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FRACTIONAL POWERS OF OPERATORS, II INTERPOLATION SPACES

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This is a continuation of an earlier paper "Fractional Powers of Operators" published in this Journal concerning fractional powers A^α, $\alpha \in C$, of closed linear operators A in Banach spaces X such that the resolvent $(\lambda + A)^{-1}$ exists for all $\lambda > 0$ and $\lambda(\lambda + A)^{-1}$ is uniformly bounded. Various integral representations of fractional powers and relationship between fractional powers and interpolation spaces, due to Lions and others, of X and domain $D(A^\alpha)$ are investigated.

In § 1 we define the space $D_p^\sigma(A)$, $0 < \sigma < \infty$, $1 \leqq p \leqq \infty$ or $p = \infty -$, as the set of all $x \in X$ such that

$$\lambda^\sigma(A(\lambda + A)^{-1})^m x \in L^p(X) ,$$

where m is an integer greater than σ and $L^p(X)$ is the L^p space of X-valued functions with respect to the measure $d\lambda/\lambda$ over $(0, \infty)$.

In § 2 we give a new definition of fractional power A^α for Re $\alpha > 0$ and prove the coincidence with the definition given in [2]. Convexity of $\| A^\alpha x \|$ is shown to be an immediate consequence of the definition. The main result of the section is Theorem 2.6 which says that if $0 < \mathrm{Re}\, \alpha < \sigma$, $x \in D_p^\sigma$ is equivalent to $A^\alpha x \in D_p^{\sigma - \mathrm{Re}\, \alpha}$. In particular, we have $D_1^{\mathrm{Re}\,\alpha} \subset D(A^\alpha) \subset D_{\infty -}^{\mathrm{Re}\,\alpha}$. For the application of fractional powers it is important to know whether the domain $D(A^\alpha)$ coincides with $D_p^{\mathrm{Re}\,\alpha}$ for some p. We see, as a consequence of Theorem 2.6, that if we have $D(A^\alpha) = D_p^{\mathrm{Re}\,\alpha}$ for an α, it holds for all Re $\alpha > 0$. An example and a counterexample are given. At the end of the section we prove an integral representation of fractional powers.

Section 3 is devoted to the proof of the coincidence of D_p^σ with the interpolation space $S(p, \sigma/m, X; p, \sigma/m - 1, D(A^m))$ due to Lions-Peetre [4]. We also give a direct proof of the fact that $D_p^\sigma(A^\alpha) = D_p^{\alpha\sigma}(A)$.

In § 4 we discuss the case in which $-A$ is the infinitesimal generator of a bounded strongly continuous semi-group T_t. A new space $C_{p,m}^\sigma$ is introduced in terms of $T_t x$ and its coincidence with D_p^σ is shown. Since $C_{\infty,m}^\sigma$, $\sigma \neq$ integer, coincides with C^σ of [2], this solves a question of [2] whether $C^\sigma = D^\sigma$ or not affirmatively. The coincidence of $C_{p,m}^\sigma$ with $S(p, \sigma/m, X; p, \sigma/m - 1, D(A^m))$ has been shown by Lions-Peetre [4]. Further, another integral representation of fractional powers is obtained.

Finally, § 5 deals with the case in which $-A$ is the infinitesimal generator of a bounded analytic semi-group T_t. Analogous results to § 4 are obtained in terms of $A^\beta T_t x$.

1. Spaces D_p^σ. Throughout this paper we assume that A is a closed linear operator with a dense domain $D(A)$ in a Banach space X and satisfies

$$\|\lambda(\lambda + A)^{-1}\| \leqq M, \qquad 0 < \lambda < \infty . \tag{1.1}$$

We defined fractional powers in [2] for operators A which may not have dense domains. It was shown, however, that if Re $\alpha > 0$, A^α is an operator in $\overline{D(A)}$ and it is determined by a restriction A_D which has a dense domain in $\overline{D(A)}$. Thus our requirement on domain $D(A)$ is not restrictive as far as we consider exponent α with positive real part. As a consequence we have

$$(\lambda(\lambda + A)^{-1})^m x \longrightarrow x, \qquad \lambda \longrightarrow \infty,\ m = 1, 2, \cdots \tag{1.2}$$

for all $x \in X$. As in [2] L stands for a bound of $A(\lambda + A)^{-1} = I - \lambda(\lambda + A)^{-1}$:

$$\|A(\lambda + A)^{-1}\| \leqq L, \qquad 0 < \lambda < \infty . \tag{1.3}$$

We will frequently make use of spaces of X-valued functions $f(\lambda)$ defined on $(0, \infty)$. By $L^p(X)$ we denote the space of all X-valued measurable functions $f(\lambda)$ such that

$$\begin{aligned} \|f\|_{L^p} &= \left(\int_0^\infty \|f(\lambda)\|^p d\lambda/\lambda\right)^{1/p} < \infty \ \text{ if } 1 \leqq p < \infty \\ \|f\|_{L^\infty} &= \sup_{0<\lambda<\infty} \|f(\lambda)\| < \infty \ \text{ if } p = \infty . \end{aligned} \tag{1.4}$$

We admit as an index $p = \infty -$. $L^{\infty-}(X)$ represents the subspace of all functions $f(\lambda) \in L^\infty(X)$ which converge to zero as $\lambda \longrightarrow 0$ and as $\lambda \longrightarrow \infty$. Since $d\lambda/\lambda$ is a Haar measure of the multiplicative group $(0, \infty)$, an integral kernel $K(\lambda/\mu)$ with $\int_0^\infty |K(\lambda)|d\lambda/\lambda < \infty$ defines a bounded integral operator in $L^p(X)$, $1 \leqq p \leqq \infty$.

Definition 1.1. Let $0 < \sigma < m$, where σ is a real number and m an integer, and p be as above. We denote by $D_{p\,m}^\sigma = D_{p,m}^\sigma(A)$ the space of all $x \in X$ such that $\lambda^\sigma(A(\lambda + A)^{-1})^m x \in L^p(X)$ with the norm

$$\|x\|_{D_{p,m}^\sigma} = \|x\|_X + \|\lambda^\sigma(A(\lambda + A)^{-1})^m x\|_{L^p(X)} . \tag{1.5}$$

$D_{\infty,1}^\sigma$ and $D_{\infty-,1}^\sigma$ coincide with D^σ and D_*^σ of [2], respectively.

It is easy to see that $D_{p,m}^\sigma$ is a Banach space. Since $(A(\lambda + A)^{-1})^m$ is uniformly bounded, only the behavior near infinity of $(A(\lambda + A)^{-1})^m x$

decides whether x belongs to $D^{\sigma}_{p,m}$ or not.

PROPOSITION 1.2. If integers m and n are greater than σ, the spaces $D^{\sigma}_{p,m}$ and $D^{\sigma}_{p,n}$ are identical and have equivalent norms.

Proof. It is enough to show that $D^{\sigma}_{p,m} = D^{\sigma}_{p,m+1}$ when $m > \sigma$. Because of (1.3) every $x \in D^{\sigma}_{p,m}$ belongs to $D^{\sigma}_{p,m+1}$. Since

$$\frac{d}{d\lambda}(\lambda^m(A(\lambda + A)^{-1})^m) = m\lambda^{m-1}(A(\lambda + A)^{-1})^{m+1},$$

we have

$$\lambda^{\sigma}(A(\lambda + A)^{-1})^m x = m\lambda^{\sigma-m}\int_0^{\lambda} \mu^{m-\sigma}\mu^{\sigma}(A(\mu + A)^{-1})^{m+1} x d\mu/\mu . \tag{1.6}$$

This shows

$$\| \lambda^{\sigma}(A(\lambda + A)^{-1})^m x \|_{L^p(X)} \leqq \frac{m}{m - \sigma}\| \lambda^{\sigma}(A(\lambda + A)^{-1})^{m+1} x \|_{L^p(X)} .$$

DEFINITION 1.3. We define D^{σ}_p, $\sigma > 0$, $1 \leqq p \leqq \infty$, as the space $D^{\sigma}_{p,m}$ with the least integer m greater than σ. We use $q^{\sigma}_p(x)$ to denote the second term of (1.5), so that D^{σ}_p is a Banach space with the norm $\| x \| + q^{\sigma}_p(x)$.

PROPOSITION 1.4. If $\mu > 0$, $\mu(\mu + A)^{-1}$ maps D^{σ}_p continuously into $D^{\sigma+1}_p$. Futhermore, if $p \leqq \infty -$, we have for every $x \in D^{\sigma}_p$

$$\mu(\mu + A)^{-1}x \to x \ (D^{\sigma}_p) \quad \text{as} \quad \mu \to \infty . \tag{1.7}$$

Proof. Let $x \in D^{\sigma}_p$. Since

$$\begin{aligned} &\| \lambda^{\sigma+1}(A(\lambda + A)^{-1})^{m+1}\mu(\mu + A)^{-1}x \| \\ &\quad \leqq \mu \| \lambda(\lambda + A)^{-1} \| \, \| A(\mu + A)^{-1} \| \, \| \lambda^{\sigma}(A(\lambda + A)^{-1})^m x \| \\ &\quad \leqq \mu ML \| \lambda^{\sigma}(A(\lambda + A)^{-1})^m x \| , \end{aligned}$$

$\mu(\mu + A)^{-1}x$ belongs to $D^{\sigma+1}_p$.

