

EXISTENCE AND GEOMETRY OF A FREE BOUNDARY PROBLEM FOR THE HEAT EQUATION

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A periodic (in t) free boundary problem for the one-dimensional heat equation is examined. The existence and regularity of the (unique) solution is established and the geometry of the free boundary is shown to be no more complicated than the geometry of the fixed boundary.

0. Introduction. Free boundary problems (and moving boundary problems) arise in a large variety of contexts and have been studied for over one hundred years. There is an extensive literature on many aspects of free boundary problems including the existence, uniqueness, regularity, and stability of solutions and the qualitative properties of the free boundary ([14], [18]). Many applications, especially to continuum mechanics, have been considered since the work in the 1860's of Helmholtz and of Kirchhoff on fluid jets and of Neuman on the Stefan problem ([9], [10], [12], [15], [17], [27], [30]). In addition, the approximation of solutions and of free boundaries using numerical methods is well established (e.g. [16]).

Among parabolic problems, Stefan problems have generated a great deal of interest and an extensive literature (e.g. [11], [30]). We will examine a periodic free boundary problem for the one-dimensional heat equation which might be considered as a free boundary problem of Stefan type ([32]) in which the known (or "fixed") boundary varies periodically in time and the free boundary is determined by a prescribed flux condition (rather than a phase-change condition). In addition, this can be viewed as a model for certain processes involving chemical reactions. Alternatively, our problem might be viewed as a model problem in which techniques which have proven useful for certain elliptic free boundary problems (e.g. [1], [2], [5]) are applied to a particular parabolic problem.

We will use a trial-free-boundary approach based on an operator method to establish the existence of a solution to our free boundary problem. Trial-free-boundary methods have been used for over 70 years with success, as illustrated, for example, by the work of Cryer

([16]). Our solution will be obtained as the limit of the fixed points of a sequence of contracting operators (similar to [2]) and the trial-free-boundary method based on this limiting process can be shown to converge (albeit slowly) to a solution as in [5].

An alternative approach for establishing the existence of a solution is to use a variational method. A natural functional to minimize in this case is the sum of the area and the heat flow (over one period). However, a minimizer of this functional does not lead to a solution of our problems. While another variational approach might be successful, we are unaware of an existence proof for our problem based on variational methods.

We will establish the regularity and, using (a variant of) the Lavrentiev principle, the uniqueness of the solution. We will then examine geometric properties of the free boundary. For certain harmonic or minimal surface free boundary problems, curves of constant gradient direction have been used to relate geometric properties of the free boundary to geometric properties of the fixed boundary (e.g. [4], [6]); these curves are related to the “nodal lines” of the Courant nodal line theorem ([13]) as well as to later work (e.g. [20], [25], [28]). We will use such curves to prove that the geometry of the free boundary is no more complicated than the geometry of the fixed boundary. To the best of our knowledge, the only previous application of this idea to parabolic free boundary problems is in the work of Friedman and Jensen ([19]).

1. Preliminaries. Given periodic functions $X^*(t)$ and $X(t)$ with period τ and, say, $X(t) > X^*(t)$ for each $t \in \mathfrak{R}$, let us denote

$$\begin{aligned}\Gamma^* &= \{(X^*(t), t) : t \in \mathfrak{R}\}, \\ \Gamma &= \{(X(t), t) : t \in \mathfrak{R}\}, \quad \text{and} \\ \Omega &= \{(x, t) : X^*(t) < x < X(t), t \in \mathfrak{R}\}.\end{aligned}$$

Let us define $U = U(\Gamma^*, \Gamma) \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to be the τ -periodic (in t) solution of the Dirichlet problem

$$\begin{aligned}U_t &= U_{xx} \quad \text{in } \Omega, \\ U(X^*(t), t) &= 1 \quad t \in \mathfrak{R}, \\ U(X(t), t) &= 0 \quad t \in \mathfrak{R}.\end{aligned}$$

We are interested in the following

Free Boundary Problem. Given Γ^* as above, find Γ (as above) such that if $U = U(\Gamma^*, \Gamma)$ then $U_x \in C^0(\Omega \cup \Gamma)$ and $U_x(X(t), t) = -1$.

We will prove that if $X^*(t)$ is Lipschitz continuous, then the free boundary problem above has a solution $\Gamma = \{(X(t), t) : t \in \mathfrak{R}\}$ with $X(t)$ Lipschitz continuous, this solution is unique, and the “geometry” of Γ is no more complicated than that of Γ^* .

2. Existence. Suppose $X^*(t)$ is a τ -periodic, Lipschitz continuous function with Lipschitz constant α . Let $K = K(\alpha, \tau)$ denote the set of τ -periodic, Lipschitz continuous functions $X(t)$ with Lipschitz constant α such that $X(t) > X^*(t)$ for $t \in \mathfrak{R}$.

