INTRINSIC TOPOLOGIES IN A TOPOLOGICAL LATTICE

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It is shown that if (L, T) is a compact connected modular topological lattice of finite dimension under a topology T, then the topology T, the interval topology of L, the complete topology of L, and the order topology of L are all the same.

There are a variety of known ways in which a lattice may be given a topology, e.g., Frink's interval topology [8], Birkhoff's order topology [4], and Insel's complete topology [9].

A lattice L is a *topological lattice* if and only if L is a Hausdorff space in which the two lattice operations are continuous.

In this paper we give some of the relationships between topological lattice and its intrinsic topologies and extend a theorem of Dyer and Shields [7] and a result of Anderson [2]. We shall finally prove the main theorem stated above.

We shall use $A \wedge B$ and $A \vee B$ for a pair of subsets A and B of a lattice L to denote the sets $\{a \wedge b \mid a \in A \text{ and } b \in B\}$ and $\{a \vee b \mid a \in A \text{ and } b \in B\}$, respectively. For a subset A of L, A^* is the closure of A. The empty set is written as $\square$.

By the *interval topology* of a lattice L, denoted by $I(L)$, we mean the topology defined by taking the closed intervals $\{a \wedge L, a \vee L \mid a \in L\}$ as a sub-base for the closed sets. It is easy to see that if (L, T) is a topological lattice and if $I(L)$ is Hausdorff, then (L, T) is compact if and only if $T = I(L)$ and L is complete.

For a net $\{x_\alpha \mid \alpha \in D\}$ in a complete lattice L, if $\limsup \{x_\alpha \mid \alpha \in D\} = \liminf \{x_\alpha \mid x_\alpha \in D\} = x$, we say that the net $\{x_\alpha\}$ order converges to x. We define a subset M of a complete lattice L to be *closed* in the *order topology* of L, denoted by $O(L)$, if and only if no net in M converges to a point outside of M.

The following two lemmas are immediate:

LEMMA 1. *If (L, T) is a compact topological lattice, and if $\{x_\alpha \mid \alpha \in D\}$ is a monotone decreasing net in L with* $\inf \{x_\alpha \mid \alpha \in D\} = a$, *then the net converges to a in T. The dual argument is also true.*

LEMMA 2. *If (L, T) is a compact topological lattice, then $T \subset O(L)$. Moreover, if $O(L)$ is also compact, then $T = O(L)$.*

By a *complete subset* C of a lattice L we shall mean a nonempty subset C of L such that for each nonempty subset S of C, S possesses both a sup S and an inf S in L, and furthermore, both sup S and inf S are in C. The smallest topology for L in which the complete subsets of L are closed is called the *complete topology* for L, and denoted by $C(L)$. It is known [9] that $C(L) \subset O(L)$, and if L is complete, then $I(L) \subset C(L)$.

The following lemma follows at once either from Lemmas 1 and 2 or from [11].

LEMMA 3. *If (L, T) is a compact topological lattice, then $I(L)$ is Hausdorff, if and only if $I(L) = C(L) = T = O(L)$.*

The *breadth* of a lattice L is the smallest integer n such that any finite subset F of L has a subset F' of at most n elements such that $\inf F = \inf F'$. It is known [4] that the breadth of L is equal to the breadth of the dual of L.

A subset M of a topological lattice L is *convex* if and only if $(M \wedge L) \cap (M \vee L) = M$ [1]. A topological lattice is *locally convex* if and only if the convex open sets form a basis for the topology. It is well known that a compact (or locally compact and connected) topological lattice is locally convex.

We shall extend a theorem of Dyer and Shields in [7] as follows:

THEOREM 1. *If L is a locally compact, locally convex topological lattice of finite breadth and U is a neighborhood of a point x in L, then there exist two elements y and z in L and a neighborhood V of x such that $V \subset [y, z] \subset U$.*

Proof. Choose neighborhoods U_0, U_1 and U_2 of x such that U_0 and U_2 are convex, U_1^* compact, and $U_0 \subset U_1^* \subset U_2 \subset U$. Again we can choose two neighborhood U_3 and U_4 of x such that $U_3 \wedge \cdots \wedge U_3$ (n times) $\subset U_0$ and $U_4 \vee \cdots \vee U_4$ (n times) $\subset U_0$, where n is the breadth of L. Setting $V = U_3 \cap U_4$ we consider the sublattice W of L generated by V. Since every element w of W can be expressed as a lattice-polynomial of finitely many elements $x_1, x_2, \cdots, x_m$ of V, we have $\inf x_i \leqq w \leqq \sup x_i$. Suppose $m > n$. By definition of breadth we can choose at most n elements x_i' from those x_i's such that $\inf x_i = \inf x_i'$. Thus $\inf x_i \in U_0$. Similarly, $\sup x_i \in U_0$. Clearly $W \subset U_0$ and $W^* \subset U_1^*$. Since W^* is a compact sublattice, W^* has a maximal element z and a minimal element y. Now consider the smallest convex subset $C(W^*) = (W^* \wedge L) \cap (W^* \vee L)$ containing W^* in L (see [1]). It is easy to see that $C(W^*) = [y, z]$. And $V \subset [y, z] \subset U_2 \subset U$. The proof is complete.

