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INEQUALITIES FOR POLYNOMIALS WITH A PRESCRIBED ZERO

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Let $\mathscr{P}_n$ denote the linear space of polynomials $p(z) = \sum_{k=0}^{n} a_k z^k$ of degree at most n. There are various ways in which we can introduce norm ($\|\ \|$) in $\mathscr{P}_n$. Given β let $\mathscr{P}_{n,\beta}$ denote the subspace consisting of those polynomials which vanish at β. Then how large can $\| p(z)/(z-\beta) \|$ be if $p(z) \in \mathscr{P}_{n,\beta}$ and $\| p(z) \| = 1$? This general question does not seem to have received much attention. Here the problem is investigated when (i) $\| p(z) \| = \max_{|z| \leqslant 1} | p(z) |$, (ii) $\| p(z) \| = (1/2\pi \int_0^{2\pi} | p(e^{i\theta}) |^2 d\theta)^{1/2}$.

It was shown by Rahman and Mohammad [1] that if $p(z) \in \mathscr{P}_{n,1}$ and $\max_{|z| \leqslant 1} | p(z) | \leqq 1$ then

$$\max_{|z| \leqslant 1} | p(z)/(z-1) | \leqq n/2 . \tag{1}$$

We observe that if $p(z) \in \mathscr{P}_{n,\beta}$ and $\max_{|z| \leqslant 1} | p(z) | = 1$ then $\max_{|z| \leqslant 1} | p(z)/(z-\beta) |$ can be greater than $n/2$ if β is arbitrary. For $n = 1$ we may simply take $p(z) = z$. When $n > 1$ we consider the polynomial

$$p(z) = (n/2)(n^2-1)^{-1/2}(1+z+z^2+\cdots+z^{n-1})(z-1+2n^{-2}) .$$

If $z = e^{i\theta}$ then for $\cos\theta \leqq 1 - 2n^{-2}$

$$| p(z) | \leqq (1/2) | (1+z+z^2+\cdots+z^{n-1})(z-1) | \leqq 1 ,$$

and also for $\cos\theta \geqq 1 - 2n^{-2}$

$$| p(z)) | \leqq n(n^2-1)^{-1/2}(n/2) | z-1+2n^{-2} | \leqq 1$$

while

$$\max_{|z|=1} | p(z)/(z-1+2n^{-2}) | = (n^2/2)(n^2-1)^{-1/2} > \frac{n}{2} .$$

We note howevever that if $p(z) \in \mathscr{P}_{n,\beta}$ and $\max_{|z| \leqslant 1} | p(z) | \leqslant 1$, then

$$\max_{|z|=1} | p(z)/(z-\beta) | \leqq (n+1)/2 . \tag{2}$$

Proof of inequality (2). Without loss of generality we may assume β to be real and nonnegative. Put $p(z) = (z-\beta)q(z)$ and write

$p^*(z) = (z-1)q(z)$. Then

$$(3) \qquad |p^*(e^{i\theta})/p(e^{i\theta})| = |(e^{i\theta}-1)/(e^{i\theta}-\beta)| \leqq 2/(1+\beta)$$

which gives us

$$(4) \qquad \max_{|z|=1} |p^*(z)| \leqq 2\,(1+\beta)^{-1} \max_{|z|=1} |p(z)| \,.$$

From inequalities (1) and (4) we obtain

$$(5) \qquad \begin{aligned} \max_{|z|=1} |q(z)| &\leqq (n/2) \max_{|z|=1} |p^*(z)| \leqq n(1+\beta)^{-1} \max_{|z|=1} |p(z)| \\ &\leqq \frac{n+1}{2} \max_{|z|=1} |p(z)| \end{aligned}$$

provided $\beta \geqq (n-1)/(n+1)$.

For $\beta \leqq (n-1)/(n+1)$ we have

$$(6) \qquad |q(e^{i\theta})| = |p(e^{i\theta})/(e^{i\theta}-\beta)| \leqq (1-\beta)^{-1} |p(e^{i\theta})| \leqq \frac{n+1}{2} |p(e^{i\theta})|$$

and hence

$$(7) \qquad \max_{|z|=1} |q(z)| \leqq \frac{n+1}{2} \max_{|z|=1} |p(z)| \,.$$

This completes the proof of inequality (2). Unfortunately, with the exception of $n = 1$ the bound $(n+1)/2$ does not appear to be sharp.

We now examine the L^2 analogue of the above problem. We prove the following theorem.

THEOREM. *If $p(z)$ is a polynomial of degree n such that $p(\beta) = 0$ where β is an arbitrary nonnegative number then*

$$(8) \quad \int_0^{2\pi} |p(e^{i\theta})/(e^{i\theta}-\beta)|^2\, d\theta \leqq \left(1+\beta^2-2\beta\cos\left(\frac{\pi}{n+1}\right)\right)^{-1} \int_0^{2\pi} |p(e^{i\theta})|^2\, d\theta \,.$$

Proof of the theorem. Let us write

$$(9) \qquad p(z)/(z-\beta) = \alpha_{n-1} z^{n-1} + \alpha_{n-2} z^{n-2} + \cdots + \alpha_1 z + \alpha_0,\ \alpha_{n-1} \neq 0 \,.$$