NOTATION. At times, we will write $\Gamma \in K$ if $\Gamma = \Sigma(X)$ and $X \in K$, where

$$\Sigma(X) = \{(X(t), t) : t \in \mathfrak{R}\}.$$

For $X_1, X_2 \in K$ (resp. $\Gamma_1, \Gamma_2 \in K$ with $\Gamma_k = \Sigma(X_k)$, $k = 1, 2$), we define the form

$$\|X_1 - X_2\| = \|\Gamma_1 - \Gamma_2\| = \max\{|X_1(t) - X_2(t)| : t \in \mathfrak{R}\}.$$

For any $X \in K$, $\varepsilon \in (0, 1)$, and $\Gamma = \Sigma(X)$, let

$$\begin{aligned} \Phi_\varepsilon(\Gamma) &= \{(x, t) : U(x, t) = \varepsilon, t \in \mathfrak{R}\}, \\ \Psi_\varepsilon(\Gamma) &= \{(x + \varepsilon, t) : (x, t) \in \Gamma\}, \quad \text{and} \\ T_\varepsilon(\Gamma) &= \Psi_\varepsilon(\Phi_\varepsilon(\Gamma)), \end{aligned}$$

where $U = U(\Gamma^*, \Gamma)$. We let ϕ_ε , ψ_ε , and t_ε be defined on K so that $\Phi_\varepsilon(\Gamma) = \Sigma(\phi_\varepsilon(X))$, $\Psi_\varepsilon(\Gamma) = \Sigma(\psi_\varepsilon(X))$, and $T_\varepsilon(\Gamma) = \Sigma(t_\varepsilon(X))$, where $\Gamma = \Sigma(X)$ and $X \in K$. Notice that

$$U(\phi_\varepsilon(X)(t), t) = \varepsilon,$$

$\psi_\varepsilon(X)(t) = X(t) + \varepsilon$, and $t_\varepsilon(X) = \psi_\varepsilon(\phi_\varepsilon(X))$ for $X \in K$.

LEMMA 1. $\phi_\varepsilon : K \rightarrow K$.

Proof. Let $X \in K$ and $\Gamma = \Sigma(X)$. Clearly $\phi_\varepsilon(X)$ is τ -periodic, since U is τ -periodic in t , and $\phi_\varepsilon(X)(t) > X^*(t)$, since $X(t) > X^*(t)$, $U(X^*(t), t) = 1$, and $U(X(t), t) = 0$. Let $|\alpha_0| \geq \alpha$, $h > 0$,

$$\begin{aligned} \Gamma_h &= \Gamma + (\alpha_0 h, h) \equiv \{(x + \alpha_0 h, t + h) : (x, t) \in \Gamma\}, \\ \Gamma_h^* &= \Gamma^* + (\alpha_0 h, h), \quad \Omega_h = \Omega + (\alpha_0 h, h), \quad \text{and} \\ U_h(x, t) &= U(x - \alpha_0 h, t - h) \end{aligned}$$

for $(x, t) \in \Omega_h$. Notice that $U_h(x, t) \geq U(x, t)$ in $\Omega \cap \Omega_h$, since $U_h = 1 \geq U$ on Γ_h^* , $U_h \geq 0 = U$ on Γ , and U and U_h both satisfy $u_t = u_{xx}$ in $\Omega_h \cap \Omega$. This means that U is (weakly) decreasing on rays $x = x_0 + \alpha_0 t$, $x \geq x_0$ (and strictly decreasing if $|\alpha_0| > \alpha$). It follows that if $(x_0, t_0) \in \Phi_\varepsilon(\Gamma)$, then $\Phi_\varepsilon(\Gamma)$ lies to the left of the cone $\{(x + x_0, t + t_0) : |t| \leq |\alpha_0 x|, x > 0\}$, since otherwise U could not be monotonic on the ray $x = x_0 + \alpha_0(t - t_0)$, $x \geq x_0$. Thus $\phi_\varepsilon(x)$ is Lipschitz continuous with Lipschitz constant α . \square

REMARK. If ∇U were continuous on $\bar{\Omega}$, we could simply have considered $\frac{\partial U}{\partial \nu}$ with $\nu = (\alpha_0, 1)/(\sqrt{\alpha_0^2 + 1})$.

LEMMA 2. *Suppose Γ^* is a Lipschitz-continuously differentiable curve. Then $U_x < 0$ on Γ^* .*

Proof. This follows from the Hopf boundary point lemma for parabolic equations (e.g. [29], p. 170).

For two curves Γ_1 and Γ_2 with $\Gamma_k = \{(X_k(t), t) : t \in \mathfrak{R}\}$, let us say $\Gamma_1 < \Gamma_2$ if $X_1(t) < X_2(t)$ for all $t \in \mathfrak{R}$.

LEMMA 3. *There exist curves $\Gamma_1, \Gamma_2 \in K$ with $\Gamma_1 < \Gamma_2$ such that $T_\varepsilon(\Gamma_1) > \Gamma_1$ and $T_\varepsilon(\Gamma_2) < \Gamma_2$ for $\varepsilon \in (0, 1)$ sufficiently small.*

Proof. Choose $\Gamma_1 = \Gamma^* + (\sigma, 0) = \{(x^*(t) + \sigma, t) : t \in \mathfrak{R}\}$ with $\sigma > 0$ small enough that $|\partial U_1 / \partial x| \gg 1$ on Γ_1 , where $U_1 = U(\Gamma^*, \Gamma_1)$, and let $\Gamma_2 = \{(x_0, t) : t \in \mathfrak{R}\}$ for x_0 sufficiently large. \square

We define $\tilde{K} = \tilde{K}(\Gamma^*, \Gamma_1, \Gamma_2)$ by

$$\tilde{K} = \{X \in K : \Gamma_1 \leq \Sigma(X) \leq \Gamma_2\}$$

and write $\Gamma \in \tilde{K}$ if $\Gamma = \Sigma(X)$ and $X \in \tilde{K}$. Notice that if $\varepsilon \in (0, 1)$ is sufficiently small, Lemma 3 implies that $T_\varepsilon : \tilde{K} \rightarrow \tilde{K}$. We may now state an existence theorem for fixed points of T_ε .