Let $p \leqq \infty -$. If $x \in D(A)$, then

$$\begin{aligned} &(A(\lambda + A)^{-1})^m \mu(\mu + A)^{-1}x \\ &\qquad = (A(\lambda + A)^{-1})^m x - (A(\lambda + A)^{-1})^m(\mu + A)^{-1}Ax \end{aligned}$$

converges to $(A(\lambda + A)^{-1})^m x$ uniformly in λ. On the other hand, $(A(\lambda + A)^{-1})^m \mu(\mu + A)^{-1}$ is uniformly bounded. Thus it follows that $(A(\lambda + A)^{-1})^m \mu(\mu + A)^{-1}x$ converges to $(A(\lambda + A)^{-1})^m x$ uniformly in λ for every $x \in X$. Since $\| \lambda^{\sigma}(A(\lambda+A)^{-1})^m \mu(\mu+A)^{-1}x \| \leqq M \| \lambda^{\sigma}(A(\lambda+A)^{-1})^m x \|$, this implies (1.7).

THEOREM 1.5. *$D_p^\sigma \subset D_q^\tau$ if $\sigma > \tau$ or if $\sigma = \tau$ and $p \leqq q$. The injection is continuous. If $q \leqq \infty-$, D_p^σ is dense in D_q^τ.*

Proof. First we prove that D_p^σ, $p < \infty$, is continuously contained in $D_{\infty-}^\sigma$.

Let $x \in D_p^\sigma$. Applying Hölder's inequality to (1.6), we obtain

$$\| \lambda^\sigma(A(\lambda + A)^{-1})^m x \| \leqq \frac{m}{((m-\sigma)p')^{1/p'}} \| \mu^\sigma(A(\mu + A)^{-1})^{m+1} x \|_{L^p(X)} ,$$

where $p' = p/(p-1)$. Hence $x \in D_\infty^\sigma$. Considering the integral over the interval (μ, λ), we have similarly

$$\| \lambda^\sigma(A(\lambda + A)^{-1})^m x \| \leqq \frac{\mu^{m-\sigma}}{\lambda^{m-\sigma}} \| \mu^\sigma(A(\mu + A)^{-1})^m x \|$$

$$+ \frac{m}{((m-\sigma)p')^{1/p'}} \left(1 - \frac{\mu^{m-\sigma}}{\lambda^{m-\sigma}}\right)\left(\int_\mu^\lambda \| \tau^\sigma(A(\tau + A)^{-1})^{m+1} x \|^p d\tau/\tau)^{1/p}\right).$$

The second term tends to zero as $\mu \to \infty$ uniformly in $\lambda > \mu$ and so does the first term as $\lambda \to \infty$. Therefore, $x \in D_{\infty-}^\sigma$.

Since $\lambda^\sigma(A(\lambda + A)^{-1})^m x \in L^p(X) \cap L^{\infty-}(X)$, it is in any $L^q(X)$ with $p \leqq q < \infty$.

If $\tau < \sigma$, D_∞^σ is contained in D_q^τ for any q. Hence every D_q^σ is contained in D_q^τ.

Let $q \leqq \infty-$. Repeated application of Proposition 1.4 shows that $D_q^{\tau+m}$ is dense in D_q^τ for positive integer m. Since D_p^σ contains some $D_q^{\tau+m}$, it is dense in D_q^τ.

2. Fractional powers.

If $x \in D_1^\sigma$, the integral

$$A_\sigma^\alpha x = \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^\infty \lambda^{\alpha-1}(A(\lambda + A)^{-1})^m x d\lambda \tag{2.1}$$

converges absolutely for $0 < \operatorname{Re}\alpha \leqq \sigma$ and represents a continuous operator from D_1^σ into X. Moreover, $A_\alpha^\sigma x$ is analytic in α for $0 < \operatorname{Re}\alpha < \sigma$.

$A_\sigma^\alpha x$ does not depend on m. In fact, substitution of (1.6) into (2.1) gives

$$A_\sigma^\alpha x = \frac{\Gamma(m)m}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^\infty \mu^{m-1}(A(\mu + A)^{-1})^{m+1} x d\mu \int_\mu^\infty \lambda^{\alpha-m-1} d\lambda$$

$$= \frac{\Gamma(m+1)}{\Gamma(\alpha)\Gamma(m+1-\alpha)} \int_0^\infty \mu^{\alpha-1}(A(\mu + A)^{-1})^{m+1} x d\mu .$$

This shows that $A_\sigma^\alpha x$ depends only on x and not on D_1^σ to which x belongs.

Obviously we have

$$A_\sigma^\alpha(\mu(\mu + A)^{-1})^{m+1}x = (\mu(\mu + A)^{-1})^{m+1}A_\sigma^\alpha x,\ x \in D_1^\alpha\ . \tag{2.2}$$

Since the left-hand side and $(\mu(\mu + A)^{-1})^{m+1}$ are continuous in X, and $(\mu(\mu + A)^{-1})^{m+1}$ is one-to-one, it follows that A_σ^α is closable in X. In view of Theorem 1.5 the smallest closed extension does not depend on σ.

DEFINITION 2.1. The fractional power A^α for $\operatorname{Re}\alpha > 0$ is the smallest closed extension of A_σ^α for a $\sigma \geqq \operatorname{Re}\alpha$.

PROPOSITION 2.2. If α is an integer $m > 0$, A^α coincides with the power A^m.

To prove the proposition we prepare a lemma.

LEMMA 2.3. *If m is an integer $m > 0$,*

$$A^m x = \operatorname*{s-lim}_{N\to\infty} m\int_0^N \lambda^{m-1}(A(\lambda + A)^{-1})^{m+1}x d\lambda\ . \tag{2.3}$$

Proof. By (1.6) we have

$$m\int_0^N \lambda^{m-1}(A(\lambda + A)^{-1})^{m+1}x = N^m(A(N + A)^{-1})^m x\ .$$

If $x \in D(A^m)$, $N^m(A(N + A)^{-1})^m x = (N(N + A)^{-1})^m A^m x$ tends to $A^m x$ as $N \to \infty$ by (1.2). Conversely if $N^m(A(N + A)^{-1})^m x = A^m(N(N + A)^{-1})^m x$ converges to an element y, $x \in D(A^m)$ and $y = A^m x$. For A^m is closed (see Taylor [5]) and $(N(N + A)^{-1})^m x$ converges to x.

Proof of Proposition 2.2. If $x \in D_1^\sigma$, $\sigma > m$, integral (2.3) converges absolutely. Therefore it follows from Lemma 2.3 that $x \in D(A^m)$ and $A^\alpha x = A^m x$. Thus A^m is an extension of A^α. Conversely if $x \in D(A^m)$, then $\mu(\mu + A)^{-1}x \in D(A^{m+1}) \subset D_\infty^{m+1}$ and we have

$$\begin{aligned} A^\alpha(\mu(\mu + A)^{-1})x &= (\mu(\mu + A)^{-1})A^m x \\ &\to A^m x \quad \text{as} \quad \mu \to \infty\ . \end{aligned}$$

Since $\mu(\mu + A)^{-1}x \to x$, it follows that $x \in D(A^\alpha)$ and $A^\alpha x = A^m x$.

The fractional power A^α defined above coincides with A_+^α defined in [2]. In fact, if $m = 1$, integeral (2.1) is the same as integral (4.2) of [2] for $n = 0$. Thus

$$A^\alpha x = A_+^\alpha x \tag{2.4}$$

holds for $0 < \operatorname{Re} < 1$ if $x \in D(A)$. If $x \in D(A^m)$, $m \geqq 1$, both sides of

(2.4) are analytic for $0 < \operatorname{Re}\alpha < m$, so that (2.4) holds there. Since $D_1^m \subset D(A^m) \subset D_{\infty -}^m$ by Lemma 2.3 and (1.2), both A^α and A_+^α are the smallest closed extension of their restrictions to $D(A^m)$, $m > \operatorname{Re}\alpha$. Thus we have $A^\alpha = A_+^\alpha$ for all $\operatorname{Re}\alpha > 0$.

Consequently we may employ all results of [2]. In particular, fractional powers satisfy additivity

$$A^{\alpha+\beta} = A^\alpha A^\beta\,, \qquad \operatorname{Re}\alpha > 0, \operatorname{Re}\beta > 0 \tag{2.5}$$

in the sense of product of operators and multiplicativity

$$(A^\alpha)^\beta = A^{\alpha\beta}\,, \qquad 0 < \alpha < \pi/\omega, \operatorname{Re}\ \beta > 0\,, \tag{2.6}$$

where ω is the minimum number such that the resolvent set of $-A$ contains the sector

$$|\arg\lambda| < \pi - \omega\,.$$

Such an operator is said to be of type $(\omega, M(\theta))$ if

$$\sup_{|\arg\lambda|=\theta} \|\lambda(\lambda + A)^{-1}\| \leqq M(\theta)\,.$$

Any operator with a dense domain which satisfies (1.1) is of type $(\omega, M(\theta))$ with $0 \leqq \omega < \pi$.

Some properties of fractional powers, however, are derived more easily through definition (2.1).