Then

$$(10) \qquad p(z) = \alpha_{n-1} z^n + (\alpha_{n-2}-\beta\alpha_{n-1})z^{n-1} + \cdots + (\alpha_0-\beta\alpha_1)z - \beta\alpha_0 \,.$$

We therefore have to consider the ratio

$$(11) \qquad R \equiv \left(\sum_{\nu=0}^{n-1} |\alpha_\nu|^2\right) \Big/ \left(|\alpha_{n-1}|^2 + \sum_{\nu=0}^{n-1} |\alpha_{\nu-1} - \beta\alpha_\nu|^2 + \beta\,|\alpha_0|^2\right) .$$

Now

$$R \leqq \left(\sum_{\nu=1}^{n-1} |\alpha_\nu|^2\right) \Big/ \left((1+\beta^2)\sum_{\nu=0}^{n-1} |\alpha_\nu|^2 - 2\beta \sum_{\nu=1}^{n-1} |\alpha_\nu| \, |\alpha_{\nu-1}|\right)$$
$$= 1/\left(1+\beta^2 - 2\beta\left(\sum_{\nu=1}^{n-1} |\alpha_\nu| \, |\alpha_{\nu-1}|\right) \Big/ \left(\sum_{\nu=0}^{n-1} |\alpha_\nu|^2\right)\right).$$

Thus we require the maximum of the function

$$f(|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{n-1}|) = \left(\sum_{\nu=1}^{n-1} |\alpha_\nu|^2\right)^{-1} \left(\sum_{\nu=1}^{n-1} |\alpha_\nu| \, |\alpha_{\nu-1}|\right) \tag{12}$$

with respect to $|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{n-1}|$. It is clear that the maximum is less than 1.

If for some ν, $\alpha_\nu = 0$ and j is the smallest positive integer such that $\alpha_{\nu-j}$, $\alpha_{\nu+j}$ are not both zero (α_{-1}, α_{-2}, etc... are to be interpreted as zero) then

$$\begin{aligned} &f(|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{\nu-1}|, 0, |\alpha_{\nu+1}|, \cdots, |\alpha_{n-1}|) \\ &\leqq f(|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{\nu-1}|, |\alpha'_\nu|, |\alpha_{\nu+1}|, \cdots, |\alpha_{n-1}|) \end{aligned} \tag{13}$$

provided

$$|\alpha'_\nu| \leqq (|\alpha_{\nu-j}| + |\alpha_{\nu+j}|)/f(|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{\nu-1}|, 0, |\alpha_{\nu+1}|, \cdots, |\alpha_{n-1}|).$$

This implies that the maximum is not attained when one or more of the numbers $|\alpha_\nu|$ are zero.

On the other hand if one or more of the numbers $|\alpha_\nu|$ are allowed to be arbitrarily large the function $f(|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{n-1}|)$ is bounded above by $(n-1)/n$.

Consider now the partial derivatives of f with respect to the variables $|\alpha_\nu|$. For a local maximum we have to find $|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{n-1}|$ such that

$$\begin{cases} \left(\sum_{\nu=0}^{n-1} |\alpha_\nu|^2\right) \dfrac{\partial f}{\partial |\alpha_0|} = |\alpha_1| - 2f|\alpha_0| = 0, \\ \left(\sum_{\nu=0}^{n-1} |\alpha_\nu|^2\right) \dfrac{\partial f}{\partial |\alpha_\mu|} = |\alpha_{\mu+1}| + |\alpha_{\mu-1}| - 2f|\alpha_\mu| = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \mu = 1, 2, \cdots, n-2, \\ \left(\sum_{\nu=0}^{n-1} |\alpha_\nu|^2\right) \dfrac{\partial f}{\partial |\alpha_{n-1}|} = |\alpha_{n-2}| - 2f|\alpha_{n-1}| = 0. \end{cases} \tag{14}$$

Let us suppose that the required local maximum is λ. Since $\lambda < 1$ we write $\lambda = \cos\gamma$ ($\gamma \neq 0$). Then from the first $n-1$ equations of the system (14) we obtain

$$|\alpha_\mu| = U_\mu(\lambda)|\alpha_0|, \qquad \mu = 1, 2, \cdots, n-1 \tag{15}$$

where $U_\mu(\lambda) = (\sin(\mu+1)\gamma)/(\sin\gamma)$ is the Chebyshev polynomial of the second kind of degree μ in λ. Using equations (15) the last equation of the system (14) gives us

$$\sin(n+1)\gamma = 0 . \tag{16}$$

The only solution of (16) which is consistent with all the numbers $|\alpha_\nu|$ being nonnegative is $\gamma = \pi/(n+1)$. Hence

$$\lambda = \cos\left(\frac{\pi}{n+1}\right) .$$

Since $\cos(\pi/(n+1)) \geqq (n-1)/n$ the required maximum of the function $f(|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{n-1}|)$ is $\cos(\pi/(n+1))$. This immediately leads to the inequality (8).

We note that the polynomial

$$p(z) = (z-\beta)\sum_{\nu=0}^{n-1} U_\nu\left(\cos\left(\frac{\pi}{n+1}\right)\right) z^\nu$$

is extremal.

References

1. Q. I. Rahman and Q. G. Mohammad, *Remarks on Schwarz's lemma,* Pacific J. Math. **23** (1967), 139-142.

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