THEOREM 1. *Let Γ^* be a Lipschitz-continuously differentiable curve and suppose $U = U(\Gamma^*, \Gamma)$ is continuously differentiable in $\Omega \cup \Gamma^*$. Then, for $\varepsilon \in (0, 1)$ sufficiently small, $t_\varepsilon : \tilde{K} \rightarrow \tilde{K}$ is a contraction. Thus, for each small $\varepsilon > 0$, there exists a unique “fixed point” $\Gamma_\varepsilon \in \tilde{K}$ of T_ε .*

Proof. Using Lemma 2 and the maximum principle, Φ_ε can be shown to be contracting as in [2]. Since Φ_ε is contracting, so is T_ε . □

LEMMA 4 (*Uniform Modulus of Continuity near Γ*). Suppose $\Gamma \in \tilde{K}$ and $V \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is a solution of $V_t = V_{xx}$ in $\Omega = \Omega(\Gamma^*, \Gamma)$, $V = 0$ on Γ , and $V = 1$ on Γ^* . If V is periodic in t with period τ , then

$$V(x, y) \leq C \text{dist}((x, t), \Gamma)$$

for $(x, t) \in \Omega$, where $C = C(\Gamma^*, \Gamma_1, \Gamma_2, \alpha) > 0$ is independent of Γ .

Proof. Let $\alpha > 0$ denote the Lipschitz constant of K . Let $(X(t_0), t_0) \in \Gamma$ and set

$$\gamma = \{(X(t_0) - \alpha(t - t_0), t) : t \in \mathfrak{R}\}.$$

Let $\gamma^* = \gamma + (X^*(t_0) - X(t_0), 0)$ and let $\omega = \omega(t_0, \alpha)$ be the region between γ^* and γ . Since the Lipschitz constant of $X^*(t)$ and $X(t)$ is $(\leq) \alpha$, $\Gamma \cap \{(x, t) : t < t_0\}$ lies to the left of γ and $\Gamma^* \cap \{(x, t) : t < t_0\}$ lies to the left of γ^* . Let $w(x, t)$ be the solution of $w_t = w_{xx}$ in ω , $w = 0$ on γ , and $w = 1$ on γ^* . Using the fact that $0 < V < 1$ in Ω and $0 < w < 1$ in ω , it is easily seen that $V \leq w$ on the parabolic boundary of the region $\Omega \cap \omega \cap \{(x, t) : t \leq t_0\}$. Therefore, the maximum principle implies that $V \leq w$ in $\Omega \cap \omega \cap \{(x, t) : t \leq t_0\}$. Now

$$w(x, t) = \phi(x + \alpha t)$$

for some $\phi \in C^2([X^*(t_0) + \alpha t_0, X(t_0) + \alpha t_0])$. Then

$$\phi''(s) = \alpha \phi'(s)$$

and so $\phi(s) = Ae^{\alpha s} + B$. Since $\phi(X^*(t_0) + \alpha t_0) = 1$ and $\phi(X(t_0) + \alpha t_0) = 0$, we see that $\phi(s) = (e^{\alpha s_1} - e^{\alpha s}) / (e^{\alpha s_1} - e^{\alpha s_0})$, where $s_0 = X^*(t_0) + \alpha t_0$ and $s_1 = X(t_0) + \alpha t_0$. Thus

$$w(x, t) = \frac{e^{\alpha(x_1 + \alpha t_0)} - e^{\alpha(x + \alpha t)}}{e^{\alpha(x_1 + \alpha t_0)} - e^{\alpha(x_0 + \alpha t_0)}},$$

where $x_0 = X^*(t_0)$ and $x_1 = X(t_0)$, and, in particular,

$$w(x, t_0) = \frac{e^{\alpha x_1} - e^{\alpha x}}{e^{\alpha x_1} - e^{\alpha x_0}}$$

for $x \in [x_0, x_1]$. Then $V(x_1 - h, t_0) \leq w(x_1 - h, t_0)$ and

$$\begin{aligned} w(x_1 - h, t_0) &= \frac{e^{\alpha x_1}}{e^{\alpha x_1} - e^{\alpha x_0}}(1 - e^{-\alpha h}) \\ &= \frac{e^{\alpha x_1}}{e^{\alpha x_1} - e^{\alpha x_0}}\alpha h + O((\alpha h)^2) \\ &\leq C_1 h \end{aligned}$$

for $h > 0$ sufficiently small and $C_1 = e^{\alpha x_1}/(e^{\alpha x_1} - e^{\alpha x_0}) + 1$, for example. Now

$$\text{dist}((x, t_0), \Gamma) \geq \text{dist}((x, t_0), \gamma)$$

and

$$\text{dist}((x_1 - h, t_0), \gamma) = \frac{1}{\sqrt{1 + \alpha^2}}h,$$

so

$$V(x - h, t_0) \leq C \text{dist}((x, t_0), \Gamma),$$

where $C = (1 + \alpha^2)^{-1/2}C_1$. Notice that $x_1 \geq x_0 + \sigma$ (recall $\Gamma_1 = \Gamma^* + (\sigma, 0)$) and so C_1 has a finite upper bound independent of Γ . □