PROPOSITION 2.4. If $0 < \operatorname{Re}\alpha < \sigma$, there is a constant $C(\alpha, \sigma, p)$ such that

$$\|A^\alpha x\| \leqq C(\alpha, \sigma, p) q_p^\sigma(x)^{\operatorname{Re}\alpha/\sigma} \|x\|^{(\sigma-\operatorname{Re}\alpha)/\sigma}, \ x \in D_p^\sigma\,. \tag{2.7}$$

Proof. Hölder's inequality gives

$$\begin{aligned}\|A^\alpha x\| &\leqq \left|\frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)}\right| \Big[\int_0^N |\lambda^{\alpha-1}|\, \|(A(\lambda + A)^{-1})^m x\|\, d\lambda \\ &\qquad + \int_N^\infty |\lambda^{\alpha-\sigma}|\, \|\lambda^\sigma (A(\lambda + A)^{-1})^m x\|\, d\lambda/\lambda\Big] \\ &\leqq \left|\frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)}\right| \left[\frac{L^m N^{\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\|x\| + \frac{N^{\operatorname{Re}\alpha-\sigma}}{((\sigma - \operatorname{Re}\alpha)p')^{1/p'}} q_p^\sigma(x)\right].\end{aligned}$$

Taking the minimum of the right-hand side when N varies $0 < N < \infty$, we obtain (2.7).

PROPOSITION 2.5. If $\mu > 0$, then

$$D_p^\sigma(A) = D_p^\sigma(\mu + A) \tag{2.8}$$

with equivalent norms.

Proof. Let $x \in D^{\sigma}_{p,m}(A)$ with $m > \sigma$. Since

$$\begin{aligned} &\| A^k(\lambda + \mu + A)^{-m}x \| \leqq C \| A^m(\lambda + \mu + A)^{-m}x \|^{k/m} \cdot \\ &\qquad \| (\lambda + \mu + A)^{-m}x \|^{(m-k)/m}, \qquad k = 1, 2, \cdots, m-1, \\ &\lambda^{\sigma}((\mu + A)(\lambda + \mu + A)^{-1})^m x \\ &\qquad = \lambda^{\sigma}(\mu^m + m\mu^{m-1}A + \cdots + A^m)(\lambda + \mu + A)^{-m}x \end{aligned}$$

belongs to $L^p(X)$. The converse is proved in the same way.

THEOREM 2.6. *Let* $0 < \operatorname{Re}\alpha < \sigma$. *Then* $x \in D^{\sigma}_p$ *if and only if* $x \in D(A^{\alpha})$ *and* $A^{\alpha}x \in D^{\sigma-\operatorname{Re}\alpha}_p$.

Proof. Let $x \in D^{\sigma}_p$ and $m > \sigma$. Clearly $x \in D(A^{\alpha})$. To estimate the integral

$$\begin{aligned} &\lambda^{\sigma-\operatorname{Re}\alpha}(A(\lambda + A)^{-1})^m A^{\alpha}x \\ &\qquad = \frac{\Gamma(m)\lambda^{\sigma-\operatorname{Re}\alpha}}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^{\infty} \mu^{\alpha-1}(A(\lambda + A)^{-1})^m (A(\mu + A)^{-1})^m x \, d\mu, \end{aligned}$$

we split it into two parts. First,

$$\begin{aligned} &\left\| \lambda^{\sigma-\operatorname{Re}\alpha} \int_0^{\lambda} \mu^{\alpha-1}(A(\lambda + A)^{-1})^m (A(\mu + A)^{-1})^m x \, d\mu \right\| \\ &\qquad \leqq \lambda^{\sigma-\operatorname{Re}\alpha} \int_0^{\lambda} \mu^{\operatorname{Re}\alpha - 1} d\mu L^m \| (A(\lambda + A)^{-1})^m x \| \\ &\qquad = L^m (\operatorname{Re}\alpha)^{-1} \lambda^{\sigma} \| (A(\lambda + A)^{-1})^m x \| \in L^p. \\ &\left\| \lambda^{\sigma-\operatorname{Re}\alpha} \int_{\lambda}^{\infty} \mu^{\alpha-1}(A(\lambda + A)^{-1})^m (A(\mu + A)^{-1})^m x \, d\mu \right\| \\ &\qquad \leqq L^m \lambda^{\sigma-\operatorname{Re}\alpha} \int_{\lambda}^{\infty} \mu^{\operatorname{Re}\alpha-\sigma} \| \mu^{\sigma}(A(\mu + A)^{-1})^m x \| \, d\mu/\mu \end{aligned}$$

also belongs to L^p because $\operatorname{Re}\alpha - \sigma < 0$.

Conversely, let $A^{\alpha}x \in D^{\sigma-\operatorname{Re}\alpha}_p$. If n is an integer greater than $\operatorname{Re}\alpha$, we have

$$\begin{aligned} \| A^{n-\alpha}(\lambda + A)^{-n} \| &\leqq C \| A^n(\lambda + A)^{-n} \|^{(n-\operatorname{Re}\alpha)/n} \| (\lambda + A)^{-n} \|^{\operatorname{Re}\alpha/n} \\ &\leqq C' \lambda^{-\operatorname{Re}\alpha} \end{aligned}$$

Thus it follows from (2.5) that

$$\begin{aligned} \lambda^{\sigma} \| (A(\lambda + A)^{-1})^{m+n}x \| &\leqq \lambda^{\sigma} \| A^{n-\alpha}(\lambda + A)^{-n} \| \, \| (A(\lambda + A)^{-1})^m A^{\alpha}x \| \\ &\leqq C' \lambda^{\sigma-\operatorname{Re}\alpha} \| (A(\lambda + A)^{-1})^m A^{\alpha}x \| \in L^p. \end{aligned}$$

This completes the proof.

As a corollary we see that if σ is not an integer, D^σ_∞ and $D^\sigma_{\infty-}$ coincide with D^σ and D^σ_* of [2], respectively.

THEOREM 2.7. *If the domain* $D(A^\alpha)$ *contains (is contained in)* $D_p^{\mathrm{Re}\,\alpha}$ *for an* $\mathrm{Re}\,\alpha > 0$, *then* $D(A^\alpha)$ *contains (is contained in)* $D_p^{\mathrm{Re}\,\alpha}$ *for all* $\mathrm{Re}\,\alpha > 0$.

Proof. By virtue of Theorem 6.4 of [2] and Proposition 2.5 we have $D(A^\alpha) = D((\mu + A)^\alpha)$ and $D_p^{\mathrm{Re}\,\alpha}(A) = D_p^{\mathrm{Re}\,\alpha}(\mu + A)$, $\mu > 0$, $\mathrm{Re}\,\alpha > 0$, so that we may assume that A has a bounded inverse without loss of generality. The theorem is obvious if we show that A^β, $-\infty < \mathrm{Re}\,\beta < \mathrm{Re}\,\alpha$, is a one-to-one mapping from $D(A^\alpha)$ and $D_p^{\mathrm{Re}\,\alpha}$ onto $D(A^{\alpha-\beta})$ and $D_p^{\mathrm{Re}\,\alpha - \mathrm{Re}\,\beta}$, respectively.

Since $D(A^\alpha) = R(A^{-\alpha})$, $\mathrm{Re}\,\alpha > 0$ ([2], Theorem 6.4), and since $A^{\beta-\alpha} = A^\beta A^{-\alpha}$ ([2], Theorem 7.3), the statemant concerning $D(A^\alpha)$ is immediate.

Let $\mathrm{Re}\,\beta < 0$. Then $x \in D_p^{\mathrm{Re}\,\alpha - \mathrm{Re}\,\beta}$ if and only if $x \in D(A^{-\beta})$ and $A^{-\beta}x \in D_p^{\mathrm{Re}\,\alpha}$. Since A^β is a bounded inverse of $A^{-\beta}$, we have $x \in D_p^{\mathrm{Re}\,\alpha - \mathrm{Re}\,\beta}$ if and only if x is in the image of $D_p^{\mathrm{Re}\,\alpha}$ by A^β. If $\mathrm{Re}\,\beta \geqq 0$, choose a number γ so that $\mathrm{Re}\,\beta < \gamma < \mathrm{Re}\,\alpha$. If $x \in D_p^{\mathrm{Re}\,\alpha - \mathrm{Re}\,\beta}$, x belongs to $D(A^{-\beta})$. Thus there is an element y such that $x = A^\beta y$. By the former part we have $A^{-\gamma}x = A^{\beta-\gamma}y \in D_p^{\mathrm{Re}\,\alpha - \mathrm{Re}\,\beta + \gamma}$. Thus y belongs to $D_p^{\mathrm{Re}\,\alpha}$. On the other hand, if $y \in D_p^{\mathrm{Re}\,\alpha}$, then $y \in D(A^\beta)$ and we have $A^{-\gamma}x = A^{\beta-\gamma}y \in D_p^{\mathrm{Re}\,\alpha - \mathrm{Re}\,\beta + \gamma}$, where $x = A^\beta y$. Then it follows from the former part that x belongs to $D_p^{\mathrm{Re}\,\alpha - \mathrm{Re}\,\beta}$.

Theorem 6.5 of [2] is obtained as a corollary.

