposition theorem. Let P stand for the orthogonal projection of $\mathscr{L}^2(\nu)$ onto $\mathfrak{M}$.

Our claim in the rest of the paper is to show that the following orthogonal decomposition

$$\widehat{\nu} = P\widehat{\nu} \oplus (I - P)\widehat{\nu}$$

of the functional $\widehat{\nu}$ corresponds to the Lebesgue–Darst decomposition of the additive set function ν. More precisely, by letting

$$\nu_s(E) := (\chi_{{}_E} \,|\, P\widehat{\nu})_\nu \qquad \text{and} \qquad \nu_a(E) := (\chi_{{}_E} \,|\, (I - P)\widehat{\nu})_\nu \tag{2}$$

for $E \in \mathscr{R}$, we obtain that $\nu = \nu_s + \nu_a$, where both ν_s and ν_a are nonnegative valued additive set functions such that ν_s is μ-singular, and that ν_a is μ-absolutely continuous.

We also notice that other functional analytic approaches treating the Lebesgue-Darst decomposition can be found in [6] and [7]. The treatment in these papers is based on the Lebesgue-type decomposition of nonnegative hermitian forms, cf. [4]. The approach contained herein does not make use of this general decomposition theorem, moreover, the only tools we employ are (more or less) elementary Hilbert space arguments.

2 Some auxiliary results

In this section we state three technical lemmas that are needed in the proof of our main theorem.

Lemma 1. *Let E be any set of $\mathscr{R}$, and let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence from $\mathscr{S}$ such that $\varphi_n \to P\chi_E$ in $\mathscr{L}^2(\nu)$ and that $\|\varphi_n\|_\mu \to 0$. Then we also have*

$$\chi_E \cdot \varphi_n \to P\chi_E \quad \text{in } \mathscr{L}^2(\nu).$$

Proof. First of all one concludes that $\|\chi_E \cdot \varphi_n - \chi_E \cdot \varphi_m\|_\nu \to 0$ and that $\|\chi_E \cdot \varphi_n\|_\mu \to 0$. Therefore the sequence $(\chi_E \cdot \varphi_n)_{n\in\mathbb{N}}$ converges in the Hilbert space $\mathscr{L}^2(\nu)$ such that the corresponding limit f belongs to $\mathfrak{M}$. In order to prove equality $P\chi_E = f$, fix a function $\psi \in \mathscr{S}$ and choose a sequence $(\psi_n)_{n\in\mathbb{N}}$ from $\mathscr{S}$ such that $\psi_n \to P\psi$ and that $\|\psi_n\|_\mu \to 0$. We can conclude just as above that $\lim\limits_{n\to\infty} \chi_E \cdot \psi_n \in \mathfrak{M}$. Therefore,

$$\begin{aligned}(P\chi_E \,|\, \psi)_\nu &= (\chi_E \,|\, P\psi)_\nu = \lim_{n\to\infty} (\chi_E \,|\, \psi_n)_\nu = \lim_{n\to\infty} (\chi_E \,|\, \chi_E \cdot \psi_n)_\nu \\ &= \lim_{n\to\infty} (P\chi_E \,|\, \chi_E \cdot \psi_n)_\nu = \lim_{n\to\infty} (\varphi_n \,|\, \chi_E \cdot \psi_n)_\nu \\ &= \lim_{n\to\infty} (\chi_E \cdot \varphi_n \,|\, \psi_n)_\nu = (f \,|\, P\psi)_\nu = (f \,|\, \psi)_\nu,\end{aligned}$$

that means that $P\chi_E - f$ is orthogonal to the dense manifold $\mathscr{S}$ of $\mathscr{L}^2(\nu)$. Consequently, $P\chi_E = f$, as it is claimed. □

Lemma 2. *Let $E, F \in \mathscr{R}$. Then following three assertions hold:*

a) *If $E \cap F = \emptyset$ then $P\chi_E \perp P\chi_F$, and likewise $(I-P)\chi_E \perp (I-P)\chi_F$ in $\mathscr{L}^2(\nu)$.*

b) *$\nu_s(E) = \|P\chi_E\|_\nu^2$ and $\nu_a(E) = \|(I-P)\chi_E\|_\nu^2$.*

c) *The functionals $P\chi_E$ and $(I-P)\chi_E$ are positive in the sense that*

$$(\varphi \,|\, P\chi_E)_\nu \geq 0 \qquad \text{and} \qquad (\varphi \,|\, (I-P)\chi_E)_\nu \geq 0$$

for all $\varphi \in \mathscr{S}, \varphi \geq 0$.