Construction of a candidate for a solution. For some $\varepsilon_0 > 0$, T_ε has a “fixed point” $\Gamma_\varepsilon \in \tilde{K}$ for all $0 < \varepsilon \leq \varepsilon_0$ provided Γ^* is sufficiently regular (Theorem 1). Since each $\Gamma \in \tilde{K}$ satisfies $\Gamma_1 \leq \Gamma \leq \Gamma_2$ and $X_1(t) \leq X(t) \leq X_2(t)$, $t \in \mathfrak{A}$, where $\Gamma = \Sigma(X)$, the Arzelà-Ascoli Theorem implies that there exists a sequence $\{\varepsilon_n\}$ converging to 0 and $X^0 \in \tilde{K}$ such that $X_{\varepsilon_n} \rightarrow X^0$ in $C^0(\mathfrak{A})$ and $\Gamma_n \rightarrow \Gamma^0$ uniformly as $n \rightarrow \infty$, where $\Gamma^0 = \Sigma(X^0)$ and $\Gamma_n = \Sigma(X_{\varepsilon_n})$. Let us denote $U(\Gamma^*, \Gamma^0)$ by U^0 , $\Omega(\Gamma^*, \Gamma^0)$ by Ω^0 , $U(\Gamma^*, \Gamma_n)$ by U_n , and $\Omega(\Gamma, \Gamma_n)$ by Ω_n .

THEOREM 2. *Suppose Γ^* is sufficiently regular that the hypotheses of Theorem 1 hold. Then $U_x^0 \in C^0(\overline{\Omega^0})$ and $U_x^0 = -1$ on Γ^0 . In particular, Γ^0 solves the free boundary problem. Further, this solution is unique.*

Proof. Notice that $\|\Gamma_n - \Gamma^0\|_\infty \equiv \max\{|X_{\varepsilon_n}(t) - X^0(t)| : t \in \mathfrak{A}\}$ goes to zero as $n \rightarrow \infty$. Using Lemma 4 and the fact that $U_n = U^0 = 1$ on Γ^* , we see that $U_n \rightarrow U^0$ uniformly on compact subsets of $\Omega^0 \cap \Gamma^*$.

Let

$$V_n(x, t) = \frac{U_n(x, t) - U_n(x - \varepsilon_n, t)}{\varepsilon_n}$$

in $\widehat{\Omega}_n = \Omega(\Gamma^* + (\varepsilon_n, 0), \Gamma_n)$. Then $V_n = (0 - \varepsilon_n)/\varepsilon_n = -1$ on Γ_n , since $T_\varepsilon(\Gamma_\varepsilon) = \Gamma_\varepsilon$ and so $\phi_\varepsilon(X_\varepsilon)(t) = X_\varepsilon(t) - \varepsilon$. Also notice $(V_n)_t = (V_n)_{xx}$ in $\widehat{\Omega}_n$ and, from the mean-value theorem,

$$V_n(x, t) = \frac{\partial}{\partial x}(U_n(\lambda_n(t), t))$$

for some $\lambda_n(t) \in (x - \varepsilon_n, x)$. Now suppose $(x, t) \in \Omega^0$ and n is large enough that $(x, t) \in \widehat{\Omega}_n$. Since, as $n \rightarrow \infty$, U_n converges uniformly on compact subsets of Ω^0 to U^0 , we see that

$$\frac{\partial}{\partial x}(U_n(\lambda_n(t), t)) \rightarrow \frac{\partial}{\partial x}(U^0(x, t)) \quad \text{as } n \rightarrow \infty$$

(e.g. [29]) and hence

$$V_n(x, t) \rightarrow \frac{\partial}{\partial x}(U^0(x, t))$$

as $n \rightarrow \infty$.

Let $\Omega_p^0 = \{(x, t) \in \Omega : U^0(x, t) < \frac{1}{2}\}$ and let Γ_p^* denote the left boundary of Ω_p^0 . Let $V \in C^2(\Omega_p^0) \cap C^0(\overline{\Omega_p^0})$ satisfy $V_t = V_{xx}$ in Ω_p^0 , $V = -1$ on Γ^0 , $V = U_x$ on Γ_p^* , and $V(x, t)$ is τ -periodic in t . Since $V_n = -1$ on Γ_n , a result analogous to Lemma 4 implies

$$|V(x, t) - V_n(x, t)| \leq k_n$$

for (x, t) an element of the right boundary of $\Omega^0 \cap \Omega_n$, where $k_n \rightarrow 0$ as $n \rightarrow \infty$. Since $V(x, t) = U_x^0(x, t)$ and $V_n(x, t) \rightarrow U_x^0(x, t)$ as $n \rightarrow \infty$, for $(x, t) \in \Gamma_p^*$, we see that $V_n(x, t) \rightarrow V(x, t)$ as $n \rightarrow \infty$, for $(x, t) \in \Gamma_p^*$. Thus

$$V_n \rightarrow V$$

uniformly on compacta in Ω_p^0 and so $V = U_x^0$. Since $V(x, t) \rightarrow -1$ as $(x, t) \in \Omega^0$ approaches Γ^0 , $U_x = -1$ on Γ^0 .