PROPOSITION 2.8. For every $\mathrm{Re}\,\alpha > 0$

$$D_1^{\mathrm{Re}\,\alpha} \subset D(A^\alpha) \subset D_{\infty-}^{\mathrm{Re}\,\alpha}\,. \tag{2.9}$$

Proof. It is enough to prove it only in the case $\alpha = 1$. The former inclusion is clear from Lemma 2.3. The latter follows from (1.2), for

$$\lambda(A(\lambda + A)^{-1})^2 x = \lambda(\lambda + A)^{-1}(1 - \lambda(\lambda + A)^{-1})Ax \to 0$$

for $x \in D(A)$ as $\lambda \to \infty$.

PROPOSITION 2.9. If there is a complex number $\mathrm{Re}\,\alpha > 0$ such that $D(A^\alpha) = D_p^{\mathrm{Re}\,\alpha}$, then $D(A^\beta) = D_p^{\mathrm{Re}\,\beta}$ for all $\mathrm{Re}\,\beta > 0$. In particular, $D(A^\alpha)$ coinsides with $D(A^\beta)$ if $\mathrm{Re}\,\alpha = \mathrm{Re}\,\beta$. Furthermore, if A has a bounded inverse, A^{it} is bounded for all real t, where A^{it} is defined in [2].

Proof. We need to prove only the last statement. Because of [2], Corollary 7.4 we have

$$A^{it} = A^{1+it}A^{-1} ,$$

Since $D(A^{1+it}) = D(A) = R(A^{-1})$, A^{it} is defined everywhere and closed, so that it is bounded.

We proved in [2] that the operator A of § 14, Example 6 has unbounded purely imaginary powers A^{it}. The above proposition shows that $D(A^{\alpha})$ cannot be the same as $D_p^{\mathrm{Re}\,\alpha}$ for any p.

However, there are also operators A for which $D(A^{\alpha})$ coincides with $D_p^{\mathrm{Re}\,\alpha}$.

Let X be $L^p(S, B, m)$, where B is a Borel field over a set S and m a measure on B, and let $A(s)$ be a measurable function on S such that

$$|\arg A(s)| \leqq \omega, \text{ a.e.} s$$

for an $0 \leqq \omega < \pi$. Define

$$Ax(s) = A(s)x(s)$$

for all $x(s) \in X$ such that $A(s)x(s) \in X$. Then it is easy to see that A is an operator of type $(\omega, M(\theta))$ if $p \leqq \infty -$, where $L^{\infty-}$ denotes the closure of $D(A)$ in L^{∞}. For this operator A we have $D(A) = D_p^1$, so that $D(A^{\alpha}) = D_p^{\mathrm{Re}\,\alpha}$ for all $\mathrm{Re}\,\alpha > 0$.

In fact, we have

$$(A(\lambda + A)^{-1})^2 x(s) = A(s)^2 x(s)/(\lambda + A(s))^2 .$$

Therefore,

$$\begin{aligned}
&\int_0^{\infty} \| \lambda(A(\lambda + A)^{-1})^2 x(s) \|^p d\lambda/\lambda \\
&\quad = \int_0^{\infty} \lambda^{p-1} d\lambda \int_S \left| \frac{A(s)^2}{(\lambda + A(s))^2} x(s) \right|^p dm(s) \\
&\quad = \int_S |x(s)|^p dm(s) \int_0^{\infty} \lambda^{p-1} \left| \frac{A(s)}{\lambda + A(s)} \right|^{2p} d\lambda \\
&\quad \sim \| Ax \|^p .
\end{aligned}$$

Any normal operator A of type $(\omega, M(\theta))$ can be represented as an operator of the above type. Therefore, it satisfies $D(A^{\alpha}) = D_2^{\mathrm{Re}\,\alpha}$ for $\mathrm{Re}\,\alpha > 0$. T. Kato [1] proved that this holds also for any maximal accretive operator A (see J.-L. Lions [3]).

Now let us complete the definition of fractional powers.

THEOREM 2.10. *Let* $0 < \mathrm{Re}\,\alpha < m$. *If there is a sequence* $N_j \to \infty$

such that

$$y = w\text{-}\lim_{j\to\infty} \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^{N_j} \lambda^{\alpha-1}(A(\lambda+A)^{-1})^m x d\lambda$$

exists, then $x \in D(A^\alpha)$ *and* $y = A^\alpha x$.

Conversely, if $x \in D(A^\alpha)$, *then*

$$A^\alpha x = s\text{-}\lim_{N\to\infty} \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^{N} \lambda^{\alpha-1}(A(\lambda+A)^{-1})^m x d\lambda\ , \tag{2.10}$$

possibly except for the case in which $\operatorname{Im}\alpha \neq 0$ *and* $\operatorname{Re}\alpha$ *is an integer.*

Proof. The former statement is obtained by modifying the proof of [2], Proposition 4.6. Since $(\mu(\mu+A)^{-1})^m x \in D_1^{\operatorname{Re}\alpha}$, we have

$$\begin{aligned} A^\alpha(\mu(\mu+A)^{-1})^m x &= c\int_0^\infty \lambda^{\alpha-1}(A(\lambda+A)^{-1})^m(\mu(\mu+A)^{-1})^m x d\lambda \\ &= (\mu(\mu+A)^{-1})^m\ w\text{-}\lim_{j\to\infty} c\int_0^{N_j} \lambda^{\alpha-1}(A(\lambda+A)^{-1})^m x d\lambda \\ &= (\mu(\mu+A)^{-1})^m y\ . \end{aligned}$$

By virtue of (1, 2), it follows that $x \in D(A^\alpha)$ and $y = A^\alpha x$.

The proof of the latter statement may be reduced to the case in which $0 < \operatorname{Re}\alpha < 1$ and $m = 1$. Suppose that $x \in D(A^\alpha)$ and an integer $m > \operatorname{Re}\alpha$. Substituting (1.6), we have

$$\begin{aligned} &\int_0^N \lambda^{\alpha-1}(A(\lambda+A)^{-1})^m x d\lambda \\ &\quad = m\int_0^N \lambda^{\alpha-m-1} d\lambda \int_0^\lambda \mu^{m-1}(A(\mu+A)^{-1})^{m+1} x d\mu \\ &\quad = \frac{m}{m-\alpha}\int_0^N \left(1 - \frac{\mu^{m-\alpha}}{N^{m-\alpha}}\right)\mu^{\alpha-1}(A(\mu+A)^{-1})^{m+1} x d\mu\ . \end{aligned}$$

Since $x \in D(A^\alpha) \subset D_{\infty-}^{\operatorname{Re}\alpha}$, it follows that

$$\left\|\int_0^N \frac{\mu^{m-\alpha}}{N^{m-\alpha}}\ \mu^{\alpha-1}(A(\mu+A)^{-1})^{m+1} x d\mu\right\| \to 0 \quad \text{as} \quad N \to \infty\ .$$

Thus the limit (2.10), if it exists, does not depend on $m > \operatorname{Re}\alpha$.

Next, let $\operatorname{Re}\alpha > 1$ and $m \geqq 2$. Since $x \in D(A^\alpha)$ belongs to $D(A)$, integration by parts yields

$$\begin{aligned} &\int_0^N \lambda^{\alpha-1}(A(\lambda+A)^{-1})^m x d\lambda \\ &\quad = \frac{\alpha-1}{m-1}\int_0^N \lambda^{\alpha-2}(A(\lambda+A)^{-1})^{m-1} Ax d\lambda - \frac{N^{\alpha-1}}{m-1}(A(N+A)^{-1})^{m-1} Ax\ . \end{aligned}$$

The second term tends to zero as $N \to \infty$ because $Ax \in D(A^{\alpha-1}) \subset D_{\infty-}^{\mathrm{Re}\alpha-1}$. Therefore, we obtain (2.10) if we can prove it when both α and m are reduced by one.

To prove (2.10) in the case $0 < \mathrm{Re}\,\alpha < 1$ and $m = 1$ we assume for a moment that A has a bounded inverse. Then $D(A^\alpha)$ is identical with the range of $A^{-\alpha}$, which may be represented by the absolulely convergent integral:

$$A^{-\alpha}x = \frac{\sin \pi x}{\pi}\int_0^\infty \lambda^{-\alpha}(\lambda + A)^{-1}x d\lambda$$

([2], Proposition 5.1). Employing the resolvent equation and (1.6), we get

$$\begin{aligned}&\frac{\Gamma(1)}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^N \lambda^{\alpha-1}A(\lambda + A)^{-1}A^{-\alpha}x d\lambda \\ &\quad = \left(\frac{\sin \pi\alpha}{\pi}\right)^2 \int_0^N \lambda^{\alpha-1}d\lambda \int_0^\infty \mu^{-\alpha}\frac{\lambda(\lambda + A)^{-1} - \mu(\mu + A)^{-1}}{\lambda - \mu}x d\mu \\ &\quad = \left(\frac{\sin \pi\alpha}{\pi}\right)^2 \int_0^N \lambda^{\alpha-1}d\lambda \int_0^\infty \mu^{-\alpha}(\lambda - \mu)^{-1}d\mu \int_\mu^\lambda A(\nu + A)^{-2}x d\nu\,.\end{aligned}$$

It is enough to show that this converges strongly to the identity, or more weakly that it simply converges, because if it converges, the limit must be $A^\alpha A^{-\alpha}x = x$.