Since compactness implies local convexity in a topological lattice, the distributivity hypothesis in Theorem 3 in [7] is not necessary.

It is remarked that the hypothesis of finite breadth in Theorem 1 can be replaced by finite dimension. The author, however, does not know how to obtain this result without using connectedness. For example, the space 2^X(X is an infinite set) has infinite breadth, but has zero dimension. And we note that the 2^X is Hausdorff in its interval topology [10]. (See [4], Problem 81).

A topological lattice is *chain-wise* connected if and only if for each pair of elements x and y with $x \leqq y$ there is a closed connected chain from x to y. It is well known [12] that a locally compact connected topological lattice is chain-wise connected.

We shall show that the hypothesis of distributivity in Anderson's result ([2], Corollary 1) can be replaced by modularity. The proof is essentially the same as in [2].

LEMMA 4. *If L is a locally compact connected modular topological lattice, then the breadth of L is less than or equal to the codimension of L.*

Proof. Suppose the codimension of L is n. If the breadth of L is $\nleqq n$, then L contains an $n+1$ element subset A, say $A = \{x_1, \cdots, x_{n+1}\}$, such that $\inf A \neq \inf B$ for any proper subset B of A. Let $b_i = \inf(A \backslash x_i)$, $i = 1, 2, \cdots, n+1$, and let $a = \inf A$. Then $b_i \neq a$, $i = 1, \cdots, n+1$, and $b_i \neq b_j\,(i \neq j)$. Let I_i be the closed interval $[a, b_i]$, $i = 1, 2, \cdots, n+1$. Now consider two mappings

$$f\colon I_1 \times \cdots \times I_{n+1} \rightarrow I_1 \vee \cdots \vee \mathrm{I}_{n+1} \subset L$$

defined by $f(a_1, \cdots, a_{n+1}) = a_1 \vee \cdots \vee a_{n+1}$, and $g\colon I_1 \vee \cdots \vee I_{n+1} \rightarrow I_1 \times \cdots \times I_{n+1}$ defined by $g(a_1 \vee \cdots \vee a_{n+1}) = (b_1 \wedge (a_1 \vee \cdots \vee a_{n+1}), \cdots, b_{n+1} \wedge (a_1 \vee \cdots \vee a_{n+1}))$. Then clearly f and g are well defined and continuous. Furthermore, $f^{-1} = g$, because by modularity we have

$$\begin{aligned} a_1 &\leqq b_1 \wedge (a_1 \vee \cdots \vee a_{n+1}) = a_1 \vee (b_1 \wedge (a_2 \vee \cdots \vee a_{n+1})) \\ &\leqq a_1 \vee (b_1 \wedge (b_2 \vee \cdots \vee b_{n+1})) \leqq a_1 \vee (b_1 \wedge x_1) = a_1 \vee a = a_1\,, \end{aligned}$$

and hence $b_1 \wedge (a_1 \vee \cdots \vee a_{n+1}) = a_1$, and similarly for $i = 2, \cdots, n+1$.

On the other hand, since such I_i is locally compact and connected in its relative topology, I_i contains a nondegenerate compact connected chain C_i, $i = 1, 2, \cdots, n+1$. The subset $C_1 \times \cdots \times C_{n+1}$ of $I_1 \times \cdots \times I_{n+1}$ has codimension $n+1$ [6]. Hence, the codimension of the closed subset $f(C_1 \times \cdots \times C_{n+1})$ of L is $n+1$. We thus have a contradiction.

LEMMA 5. *If (L, T) is a compact topological lattice of finite breadth, then $I(L)$ is Hausdorff.*

Proof. For two distinct elements x and y of L, choose T-neighborhoods U and V of x and y, respectively, such that $U \cap V = \square$. For each element z of $L \backslash \{x, y\}$, choose *a* T-neighborhood W of z such that $W \cap \{x, y\} = \square$. By Theorem 1, we can find T-neighborhoods U', V' and W' of x, y and z, respectively, and closed intervals $[x_1, x_2]$, $[y_1, y_2]$ and $[z_1, z_2]$ such that $U' \subset [x_1, x_2] \subset U$, $V' \subset [y_1, y_2] \subset V$ and $W' \subset [z_1, z_2] \subset W$. Clearly the family $\mathscr{W} = \{U', V', W' \mid z \in L \backslash \{x, y\}\}$ is an open covering of L. So there is a finite sub-family of $\mathscr{W}$ which covers L. Therefore, there is a finite family of closed intervals whose union is L such that no interval contains both x and y. It follows by Proposition 1 in [10] that $I(L)$ is Hausdorff.

Summarizing Lemmas 4, 5 and 3, we have the following main theorem:

THEOREM 2. *If (L, T) is a compact, connected, modular topological lattice of finite codimension, then $I(L) = C(L) = T = O(L)$.*

COROLLARY 1. *If (L, T) is a compact topological lattice of finite breadth, then $I(L) = C(L) = T = O(L)$.*

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