Turning to the uniqueness question, we will show, using an adaptation of the Lavrentiev principle ([23], [24]) that the existence of two distinct solutions $\Gamma^0, \Gamma^1 \in K$ leads to a contradiction. Assuming $\Gamma^0 \geq \Gamma^1$ is false, let $\sigma > 0$ be the least number such that $\Gamma^1 \leq \Gamma^2 \equiv \Gamma^0 + (\sigma, 0)$. Then two applications of the maximum principle show that

$$U(\Gamma^*, \Gamma^1) < U(\Gamma^*, \Gamma^2) < U(\Gamma^* + (\sigma, 0), \Gamma^0 + (\sigma, 0))$$

throughout $\Omega(\Gamma^* + (\sigma, 0), \Gamma^1)$, where the inequalities reduce to equality at any point $p_0 \in \Gamma^1 \cap \Gamma^2$. Using the regularity results of §3 (which

apply to both Γ^0 and Γ^1) and the Hopf boundary point lemma, we see that

$$\begin{aligned} -1 &= U_x(\Gamma^*, \Gamma^1)(p_0) > U_x(\Gamma^* + (\sigma, 0), \Gamma + (\sigma, 0))(p_0) \\ &= U_x(\Gamma^*, \Gamma^0)(p_0 - (\sigma, 0)) = -1. \end{aligned} \quad \square$$

From now on, we will write Γ^0 as Γ .

3. Regularity of the free boundary. In this section, we will let $\Gamma^* = \{(X^*(t), t) : t \in \mathfrak{R}\}$, where X^* is a τ -periodic, Lipschitz continuous function with Lipschitz constant α and we will let $\Gamma \in K$ be the solution of the free boundary problem, whose existence follows from Theorem 2. We will let $U = U(\Gamma^*, \Gamma)$ and $\Omega = \Omega(\Gamma^*, \Gamma)$.

LEMMA 5. *The level curves of U are Lipschitz continuous with Lipschitz constant α .*

Proof. This follows from Lemma 1. □

LEMMA 6. *U_t and U_{xx} are uniformly bounded in Ω .*

Proof. Recall $U(X_\varepsilon(t), t) = \varepsilon$ for all $0 < \varepsilon \leq \varepsilon_0$. If we differentiate with respect to t , we obtain

$$U_x(X_\varepsilon(t), t)X'_\varepsilon(t) + U_t(X_\varepsilon(t), t) = 0$$

and so

$$|U_t(X_\varepsilon(t), t)| = |X'_\varepsilon(t)| |U_x(X_\varepsilon(t), t)| \leq \alpha |U_x(X_\varepsilon(t), t)|.$$

Now $U_x(x, t) \rightarrow -1$ as $(x, t) \rightarrow \Gamma$ and U_x is bounded in $\bar{\Omega}$, so $|U_t|$ is bounded in Ω . Since $U_{xx} = U_t$, U_{xx} is also bounded. □

LEMMA 7. *U_{xt} is uniformly bounded in $\Omega_p = \{(x, t) : U(x, t) < \frac{1}{2}\}$.*

Proof. We will assume that Γ^* is as regular as we wish, and prove the lemma in Ω , since otherwise we could replace U by $V(x, t) = 2U(x, t)$ and Ω by Ω_p and notice that the left boundary of Ω_p is smooth. Let us define

$$V_h(x, t) = U_x(x, t+h) - U_x(x, t)$$

in $\Omega_h = \Omega \cap (\Omega - (0, h))$. Notice that $\Omega_h \rightarrow \Omega$ as $h \rightarrow 0+$. Let $\Gamma_h = \Sigma(X_h)$ and $\Gamma_h^* = \Sigma(X_h^*)$, where $X_h(t) = \min\{X(t), X(t+h)\}$ and $X_h^*(t) = \max\{X^*(t), X^*(t+h)\}$. Then

$$|V_h(X_h(t), t)| = |U_x(X_h(t), t+h) - U_x(X_h(t), t)|.$$

Suppose, say, that $X(t+h) \geq X(t)$ and so $X_h(t) = X(t)$. Then

$$\begin{aligned} |V_h(X_h(t), t)| &= |U_x(X(t), t+h) - U_x(X(t), t)| \\ &= |U_x(X(t), t+h) + 1| \\ &= |U_x(X(t), t+h) - U_x(X(t+h), t+h)| \\ &= |U_{xx}(\lambda(t), t+h)| |X(t+h) - X(t)| \\ &\leq M\alpha|h| \end{aligned}$$

where $|U_{xx}| \leq M$ in Ω and $\lambda(t) \in (X(t), X(t+h))$. If $X(t+h) < X(t)$, a similar argument yields the same estimate. Now U is smooth in $\Omega \cup \Gamma^*$ (by our assumption of regularity of Γ^*) and so there exists M_1 such that $|U_{xt}| \leq M_1$ on $\{(x, t) : \frac{1}{6} < U(x, t) < 1\}$. Since $V_h(x, t)/h$ is the difference quotient for $U_{xt}(x, t)$, we see that for $M_2 > M_1$ and $h > 0$ sufficiently small,

$$|V_h(X_h^*(t), t)| \leq M_2|h|.$$

Since $(V_h)_t = (V_h)_{xx}$ in Ω_h and $|V_h(x, t)| \leq M_3|h|$ on $\partial(\Omega_h)$, where $M_3 = \max\{M\alpha, M_2\}$, we obtain

$$\left| \frac{U_x(x, t+h) - U_x(x, t)}{h} \right| \leq M_3 \quad \text{in } \Omega_h$$

for any h small enough. It follows that $|U_{xt}| \leq M_3$ in Ω . □

LEMMA 8. U_t is continuous on $\Omega \cup \Gamma$.