First of all, we have

$$\begin{aligned}I_1 &= \int_0^N \lambda^{\alpha-1}d\lambda \int_0^\lambda \mu^{-\alpha}(\lambda - \mu)^{-1}d\mu \int_\mu^\lambda A(\nu + A)^{-2}x d\nu \\ &= \int_0^N A(\nu + A)^{-2}x d\nu \int_\nu^N \lambda^{\alpha-1}d\lambda \int_0^\nu \mu^{-\alpha}(\lambda - \mu)^{-1}d\mu\,.\end{aligned}$$

Changing variables by $\lambda = \nu l$, $\mu = \nu m$ and integrating by parts with respect to ν, we obtain

$$\begin{aligned}I_1 &= \int_1^\infty l^{\alpha-1}dl \int_0^1 m^{-\alpha}(l - m)^{-1}dm x \\ &\quad - \int_0^N A(\nu + A)^{-1}x d\nu N^\alpha \nu^{-\alpha-1}\int_0^1 m^{-\alpha}(N\nu^{-1} - m)^{-1}dm \\ &= c_1 x - \int_0^1 A(Nn + A)^{-1}x n^{-\alpha-1}dn \int_0^1 m^{-\alpha}(n^{-1} - m)^{-1}dm\,.\end{aligned}$$

Since $n^{-\alpha-1}\int_0^1 m^{-\alpha}(n^{-1} - m)^{-1}dm$ is absolutely integrable in n and since $A(Nn + A)^{-1}x = x - Nn(Nn + A)^{-1}x$ tends to zero as $N \to \infty$, the second term converges to zero as $N \to \infty$.

Next we write

$$\begin{aligned}
&\int_0^N \lambda^{\alpha-1}d\lambda\int_\lambda^\infty \mu^{-\alpha}(\lambda-\mu)^{-1}d\mu\int_\mu^\lambda A(\nu+A)^{-2}xd\nu \\
&\quad = \int_0^N A(\nu+A)^{-2}xd\nu\int_0^\nu \lambda^{\alpha-1}d\lambda\int_\nu^\infty \mu^{-\alpha}(\mu-\lambda)^{-1}d\mu \\
&\qquad + \int_N^\infty A(\nu+A)^{-2}xd\nu\int_0^N \lambda^{\alpha-1}d\lambda\int_\nu^\infty \mu^{-\alpha}(\mu-\lambda)^{-1}d\mu \\
&\quad = I_2 + I_3 .
\end{aligned}$$

Changing variables as above, we have

$$\begin{aligned}
I_2 &= \int_0^N A(\nu+A)^{-2}xd\nu\int_0^1 l^{\alpha-1}dl\int_1^\infty m^{-\alpha}(m-l)^{-1}dm \\
&= c_2N(N+A)^{-1}x \longrightarrow c_2x \quad \text{as} \quad N\longrightarrow\infty .
\end{aligned}$$

Finally,

$$I_3 = \int_1^\infty m^{-\alpha}dm\int_0^1 l^{\alpha-1}(m-l)^{-1}dl\int_N^{mN} A(\nu+A)^{-2}xd\nu$$

tends to zero as $N\longrightarrow\infty$ because $\int_N^{mN} A(\nu+A)^{-2}xd\nu = mN(mN+A)^{-1}x - N(N+A)^{-1}x$ tends to zero and $m^{-\alpha}\int_0^1 l^{\alpha-1}(m-l)^{-1}dl$ is absolutely integrable.

Next suppose that A has not necessarily a bounded inverse. We have, for $\mu > 0$,

$$\begin{aligned}
&(A^\alpha - (\mu+A)^\alpha)(\mu+A)^{-\alpha}x \\
&= \frac{\sin\pi\alpha}{\pi}\Big(\int_0^\mu \lambda^{\alpha-1}A + \int_\mu^\infty(\lambda^{\alpha-1}A - (\lambda-\mu)^{\alpha-1}(\mu+A)\Big)(\lambda+A)^{-1}(\mu+A)^{-\alpha}xd\lambda
\end{aligned}$$

because the integral is absolutely convergent and the equality holds for all $x\in D(A)$ which is dense in X. This shows together with the above that

$$\begin{aligned}
A^\alpha(\mu+A)^{-\alpha}x &= (\mu+A)^\alpha(\mu+A)^{-\alpha}x \\
&\quad + \operatorname*{s-lim}_{N\to\infty}\frac{\sin\pi\alpha}{\pi}\Big(\int_0^\mu \lambda^{\alpha-1}A + \int_\mu^N(\lambda^{\alpha-1}A - (\lambda-\mu)^{\alpha-1}(\mu+A)\Big) \\
&\qquad\qquad \cdot(\lambda+A)^{-1}(\mu+A)^{-\alpha}xd\lambda \\
&= \operatorname*{s-lim}_{N\to\infty}\frac{\Gamma(1)}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^N \lambda^{\alpha-1}A(\lambda+A)^{-1}(\mu+A)^{-\alpha}xd\lambda .
\end{aligned}$$

3. Interpolation spaces. Let X and Y be Banach spaces contained in a Hausdorff vector space Z. Lions and Peetre [4] defined the mean space $S(p, \theta, X; p, \theta - 1, Y)$, $1 \leqq p \leqq \infty, 0 < \theta < 1$, of X and Y as the space of the means

$$x = \int_0^\infty u(\lambda) d\lambda/\lambda \,, \tag{3.1}$$

where $u(\lambda)$ is a Z-valued function such that

$$\lambda^\theta u(\lambda) \in L^p(X) \text{ and } \lambda^{\theta-1} u(\lambda) \in L^p(Y) \,. \tag{3.2}$$

$S(p, \theta, X; p, \theta - 1, Y)$ is a Banach space with the norm

$$\begin{aligned} \| x \|_{S(p,\theta,X,p,\theta-1,Y)} \\ = \inf \left\{ \max (\| \lambda^\theta u(\lambda) \|_{L^p(X)}, \| \lambda^{\theta-1} u(\lambda) \|_{L^p(Y)}); x = \int_0^\infty u(\lambda) d\lambda/\lambda \right\} . \end{aligned} \tag{3.3}$$

Theorem 3.1. *$S(p, \theta, X; p, \theta - 1, D(A^m))$, $0 < \theta < 1$, $1 \leqq p \leqq \infty$, coincides with $D_p^{\theta m}(A)$.*

Proof. By virtue of Proposition 2.5, we may assume that A has a bounded inverse without loss of generality. In particular, $D(A^m)$ is normed by $\| A^m x \|$. Further, if we change the variable by $\lambda' = \lambda^{1/m}$, condition (3.2) becomes

$$\lambda^{m\theta} u(\lambda) \in L^p(X) \text{ and } \lambda^{m(\theta-1)} A^m u(\lambda) \in L_p(X) \,. \tag{3.4}$$

Suppose $x \in D_p^\sigma$ and define

$$u(\lambda) = c\lambda^m A^m (\lambda + A)^{-2m} x \,,$$

where $c = \Gamma(2m)/(\Gamma(m))^2$. Then

$$\lambda^\sigma u(\lambda) = c(\lambda(\lambda + A^{-1})^m \lambda^\sigma (A(\lambda + A)^{-1})^m x \in L^p(X)$$

and

$$\lambda^{\sigma - m} A^m u(\lambda) = c\lambda^\sigma (A(\lambda + A)^{-1})^{2m} x \in L^p(X) \,.$$

Thus $u(\lambda)$ satisfies (3.4) with $\sigma = m\theta$. Moreover, it follows from Lemma 2.3 that

$$\begin{aligned} \int_0^\infty u(\lambda) d\lambda/\lambda &= \frac{\Gamma(2m)}{(\Gamma(m))^2} \int_0^\infty \lambda^{m-1} (A(\lambda + A)^{-1})^{2m} A^{-m} x \\ &= x \,. \end{aligned}$$

Therefore, x belongs to $S(p, \sigma/m, X; p, \sigma/m - 1, D(A^m))$.

Conversely, let $x \in S(p, \sigma/m, X; p, \sigma/m - 1, D(A^m))$ so that x is represented by integral (3.1) with an integrand satisfying (3.4). Then

$$\lambda^\sigma (A(\lambda + A)^{-1})^m x = (A(\lambda + A)^{-1})^m \lambda^\sigma \int_\lambda^\infty \mu^{-\sigma} \mu^\sigma u(\lambda) d\mu/\mu$$
$$+ (\lambda(\lambda + A)^{-1})^m \lambda^{\sigma - m} \int_0^\lambda \mu^{m-\sigma} \mu^{\sigma - m} A^m u(\lambda) d\mu/\mu .$$

Since both $(A(\lambda + A)^{-1})^m$ and $(\lambda(\lambda + A)^{-1})^m$ are uniformly bounded, $\lambda^\sigma(A(\lambda + A)^{-1})^m x$ belongs to $L^p(X)$, that is, $x \in D_p^\sigma$.

THEOREM 3.2. *Let A be an operator of type $(\omega, M(\theta))$. Then*

$$D_p^\sigma(A^\alpha) = D_p^{\sigma\alpha}(A) , \qquad 0 < \alpha < \pi/\omega, \sigma > 0 .$$

Proof. It is sufficient to prove it in the case $0 < \alpha < 1$, because otherwise we have $A = (A^\alpha)^{1/\alpha}$ with $0 < 1/\alpha < 1$ (see (2.6)). In view of Theorem 2.6 we may also assume that σ is sufficiently small.