Proof. Statement a) is an easy consequence of Lemma 1. To prove b), let $E \in \mathscr{R}$ and choose a sequence $(\varphi_n)_{n\in\mathbb{N}}$ from $\mathscr{S}$ such that $\varphi_n \to P\chi_E$ and that $\|\varphi_n\|_\mu^2 \to 0$. Then, due to Lemma 1 we conclude that

$$\begin{aligned}\nu_s(E) &= (P\chi_E \,|\, \widehat{\nu})_\nu = \lim_{n\to\infty} (\chi_E \cdot \varphi_n \,|\, \widehat{\nu})_\nu = \lim_{n\to\infty} \int_T \chi_E \cdot \varphi_n \, d\nu \\ &= \lim_{n\to\infty} (\varphi_n \,|\, \chi_E)_\nu = (P\chi_E \,|\, \chi_E)_\nu = \|P\chi_E\|_\nu^2.\end{aligned}$$

The second identity of b) follows from the Parseval formula:

$$\nu_a(E) = \nu(E) - \nu_s(E) = \|\chi_E\|_\nu^2 - \|P\chi_E\|_\nu^2 = \|(I-P)\chi_E\|_\nu^2.$$

Finally, if $\varphi \in \mathscr{S}$ is nonnegative, then there is are two finite systems $(c_\alpha)_{\alpha\in A}$ of nonnegative numbers, and $(E_\alpha)_{\alpha\in A}$ of some sets from $\mathscr{R}$ such that $\varphi = \sum\limits_{\alpha\in A} c_\alpha \chi_{E_\alpha}$. Then, according to statement a),

$$(\varphi \,|\, P\chi_E)_\nu = \sum_{\alpha\in A} c_\alpha (\chi_{E_\alpha} \,|\, P\chi_E)_\nu = \sum_{\alpha\in A} c_\alpha (P\chi_{E\cap E_\alpha} \,|\, P\chi_{E\cap E_\alpha})_\nu \geq 0.$$

The second identity of c) is proved analogously. □

The last result of this section states that each functional of $\mathscr{L}^2(\nu)$ which is positive in the sense of Lemma 2 can be approximated by nonnegative $\mathscr{R}$-simple functions (with respect to the norm of $\mathscr{L}^2(\nu)$, of course):

Lemma 3. *Assume that $f \in \mathscr{L}^2(\nu)$ is positive in the sense that $(\varphi \,|\, f)_\nu \geq 0$ for all $\varphi \in \mathscr{S}$ with $\varphi \geq 0$. Then there is a sequence $(\psi_n)_{n\in\mathbb{N}}$ of nonnegative $\mathscr{R}$-simple functions such that $\psi_n \to f$ in $\mathscr{L}^2(\nu)$.*

Proof. Let $(\varphi_n)_{n\in\mathbb{N}}$ be any sequence from $\mathscr{S}$ that converges to f in $\mathscr{L}^2(\nu)$. For fixed integer n, let φ_n^+ (resp., φ_n^-) denote the positive (resp., the negative) part of φ_n. Clearly, that both φ_n^+ and φ_n^- are nonnegative $\mathscr{R}$-simple functions, and that the sequences $(\varphi_n^+)_{n\in\mathbb{N}}$, $(\varphi_n^-)_{n\in\mathbb{N}}$ also converge in $\mathscr{L}^2(\nu)$. Let f^+ and f^- stand for the corresponding limit vectors. Since $\varphi_n = \varphi_n^+ - \varphi_n^-$ for all integer n, it suffices to show that $f^- = 0$. Indeed, since $\varphi_n^- \geq 0$, we have that $(\varphi_n^- \,|\, f)_\nu \geq 0$. Consequently,

$$0 \leq (f^- \,|\, f)_\nu = \lim_{n\to\infty} (\varphi_n^- \,|\, \varphi_n)_\nu = \lim_{n\to\infty} (\varphi_n^- \,|\, -\varphi_n^-)_\nu = -\|f^-\|_\nu^2 \leq 0,$$

which means just that $f^- = 0$, i.e., $\lim\limits_{n\to\infty} \varphi_n^+ = f$. □

3 The Lebesgue–Darst decomposition theorem

We are now in position to prove the main result of the paper, the Lebesgue–Darst decomposition theorem.

Theorem 4. *Assume that μ and ν are nonnegative valued bounded additive set functions on the ring $\mathscr{R}$. Then*

$$\nu = \nu_s + \nu_a$$

is according to the Lebesgue-Darst decomposition, that is ν_s and ν are both nonnegative valued additive set functions such that ν_s is μ-singular, and ν_a is μ-absolutely continuous.

PROOF. The nonnegativity of the set functions in question is clear from Lemma 2 b). We prove first the absolute continuity of ν_a: consider a sequence $(E_n)_{n\in\mathbb{N}}$ from $\mathscr{R}$ such that $\mu(E_n)\to 0$. We need to show that $\nu_a(E_n)\to 0$ as well. According to the boundedness of ν, the sequence $(\nu_a(E_n))_{n\in\mathbb{N}}$ is also bounded. Assume indirectly that there is a subsequence $(E_{n_k})_{k\in\mathbb{N}}$ such that

$$\nu_a(E_{n_k})\to\alpha\neq 0.$$

According to the boundedness of $(\chi_{E_n})_{n\in\mathbb{N}}$ in $\mathscr{L}^2(\nu)$, we may also assume that $(\chi_{E_{n_k}})_{k\in\mathbb{N}}$ converges weakly in $\mathscr{L}^2(\nu)$, namely to a vector $\chi\in\mathscr{L}^2(\nu)$. Hence the pair $(0,\chi)$ belongs to the weak closure of the linear relation J defined in (1). Since weak and norm closures of a linear manifold in a normed space are the same, we obtain that $(0,\chi)\in\overline{J}$ as well, and therefore that $\chi\in\mathfrak{M}$. Consequently,

$$\alpha=\lim_{k\to\infty}\nu_a(E_{n_k})=\lim_{k\to\infty}(\chi_{E_{n_k}}\,|\,(I-P)\widehat{\nu})_\nu=(\chi\,|\,(I-P)\widehat{\nu})_\nu=0,$$

which is a contradiction.