Proof. Let us begin by defining U_t on Γ . Set $f_\varepsilon = U_t(X(t) - \varepsilon, t)$ for $\varepsilon > 0$ small. Notice that $f_\varepsilon \in C^0(\mathfrak{R})$ and

$$\begin{aligned} |f_\varepsilon(t) - f_\beta(t)| &= |U_t(X(t) - \varepsilon, t) - U_t(X(t) - \beta, t)| \\ &\leq |U_{xt}(\lambda_{\varepsilon, \beta}(t))| |\varepsilon - \beta| \leq M_3|\varepsilon - \beta|, \end{aligned}$$

where $X(t) - \varepsilon < \lambda_{\varepsilon, \beta}(t) < X(t) - \beta$ if $\varepsilon > \beta$. Then $f_\varepsilon(t)$ converges uniformly as $\varepsilon \rightarrow 0+$ to a function $f \in C^0(\mathfrak{R})$. Let us define $U_t(X(t), t) = f(t)$.

Now we wish to show that $U_t \in C^0(\Omega \cup \Gamma)$. Let $(x_0, t_0) = (X(t_0), t_0) \in \Gamma$ and let $(x, t) \in \Omega$. Then

$$\begin{aligned} |U_t(x, t) - U_t(x_0, t_0)| &\leq |U_t(x_0, t_0) - U_t(X(t), t)| + |U_t(X(t), t) - U_t(x, t)| \\ &\leq |f(t_0) - f(t)| + M_3|x - X(t)| \\ &\leq \xi(t - t_0) + M_3(|x - x_0| + |x_0 - X(t)|) \\ &\leq \xi(t - t_0) + M_3|x - x_0| + M_3\alpha|t - t_0| \rightarrow 0 \end{aligned}$$

as $t \rightarrow t_0, x \rightarrow x_0$, where $\xi(\tau) \rightarrow 0$ as $\tau \rightarrow 0$. □

LEMMA 9. Γ is C^1 (i.e. $X \in C^1(\mathfrak{R})$).

Proof. Since $U(X_\varepsilon(t), t) = \varepsilon$, $X'_\varepsilon(t) = -U_t(X_\varepsilon(t), t)/U_x(X_\varepsilon(t), t)$. Since $U_x \in C^0(\Omega \cup \Gamma)$, $U_x = -1$ on Γ , and U_t is bounded and continuous, we have

$$X'_\varepsilon \rightarrow -U_t(X(t), t)$$

uniformly in t as $\varepsilon \rightarrow 0+$. Now

$$\begin{aligned} X(t+h) - X(t) &= \lim_{\varepsilon \rightarrow 0} (X_\varepsilon(t+h) - X_\varepsilon(t)) \\ &= \lim_{\varepsilon \rightarrow 0+} \int_t^{t+h} X'_\varepsilon(s) ds = - \int_t^{t+h} U_t(X(s), s) ds \end{aligned}$$

and so

$$X'(t) = \lim_{h \rightarrow 0} -\frac{1}{h} \int_t^{t+h} U_t(X(s), s) ds = -U_t(X(t), t).$$

Since $U_t \in C^0(\Omega \cup \Gamma)$, $X' \in C^0(\mathfrak{R})$. □

THEOREM 3. Γ is C^∞ and $U \in C^\infty(\Omega \cup \Gamma)$.

Proof. This follows from Lemma 9 and [22]. □

4. Geometry of the free boundary. As in the previous section, we let $\Gamma^* = \{(x^*(t), t) : t \in \mathfrak{R}\}$ and $\Gamma = \{(X(t), t) : t \in \mathfrak{R}\}$ be the solution of the free boundary problem, $U = U(\Gamma^*, \Gamma)$, and $\Omega = \Omega(\Gamma^*, \Gamma)$. We will compare the geometry of Γ to that of Γ^* .

Let $\nu = a\vec{i} + b\vec{j}$ with $a > 0$ and suppose that $p_0 = (x_0, t_0) = (X(t_0), t_0) \in \Gamma$ is a ν -minimum of Γ (i.e. $h(t) = (X(t), t) \cdot \nu$ has a strict local minimum at t_0 or, for some $\delta > 0$, if $|t - t_0| < \delta$, then $h(t) \geq h(t_0)$ and for some $t_1, t_2 \in (t_0 - \delta, t_0 + \delta)$ with $t_1 < t_0 < t_2$, $h(t_1) > h(t_0)$, and $h(t_2) > h(t_0)$). Then $U_\lambda(p_0) = 0$ if $\lambda = -b\vec{i} + a\vec{j}$. Also $U_\lambda(X(t), t) > 0$ if $t > t_0$ is near t_0 and $U_\lambda \leq 0$ if $t < t_0$ is near t_0 . Here $U_\lambda(x, t) = \nabla U \cdot \lambda = -bU_x(x, t) + aU_t(x, t)$.

LEMMA 10. For any direction λ and any compact $M \subset \Omega$, there are only a finite number of points in M at which $U_\lambda = 0$ and $(U_\lambda)_x = 0$ simultaneously.