By [2] Proposition 10.2 we have

$$\lambda^\sigma A^\alpha(\lambda + A^\alpha)^{-1} x = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{\lambda^{\sigma+1} \tau^{\alpha - \alpha\sigma}}{\lambda^2 + 2\lambda\tau^\alpha \cos \pi\alpha + \tau^{2\alpha}} \tau^{\alpha\sigma} A(\tau + A)^{-1} x d\tau/\tau .$$

Since the kernel

$$\frac{(\lambda^{-1}\tau^\alpha)^{1-\sigma}}{1 + 2(\lambda^{-1}\tau^\alpha) \cos \pi\alpha + (\lambda^{-1}\tau^\alpha)^2} , \qquad 0 < \sigma < 1 ,$$

defines a bounded integral operator in $L^p(X)$, $D_p^{\sigma\alpha}(A)$ is contained in $D_p^\sigma(A^\alpha)$.

If $\alpha = 1/m$ with an odd integer m, we have conversely

$$D_p^\sigma(A^{1/m}) \subset D_p^{\sigma/m}(A) .$$

In fact, let $x \in D_p^\sigma(A^{1/m})$. Since

$$\lambda^\sigma A(\lambda^m + A)^{-1} = \lambda^\sigma \prod_{i=1}^m (A^{1/m}(\varepsilon_i \lambda + A^{1/m})^{-1}) x ,$$

where ε_i are roots of $(-\varepsilon)^m = -1$ with $\varepsilon_1 = 1$, and since

$$A^{1/m}(\varepsilon_i \lambda + A^{1/m})^{-1}, \qquad i = 2, \cdots, m,$$

are uniformly bounded, $\lambda^\sigma A(\lambda^m + A)^{-1} x \in L^p(X)$. Changing the variable by $\lambda' = \lambda^m$, we get $\lambda^{\sigma/m} A(\lambda + A)^{-1} x \in L^p(X)$.

In a general case choose an odd number m such that $0 < 1/m < \alpha$. Since $A^{1/m} = (A^\alpha)^{1/(\alpha m)}$, we have

$$D_p^{\alpha\sigma}(A) \subset D_p^\sigma(A^\alpha) \subset D_p^{\alpha\sigma m}(A^{1/m}) \subset D_p^{\alpha\sigma}(A) .$$

Another less computational proof will be obtained from the Lions-Peetre theory and Proposition 2.8.

4. Infinitesimal generators of bounded semi-groups. Throughout this section we assume that T_t, $t \geqq 0$, is a bounded strongly continuous semi-group of operators in X and $-A$ is its infinitesimal generator:

$$T_t = \exp(-tA), \qquad \| T_t \| \leqq M . \tag{4.1}$$

A is an operator of type $(\pi/2, M(\theta))$.

DEFINITION 4.1. Let $0 < \sigma < m$, where σ is a real number and m an integer, and let $1 \leqq p \leqq \infty$. We denote by $C^{\sigma}_{p,m} = C^{\sigma}_{p,m}(A)$ the set of all elements $x \in X$ such that

$$t^{-\sigma}(I - T_t)^m x \in L^p(X) . \tag{4.2}$$

As is easily seen, $C^{\sigma}_{p,m}$ is a Banach space with the norm

$$\| x \|_{C^{\sigma}_{p,m}} = \| x \| + \| t^{-\sigma}(I - T_t)^m x \|_{L^p(X)} .$$

Since $(I - T_t)^m$ is uniformly bounded, condition (4.2) is equivalent to that $t^{-\sigma}(I - T_t)^m x$ belongs to $L^p(X)$ near the origin. In particular, we have

$$C^{\sigma}_{p,m}(A) = C^{\sigma}_{p,m}(\mu + A), \ \mu > 0 . \tag{4.3}$$

$C^{\sigma}_{\infty,1}$ and $C^{\sigma}_{\infty-,1}$ coincide with C^{σ} and C^{σ}_{*} of [2], respectively, and $C^{\sigma}_{\infty,1}$ consists of all elements x such that $T_t x$ is (weakly) uniformly Hölder continuous with exponent σ.

PROPOSITION 4.2. If $x \in C^{\sigma}_{p,m}$, then x belongs to $D(A^{\alpha})$ for all $0 < \operatorname{Re}\alpha < \sigma$, and

$$A^{\alpha}x = \frac{1}{K_{\alpha,m}} \int_0^{\infty} t^{-\alpha-1}(I - T_t)^m x dt, \qquad 0 < \operatorname{Re}\alpha < \sigma , \tag{4.4}$$

where

$$K_{\alpha,m} = \int_0^{\infty} t^{-\alpha-1}(1 - e^{-t})^m dt .$$

Proof. If $0 < \operatorname{Re}\alpha < \sigma$, the right-hand side of (4.4) converges absolutely and represents an analytic function of α.

If $x \in D(A)$, then we have by [2] Proposition 11.4

$$\int_0^{\infty} t^{-\alpha-1}(I - T_t)^m x dt$$

$$= \sum_{k=1}^{m} (-1)^{k+1}\binom{m}{k}\int_0^\infty t^{-\alpha-1}(I - T_{kt})x dt$$

$$= \Gamma(-\alpha) \sum_{k=1}^{m} (-1)^{k+1}\binom{m}{k} k^\alpha A^\alpha x, \qquad 0 < \operatorname{Re}\alpha < 1 .$$

The coefficient of $A^\alpha x$ does not depend on A. Taking $A = 1$, we see that it is equal to $K_{\alpha,m}$.

Next let $0 < \operatorname{Re}\alpha < \min(\sigma, 1)$ and $x \in C^\sigma_{p,m}$. Then integral (4.4) with x replaced by $\mu(\mu + A)^{-1}x$, $\mu > 0$, exists and converges to the integral (4.4) as $\mu \to \infty$. Thus $A^\alpha \mu(\mu + A)^{-1}x$ converges to the integral (4.4). Since A^α is closed and $\mu(\mu + A)^{-1}x \to x$ as $\mu \to \infty$, it follows that $x \in D(A^\alpha)$ and (4.4) holds.

In the general case the assertion is obtained by [2], Proposition 8.4 or by repeating an argument as above.

Lions and Peetre [4] gave another proof when α is an integer.

THEOREM 4.3. *$C^\sigma_{p,m}$ coincides with D^σ_p with equivalent norms.*

Proof. First we note that

$$(I - T_t)x = AI_t x , \qquad x \in X , \tag{4.5}$$

where

$$I_t x = \int_0^t T_s x ds . \tag{4.6}$$

Obviously we have

$$\| I_t \| \leqq M_t , \qquad t > 0 . \tag{4.7}$$

Let $x \in C^\sigma_{p\ m}$. Then $(\lambda + A)^{-m}x$, $\lambda > 0$, belongs to $C^{\sigma+m}_{p,2m}$ since

$$t^{-\sigma-m} \| (I - T_t)^{2m}(\lambda + A)^{-m}x \|$$
$$\leqq t^{-m} \| I_t^m \| \, \| (A(\lambda + A)^{-1})^m \| \, t^{-\sigma} \| (I - T_t)^m x \| .$$

Hence we have by Proposition 4.2

$$(A(\lambda + A)^{-1})^m x = c\int_0^\infty t^{-m-1}(I - T_t)^{2m}(\lambda + A)^{-m}x$$
$$= c\int_0^{1/\lambda} (A(\lambda + A)^{-1})^m t^{-m-1} I_t^m (I - T_t)^m x dt$$
$$+ c\int_{1/\lambda}^\infty (\lambda + A)^{-m} t^{-m-1} (I - T_t)^{2m} x dt ,$$

where $c = K^{-1}_{m,2m}$. Therefore,

$$\lambda^\sigma \| (A(\lambda + A)^{-1})^m x \| \leqq cL^m M^m \lambda^\sigma \int_0^{1/\lambda} t^\sigma t^{-\sigma} \| (I - T_t)^m x \| dt/t$$
$$+ cM^m (2M)^m \lambda^{\sigma-m} \int_{1/\lambda}^\infty t^{\sigma-m} t^{-\sigma} \| (I - T_t)^m x \| dt/t .$$

This shows that $x \in D^{\sigma}_{p,m}$.

Conversely, let $x \in D^{\sigma}_{p,m}$. Since

$$(A(\lambda + A)^{-1})^{2m} I^m_t x = (\lambda + A)^{-m}(I - T_t)^m (A(\lambda + A)^{-1})^m x ,$$

it follows that $I^m_t x \in D^{\sigma+m}_{p,2m}$. Thus by Proposition 2.2 we get

$$\begin{aligned}(I - T_t)^m x = A^m I^m_t x &= c\int_0^{\infty} \lambda^{m-1}(A(\lambda + A)^{-1})^{2m} I^m_t x \\ &= c\int_0^{1/t} I^m_t \lambda^{m-1}(A(\lambda + A)^{-1})^{2m} x d\lambda \\ &\quad + c\int_{1/t}^{\infty} (I - T_t)^m \lambda^{m-1}(\lambda + A)^{-m}(A(\lambda + A)^{-1})^m x d\lambda ,\end{aligned}$$

where $c = \Gamma(2m)/(\Gamma(m))^2$. By the same computation as above we conclude that $x \in C^{\sigma}_{p,m}$.