In order to prove the μ-singularity of ν_s fix a nonnegative valued additive set function ϑ on $\mathscr{R}$ such that $\vartheta\le\mu$ and $\vartheta\le\nu$. We need to show that $\vartheta=0$. First of all observe that

$$\varphi\mapsto\int_T\varphi\,d\vartheta,\qquad(\varphi\in\mathscr{S}),\tag{3}$$

defines a continuous linear functional on the dense linear manifold $\mathscr{S}$ of $\mathscr{L}^2(\nu)$. Therefore, thanks to the Riesz representation theorem, there is a (unique) vector $\widehat{\vartheta}$ in $\mathscr{L}^2(\nu)$ such that

$$(\varphi\,|\,\widehat{\vartheta})_\nu=\int_T\varphi\,d\vartheta,\qquad(\varphi\in\mathscr{S}).$$

We show first that $\widehat{\vartheta}\in\mathfrak{M}$. Let $E\in\mathscr{R}$, and choose a sequence $(\psi_n)_{n\in\mathbb{N}}$ of nonnegative $\mathscr{R}$-simple functions tending to $(I-P)\chi_E$ in $\mathscr{L}^2(\nu)$. The existence of such a sequence is due to Lemma 2 c) and Lemma 3. Since $0\le\vartheta\le\nu_s$ by assumption, we obtain that

$$\begin{aligned}0\le(\chi_E\,|\,(I-P)\widehat{\vartheta})_\nu&=((I-P)\chi_E\,|\,\widehat{\vartheta})_\nu=\lim_{n\to\infty}(\psi_n\,|\,\widehat{\vartheta})_\nu\\&\le\lim_{n\to\infty}(\psi_n\,|\,P\widehat{\nu})_\nu=((I-P)\chi_E\,|\,P\widehat{\nu})_\nu=0,\end{aligned}$$

which means that $(I-P)\widehat{\vartheta} \in \{\chi_{_E} \mid E \in \mathscr{R}\}^{\perp} = \{0\}$, i.e., $\widehat{\vartheta} \in \mathfrak{M}$. On the other hand, since $\vartheta \leq \mu$, the functional in (3) is continuous also with respect to the norm $\|\cdot\|_\mu$. Therefore, according again to the Riesz representation theorem, there is a (unique) vector $\Theta \in \mathscr{L}^2(\mu)$ such that

$$(\varphi \,|\, \Theta)_\mu = \int\limits_T \varphi \; d\vartheta = (\varphi \,|\, \widehat{\vartheta})_\nu, \qquad (\varphi \in \mathscr{S}).$$

Finally, by considering a sequence $(\varphi_n)_{n\in\mathbb{N}}$ from $\mathscr{S}$ such that $\varphi_n \to \widehat{\vartheta}$ and that $\|\varphi_n\|_\mu \to 0$, it follows that

$$\|\widehat{\vartheta}\|_\nu^2 = \lim_{n\to\infty} (\varphi_n \,|\, \widehat{\vartheta})_\nu = \lim_{n\to\infty} (\varphi_n \,|\, \Theta)_\mu = 0,$$

which completes the proof. □

References

[1] R. B. Darst, *A decomposition of finitely additive set functions*, J. for Angew. Math., **210** (1962), 31–37.

[2] N. Dunford and J. Schwarz, *Linear operators*, Interscience, New York, 1958.

[3] S. Hassi, Z. Sebestyén, H. S. V. de Snoo, and F. H. Szafraniec, *A canonical decomposition for linear operators and linear relations*, Acta Math. Hungar., **115** (2007), 281–307.

[4] S. Hassi, Z. Sebestyén, and H. de Snoo, *Lebesgue type decompositions for nonnegative forms*, J. Funct. Anal., **257** (2009), 3858–3894.

[5] F. Riesz and B. Sz.-Nagy, *Leçons d'analyse fonctionnelle*, Académie des Sciences de Hongrie, Akadémiai Kiadó, Budapest, 1952.

[6] Z. Sebestyén, Zs. Tarcsay and T. Titkos, *Lebesgue decomposition theorems*, Acta Sci. Math. (Szeged), **79** (2013), 219–233.

[7] T. Titkos, *Lebesgue decomposition of contents via nonnegative forms*, Acta Math. Hungar., **140** (2013), 151–161.