Proof. Let $\phi(x, t) = U_\lambda(x, t)$. Then $\phi_t = \phi_{xx}$ in Ω . Now let $p_0 = (x_0, t_0) \in M$. Suppose first that $\phi(p_0) \neq 0$ or $\phi_x(p_0) \neq 0$. Then we can find a (closed) rectangle N centered at p_0 such that either $\phi(x, t) \neq 0$ or $\phi_x(x, t) \neq 0$ for each $(x, t) \in N$. Suppose now that

$\phi(p_0) = \phi_x(p_0) = 0$. Let N be the rectangle given in Theorem B of [8], where $\delta, \varepsilon > 0$. Let $z(t)$ denote the number of zeros of $\phi(\cdot, t)$ in the interval $I = [x_0 - \varepsilon, x_0 + \varepsilon]$ and let $k(t)$ denote the number of common zeros of $\phi(\cdot, t)$ and $\phi_x(\cdot, t)$ in I , for $t \in J = [t_0 - \delta, t_0 + \delta]$. Using [8], we see that $z(t)$ is nonincreasing and $z(t_1) \geq z(t_2) + k(t_2)$ when $t_1, t_2 \in J$ with $t_1 < t_2$. Since $z(t_0 - \delta)$ is finite, a simple counting argument, together with the maximum principle, shows that $S = \sum_{t \in J} k(t) < \infty$; in fact, $S \leq \frac{1}{2}z(t_0 - \delta)$. Hence, the number of points in N at which $\phi = \phi_x = 0$ is finite. Since M can be covered by a finite number of rectangles of the two types above, which have either none or a finite number of points at which $\phi = \phi_x = 0$, we are done. \square

LEMMA 11. *Let $p_0 \in \Omega$ be a point of the set $\sigma = \{(x, t) \in \Omega : U_\lambda(x, t) = 0\}$ at which $\nabla U_\lambda \neq \vec{0}$. Then σ is (or can be extended to be) a smooth (i.e. C^∞) arc near p_0 .*

Proof. The proof follows from the implicit function theorem. \square

LEMMA 12. *Let σ be any directed arc such that $U_\lambda = 0$ on σ and $U_\lambda > 0$ locally to the right of σ . Then the map $f(p) = -bU(p) + aU_x(p)$ is strictly increasing on σ .*

Proof. Let σ be parametrized by $p(s) = (x(s), t(s))$ with $|p'(s)| \neq 0$ (except possibly at a finite number of points). Let $s_1 \in \mathfrak{R}$ and $(x_1, t_1) = p(s_1) \in \gamma$. Let us set $F(s) = f(x(s), t(s))$. Then $F'(s) = (-bU_t(p(s)) + aU_{xt}(p(s)))t'(s)$. If $t'(s_1) \geq 0$, then $(-bU_x + aU_t)_x \geq 0$ and so $-bU_t + aU_{xt} = -bU_{xx} + aU_{xt} \geq 0$. If $t'(s_1) < 0$, then $(-bU_x + aU_t)_x \leq 0$ and so $-bU_t + aU_{xt} \leq 0$. In either case, $F'(s_1) \geq 0$. Since s_1 was arbitrary, we see that $F(s)$ is (weakly) increasing. To see that $F(s)$ is strictly increasing, suppose $F'(s) = 0$ for s in an open interval I . There are only a finite number of points in any compact subset of Ω at which $U_\lambda = (U_\lambda)_x = 0$. We may thus assume that $U_\lambda \neq 0$ on $p(I)$; this implies $t' \equiv 0$ on I . Then σ is horizontal (near $p(I)$) and so $(U_\lambda)_x \equiv 0$ on $p(I)$, a contradiction. \square

Let us now define a directed arc $\gamma = \gamma(\nu, p_0)$ from p_0 into Ω such that $U_\lambda = 0$ on γ and $U_\alpha > 0$ locally to the right of γ . Let $D = \{(x, t) \in \Omega : U_\lambda(x, t) > 0\}$ and let D^0 be the component of D whose boundary contains the upper portion (i.e. $t > t_0$) of some neighborhood of p_0 in Γ . Let $\gamma(\nu, p_0)$ be the component of $\partial D^0 \cap (\Omega \cup \{(x, t) \in \Gamma : U_\lambda(x, t) = 0\})$ which contains p_0 . We give

$\gamma = \gamma(\nu, p_0)$ the direction which keeps D^0 locally to the right of γ at every point of $\gamma \cap \Omega$. In a neighborhood of each point $p \in \gamma$ with $\nabla U_\lambda(p) \neq \vec{0}$, γ is a smooth arc with $U_\lambda < 0$ to the left of γ . \square