In particular, $C^{\sigma}_{p,m}$ does not depend on m. We denote $C^{\sigma}_{p,m}$ with the least $m > \sigma$ by C^{σ}_p. Because of Theorem 2.6, C^{σ}_{∞} coincides with C^{σ} of [2] if σ is not an integer.

THEOREM 4.4. *Let $0 < \operatorname{Re} \alpha < m$. If there is a sequence $\varepsilon_j \to 0$ such that*

$$y = w\text{-}\lim_{j\to\infty} \frac{1}{K_{\alpha,m}} \int_{\varepsilon_j}^{\infty} t^{-\alpha-1}(I - T_t)^m x dt \tag{4.8}$$

exists, then $x \in D(A^{\alpha})$ and $y = A^{\alpha}x$.

Conversely, if $x \in D(A^{\alpha})$, then

$$A^{\alpha}x = s\text{-}\lim_{\varepsilon\to 0} \frac{1}{K_{\alpha,m}} \int_{\varepsilon}^{\infty} t^{-\alpha-1}(I - T_t)^m x dt . \tag{4.9}$$

Proof. The former part is proved in the same way as Theorem 2.10.

To prove the latter part, let us assume for a moment that T_t satisfies

$$\| T_t \| \leqq M e^{-\mu t}, \qquad t > 0 ,$$

for a $\mu > 0$. Then A^{α} is the inverse of $A^{-\alpha}$ which can be represented by the absolutely convergent integral

$$A^{-\alpha}x = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} s^{\alpha-1} T_s x ds \tag{4.10}$$

([2], Theorem 7.3 and Proposition 11.1).

Now it is enough to prove that

$$\frac{1}{K_{\alpha,m}\Gamma(\alpha)} \int_{\varepsilon}^{\infty} t^{-\alpha-1}(I - T_t)^m dt \int_0^{\infty} s^{\alpha-1} T_s x ds$$

converges strongly as $\varepsilon \to 0$, because the limit must coincide with $A^{\alpha}A^{-\alpha}x = x$.

We have

$$I_{\varepsilon} = \int_{\varepsilon}^{\infty} t^{-\alpha-1}(I - T_t)^m dt \int_0^{\infty} s^{\alpha-1} T_s x ds$$
$$= \sum_{k=1}^{m} (-1)^{k+1}\binom{m}{k} k^{\alpha} \int_{k\varepsilon}^{\infty} t^{-\alpha-1}(I - T_t)dt \int_0^{\infty} s^{\alpha-1} T_s x ds .$$

Now

$$\int_{k\varepsilon}^{\infty} t^{-\alpha-1} T_t dt \int_0^{\infty} s^{\alpha-1} T_s x ds$$
$$= \int_{k\varepsilon}^{\infty} t^{-\alpha-1} dt \int_t^{\infty} (s - t)^{\alpha-1} T_s x ds$$
$$= \int_{k\varepsilon}^{\infty} T_s x ds \int_{k\varepsilon}^{s} t^{-\alpha-1}(s - t)^{\alpha-1} dt$$
$$= \frac{1}{\alpha(k\varepsilon)^{\alpha}} \int_{k\varepsilon}^{\infty} (s - k\varepsilon)^{\alpha} T_s x ds/s .$$

Furthermore,

$$\sum_{k=1}^{m} (-1)^{k+1}\binom{m}{k} k^{\alpha} \int_{k\varepsilon}^{\infty} t^{-\alpha-1} dt \int_0^{\infty} s^{\alpha-1} T_s x ds$$
$$= \frac{1}{\alpha\varepsilon^{\alpha}} \int_0^{\infty} s^{\alpha-1} T_s x ds ,$$

so that we obtain

$$I_{\varepsilon} = \frac{1}{\alpha\varepsilon^{\alpha}} \sum_{k=0}^{m} (-1)^{k}\binom{m}{k} \int_{k\varepsilon}^{\infty} (s - k\varepsilon)^{\alpha} T_s x ds/s .$$

Since $T_s x \to x$ as $s \to 0$, it follows that

$$\frac{1}{\alpha\varepsilon^{\alpha}} \sum_{k=0}^{m} (-1)^{k}\binom{m}{k} \int_{k\varepsilon}^{m\varepsilon} (s - k\varepsilon)^{\alpha} T_s x ds/s$$
$$= \frac{1}{\alpha} \sum_{k=0}^{m} (-1)^{k}\binom{m}{k} \int_{k}^{m} (s - k)^{\alpha} T_{\varepsilon s} x ds/s$$
$$\to \frac{1}{\alpha} \sum_{k=0}^{m} (-1)^{k}\binom{m}{k} \int_{k}^{m} (s - k)^{\alpha} ds/s x \text{ as } \varepsilon \to 0 .$$

On the other hand, the Taylor expansion up to order m gives

$$f_{\varepsilon}(s) = \sum_{k=0}^{m} (-1)^{k}\binom{m}{k}(s - k\varepsilon)^{\alpha}$$
$$= \sum_{k=0}^{m} (-1)^{k}\binom{m}{k} \frac{\alpha(\alpha - 1) \cdots (\alpha - m + 1)}{m!} (s - k'\varepsilon)^{\alpha-m}(-k\varepsilon)^{m} ,$$

where $0 < k' < k$. Hence we have

$$\frac{1}{\alpha\varepsilon^{\alpha}}\int_{m\varepsilon}^{\infty} f_{\varepsilon}(s)T_s x ds/s$$
$$= \frac{(\alpha-1)\cdots(\alpha-m+1)}{m!}\sum_{k=0}^{m}(-1)^{k+m}\binom{m}{k}k^m\int_m^{\infty}(s-k')^{\alpha-m}T_{\varepsilon s}x ds/s .$$

Since $(s-k')^{\alpha-m}s^{-1}$ is absolutely integrable, this converges to a constant times x as $\varepsilon \to 0$.

To prove (4.9) in the general case, it is sufficient to show that

$$(4.11) \qquad (A^{\alpha}-(\mu+A)^{\alpha})(\mu+A)^{-\alpha}x$$
$$= \frac{1}{K_{\alpha,m}}\int_0^{\infty} t^{-\alpha-1}\{(I-T_t)^m-(I-e^{-\mu t}T_t)^m\}(\mu+A)^{-\alpha}x dt ,$$
$$\mu > 0,\ x \in X ,$$

and that the integral converges absolutely.

By Theorem 2.6, (4.5) and a similar decomposition of $I-e^{-\mu t}T_t$ we have

$$(I-T_t)^m(I-e^{-\mu t}T_t)^n x = O(t^{\sigma}),\ x \in C_{\infty}^{\sigma},\ m+n > \sigma .$$

Since $(\mu+A)^{-\alpha}x \in D(A^{\alpha}) \subset C_{\infty}^{\mathrm{Re}\,\alpha}$, it follows that

$$\{(I-T_t)^m-(I-e^{-\mu t}T_t)^m\}x$$
$$= (e^{-\mu t}-1)T_t\{(I-T_t)^{m-1}+\cdots+(I-e^{-\mu t}T_t)^{m-1}\}x$$
$$= O(t^{\min(\mathrm{Re}\,\alpha, m-1)+1}) .$$

This shows that integral (4.11) is absolutely convergent. (4.11) is valid for all $x \in D(A)$ which is dense in X. Therefore, (4.11) holds for all $x \in X$.

5. Infinitesimal generators of bounded analytic semi-groups. Let T_t be a semi-group of operators analytic in a sector $|\arg t| < \pi/2-\omega$, $0 \leqq \omega < \pi/2$, and uniformly bounded in each smaller sector $|\arg t| \leqq \pi/2-\omega-\varepsilon$, $\varepsilon > 0$. We call such a semi-group a bounded analytic semi-group.

It is known that the negative of an operator A generates a bounded analytic semi-group if and only if A is of type $(\omega, M(\theta))$ for some $0 \leqq \omega < \pi/2$. A bounded strongly continuous semi-group T_t has a bounded analytic extension if there is a complex number $\mathrm{Re}\,\alpha > 0$ such that

$$(5.1) \qquad \| A^{\alpha}T_t \| \leqq Ct^{-\mathrm{Re}\,\alpha},\ t > 0,$$

with a constant C independent of t. Conversely, if T_t is bounded analytic, (5.1) holds for all $\mathrm{Re}\,\alpha > 0$ ([2], Theorems 12.1 and 12.2).

We assume throughout this section that $-A$ is the infinitesimal generator of a bounded analytic semi-group T_t.

DEFINITION 5.1. Let $0 < \sigma < \mathrm{Re}\,\beta$ and $1 \leqq p \leqq \infty$. We denote by $B^\sigma_{p,\beta} = B^\sigma_{p,\beta}(A)$ the set of all $x \in X$ such that

$$t^{\mathrm{Re}\beta-\sigma} A^\beta T_t x \in L^p(X). \tag{5.2}$$

$B^\sigma_{p,\beta}$ is a Banach space with the norm

$$\| x \|_{B^\sigma_{p,\beta}} = \| x \| + \| t^{\mathrm{Re}\beta-\sigma} A^\beta T_t x \|_{L^p(X)} .$$

PROPOSITION 5.2. Let $0 < \mathrm{Re}\,\alpha < \sigma$. Then every $x \in B^\sigma_{p,\beta}$ belongs to $D(A^\alpha)$ and

$$A^\alpha x = \frac{1}{\Gamma(\beta-\alpha)} \int_0^\infty t^{\beta-\alpha-1} A^\beta T_t x dt , \tag{5.3}$$

where the integral converges absolutely.

Proof. Since $A^\beta T_t x$ is of order $t^{\sigma-\mathrm{Re}\beta}$ as $t \to 0$ and of order $t^{-\mathrm{Re}\beta+\varepsilon}$ as $t \to \infty$ in the sense of $L^p(X)$, the integral converges absolutely for $0 < \mathrm{Re}\,\alpha < \sigma$.

To prove (5.3), first let $x \in D(A^\beta)$. Then it follows from [2], Proposition 11.1 and Theorem 7.3 that

$$\begin{aligned} &\frac{1}{\Gamma(\beta-\alpha)} \int_0^\infty t^{\beta-\alpha-1} A^\beta T_t x dt \\ &= s\text{-}\lim_{\varepsilon \to 0} \frac{1}{\Gamma(\beta-\alpha)} \int_0^\infty t^{\beta-\alpha-1} e^{-\varepsilon t} T_t A^\beta x dt \\ &= s\text{-}\lim_{\varepsilon \to 0} (\varepsilon + A)^{\alpha-\beta} A^\beta x \\ &= s\text{-}\lim_{\varepsilon \to 0} A^{\beta-\alpha}(\varepsilon + A)^{\alpha-\beta} A^\alpha x . \end{aligned}$$

Because of [2], Propositions 6.2 and 6.3, $A^{\beta-\alpha}(\varepsilon + A)^{\alpha-\beta}$ converges strongly to the identity on $\overline{R(A)}$ as $\varepsilon \to 0$. Since $A^\alpha X$ is contained in $\overline{R(A)}$ ([2], Proposition 4.3), (5.3) holds for all $x \in D(A^\beta)$. In the general case (5.3) is proved by approximating $x \in B^\sigma_{p,\beta}$ by $(\mu(\mu + A)^{-1})^m x$, $m > \mathrm{Re}\,\beta$, which belongs to $D(A^\beta)$.

THEOREM 5.3. *$B^\sigma_{p,\beta}$ coincides with D^σ_p. In particular, $B^\sigma_{p,\beta}$ does not depend on β.*

Proof. Let $x \in B^\sigma_{p,\beta}$. If m is an integer greater than $\mathrm{Re}\,\beta$, x belongs to $B^\sigma_{p,m}$, for

$$t^{m-\sigma}A^m T_t x = t^{m-\beta}A^{m-\beta}T_{t/2} \cdot t^{\beta-\sigma}A^\beta T_{t/2}x$$

and $t^{m-\beta}A^{m-\beta}T_{t/2}$ is uniformly bounded. Since

$$t^{m-\sigma}A^{2m}T_t(\lambda + A)^{-m}x = (A(\lambda + A)^{-1})^m t^{m-\sigma}A^m T_t x ,$$

$(\lambda + A)^{-m}x$ belongs to $B^{\sigma+m}_{p,2m}$. Hence it follows from Proposition 5.2 that

$$\begin{aligned} A^m(\lambda + A)^{-m}x &= c\int_0^\infty t^m A^{2m} T_t(\lambda + A)^{-m}x dt/t \\ &= c(A(\lambda + A)^{-1})^m \int_0^{1/\lambda} t^m A^m T_t x dt/t \\ &\quad + c(\lambda + A)^{-m}\int_{1/\lambda}^\infty t^m A^{2m} T_t x dt/t , \end{aligned}$$

where $c = \Gamma(m)^{-1}$. The rest of the proof is the same as that of Theorem 4.3.

Conversely, assume that $x \in D^\sigma_{p,m} = D^\sigma_{p,2m}$. Since $T_t x$, $t > 0$, belongs to any $D^\sigma_{p,m}$, we have by (2.1)

$$\begin{aligned} A^\beta T_t x &= c\int_0^\infty \lambda^{\beta-1}(A(\lambda + A)^{-1})^{2m} T_t x d\lambda \\ &= c\, T_t \int_0^{1/t} \lambda^{\beta-1}(A(\lambda + A)^{-1})^{2m} x d\lambda \\ &\quad + cA^m T_t \int_{1/t}^\infty \lambda^{\beta-1}(\lambda + A)^{-m}(A(\lambda + A)^{-1})^m x d\lambda , \end{aligned}$$

where $c = \Gamma(2m)/(\Gamma(\beta)\Gamma(2m - \beta))$. Arguing as before, we get $x \in B^\sigma_{p,\beta}$.

THEOREM 5.4 *Let $0 < \operatorname{Re}\alpha < \operatorname{Re}\beta$. If*

$$y = w\text{-}\lim_{\varepsilon_j \to 0} \frac{1}{\Gamma(\beta - \alpha)} \int_{\varepsilon_j}^\infty t^{\beta-\alpha-1} A^\beta T_t x dt \tag{5.4}$$

exists, then $x \in D(A^\alpha)$ and $y = A^\alpha x$. If $x \in D(A^\alpha)$, then

$$A^\alpha x = s\text{-}\lim_{\varepsilon \to 0} \frac{1}{\Gamma(\beta - \alpha)} \int_\varepsilon^\infty t^{\beta-\alpha-1} A^\beta T_t x dt . \tag{5.5}$$

Proof. The former part is proved in the same way as Theorem 2.10. Let us prove the latter assuming that $\mu - A$ generates a bounded analytic semi-group for a $\mu > 0$. $D(A^\alpha)$ is the same as the range $R(A^{-\alpha})$ in this case, and we have $A^\beta T_t A^{-\alpha}x = A^{\beta-\alpha}T_t x$ by the additivity of fractional powers. So it is sufficient to prove the following:

$$x = s\text{-}\lim_{\varepsilon \to 0} \frac{1}{\Gamma(\beta)} \int_\varepsilon^\infty t^{\beta-1} A^\beta T_t x dt, \qquad x \in X , \tag{5.6}$$

when $\operatorname{Re}\beta > 0$.

First we note that if $\operatorname{Re}\alpha > 0$, then

$$t^{\alpha}A^{\alpha}T_t x \to 0 \text{ as } t \to 0 \text{ or as } t \to \infty \tag{5.7}$$

for each $x \in X$, because (5.7) holds for $x \in D(A)$ and $t^{\alpha}A^{\alpha}T_t$ is uniformly bounded.

Let β be equal to an integer m. Since $d/dtA^{\beta}T_t x = -A^{\beta+1}T_t x$, we have, by integrating by parts,

$$\int_{\varepsilon}^{\infty} t^{m-1}A^m T_t x dt$$
$$= \varepsilon^{m-1}A^{m-1}T_{\varepsilon}x + (m-1)\int_{\varepsilon}^{\infty} t^{m-2}A^{m-1}T_t x dt .$$

(5.7) shows that the first term tends to zero as $\varepsilon \to 0$ if $m > 1$. When $m = 1$, we have

$$\int_{\varepsilon}^{\infty} AT_t x dt = T_{\varepsilon}x \to x \text{ as } \varepsilon \to 0 .$$

Thus (5.6) holds if β is an integer.

If β is not an integer, take an integer $m > \operatorname{Re}\beta$. We have

$$A^{\beta}T_t x = A^{\beta-m}A^m T_t x$$
$$= \frac{1}{\Gamma(m-\beta)}\int_t^{\infty}(s-t)^{m-\beta-1}A^m T_s x ds, \qquad t > 0 ,$$

by [2], Proposition 11.1. Therefore,

$$\frac{1}{\Gamma(\beta)}\int_{\varepsilon}^{\infty} t^{\beta-1}A^{\beta}T_t x dt$$
$$= \frac{1}{\Gamma(\beta)\Gamma(m-\beta)}\int_{\varepsilon}^{\infty} A^m T_s x ds \int_{\varepsilon}^{s} t^{\beta-1}(s-t)^{m-\beta-1}dt$$
$$= \frac{1}{\Gamma(m)}\int_{\varepsilon}^{\infty} s^{m-1}A^m T_s x ds$$
$$- \frac{\varepsilon^m}{\Gamma(\beta)\Gamma(m-\beta)}\int_1^{\infty} A^m T_{\varepsilon\sigma}x d\sigma \int_0^1 \tau^{\beta-1}(\sigma-\tau)^{m-\beta-1}d\tau .$$

The first term tends to x as $\varepsilon \to 0$. The second term converges to zero, because

$$\int_1^{\infty} \sigma^{-m}d\sigma \int_0^1 \tau^{\beta-1}(\sigma-\tau)^{m-\beta-1}d\tau$$

is absolutely convergent and $(\varepsilon\sigma)^m A^m T_{\varepsilon\sigma}x$ tends to zero as $\varepsilon \to 0$.

The proof in the general case is obtained from the absolutely convergent integral representation:

$$(A^{\alpha} - (\mu + A)^{\alpha})(\mu + A)^{-\alpha}x$$
$$= \frac{1}{\Gamma(\beta - \alpha)} \int_0^{\infty} t^{\beta-\alpha-1}(A^{\beta} - e^{-\mu t}(\mu + A)^{\beta})T_t(\mu + A)^{-\alpha}x dt .$$

The absolute convergence follows from [2], Propositions 6.2 and 6.3.

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