LEMMA 13. *The set $\gamma = \gamma(\nu, p_0)$ is a Jordan arc.*

Proof. Since γ cannot intersect itself (Lemma 12 or the maximum principle, since U_λ cannot vanish identically on an open subset of Ω), the only possible difficulty would have to occur at a point $p_3 = (x_3, t_3) \in \gamma$ with $\nabla U_\lambda(p_3) = \vec{0}$. By Theorem B of [8] and Lemma 10, we may assume that in the rectangle $N = [x_3 - \varepsilon, x_3 + \varepsilon] \times [t_3 - \delta, t_3 + \delta]$ (for some $\varepsilon, \delta > 0$), $U_\lambda(\cdot, t_3 + \delta)$ has at most one zero in $[x_3 - \varepsilon, x_3 + \varepsilon]$, $U_\lambda(\cdot, t_3 - \delta)$ has at least two zeros in $[x_3 - \varepsilon, x_3 + \varepsilon]$, and $U_{\lambda x}(p) \neq 0$ if $p_3 \neq p \in N \cap \gamma$. Either $U_\lambda(\cdot, t_3)$ changes signs at x_3 or it does not. First if, say, $U_\lambda(x, t_3) > 0$ for $x \neq x_3$ with $(x, t_3) \in N$, then $U_\lambda(x, t) > 0$ if $(x, t) \in N$ with $t > t_3$ and there exist (unique) $x_L, x_R \in C^0([t_3 - \delta, t_3])$ such that $x_L(t_3) = x_R(t_3) = x_3$, $x_3 - \varepsilon \leq x_L(t) < x_R(t) \leq x_3 + \varepsilon$ for all $t \in [t_3 - \delta, t_3]$, $U_\lambda(x_L(t), t) = U_\lambda(x_R(t), t) = 0$ for $t \in [t_3 - \delta, t_3]$, and $U_\lambda(x, t) > 0$ if $(x, t) \in N$, $t < t_3$, and $x \notin [x_L(t), x_R(t)]$. Then $\gamma \cap N = \{(x_R(t), t)\} \cup \{(x_L(t), t)\}$. Second, if $U_\lambda(x, t_3) < 0$ when $x_3 - \varepsilon < x < x_3$ and $U_\lambda(x, t_3) > 0$ when $x_3 < x < x_3 + \varepsilon$, for example, then there exists $x_R \in C^0([t_3 - \delta, t_3 + \delta])$ such that $U_\lambda(x_R(t), t) = 0$ for $t \in [t_3 - \delta, t_3 + \delta]$ and $U_\lambda(x, t) < 0$ if $(x, t) \in N$ and $x > x_R(t)$. Then $\gamma \cap N = \{(x_R(t), t)\}$. In each case, γ is a Jordan arc in a neighborhood of p_3 . \square

REMARK. (a) γ is double-point free.

(b) If γ enters Ω at a point $p_0 \in \Gamma$, then γ can never return to Γ , since $f(p) = -a$ on Γ .

LEMMA 14. *Let γ be as above. Then*

(a) *γ cannot have a strict local $(0, 1)$ -minimum in Ω .*

(b) *γ cannot have more than one strict local $(0, 1)$ -maximum in Ω .*

Proof. Notice that $\phi \equiv -bU_x + aU_t$ satisfies $\phi_t = \phi_{xx}$ in Ω and $\phi = 0$ on γ . Suppose γ has a strict local $(0, 1)$ -minimum at (x_0, t_0) . Then, for some sufficiently small $\varepsilon > 0$, there is a connected, simply connected region ω bounded below by an arc of γ and above by an interval of $t = t_0 + \varepsilon$ such that $(x_0, t_0) \in \partial\omega$. Since $\phi = 0$ on $\partial\omega \cap \gamma$, which is the parabolic boundary of ω , the maximum

principle implies that $\phi \equiv 0$ and so $\phi \equiv 0$ in Ω , a contradiction. Conclusion (b) follows immediately from (a). \square

COROLLARY. *Let $(x(s), t(s))$ be a parametrization of γ . Then $t(s)$ is monotonic or there exists s_0 such that $t(s)$ is increasing on $s \leq s_0$ and decreasing on $s \geq s_0$.*

LEMMA 15. *Suppose Γ^* is not a vertical line, $p_0 = (x_0, t_0) \in \Gamma$ is a ν -minimum of Γ , where $\nu = a\vec{i} + b\vec{j}$, and $\gamma = \gamma(\nu, p_0)$ begins at p_0 . Then, for some $k > 0$, γ lies between $t = t_0 + k$ and $t = t_0 - k$.*

Proof. Let $(x(s), t(s))$ be an arclength parametrization of γ and suppose $t(s) \rightarrow \infty$ as $s \rightarrow \infty$. From the corollary above, we see that $t(s)$ is an increasing function and γ can be represented as $x = \psi(t)$, $t \geq t_0$. For each natural number n , let

$$\gamma_n = \{(x, t - n\tau) : (x, t) \in \gamma\}$$

and notice that γ_n is represented by

$$x = \psi_n(t) \equiv \psi(t + n\tau), \quad t \geq t_0 - n\tau.$$

Now the γ_n do not intersect, since if $(x, t) \in \gamma_n \cap \gamma_m$, then $(x, t + n\tau)$ and $(x, t + m\tau)$ are in γ and Lemma 12 implies $m = n$. This implies $m = n$. This implies $\psi_{n+1}(t) < \psi_n(t)$ in the domain $t \geq t_0$. For any fixed $t \geq t_0$, $(\psi_n(t))$ is a decreasing sequence bounded below by $X^*(t)$ (since $\gamma_n \subset \Omega$ for each n) and so $\tilde{\psi}(t) = \lim_{n \rightarrow \infty} \psi_n(t)$ exists. Now $\tilde{\psi}(t) = X^*(t)$ for all $t \geq t_0$ by Theorem A of [8] and so $\gamma_n \rightarrow \Gamma^*$ as $n \rightarrow \infty$. Since $-bU + aU_x$ is increasing on γ_n for each n , it must be nondecreasing on Γ^* . Since Γ^* , U , and U_x are all τ -periodic in t , we see that $-bU + aU_x$ is constant on Γ^* . Now $U = 1$ on Γ^* and so U_x is constant on Γ^* . Thus the τ -periodic function $g \equiv U_x$ is a solution of the boundary value problem

$$\begin{aligned} g_t &= g_{xx} && \text{in } \Omega \\ g &= -1 && \text{on } \Gamma \\ g &= C && \text{on } \Gamma^*. \end{aligned}$$

Then $U_x(x, t) = (C + 1)U - 1$ in $\bar{\Omega}$ and so $U_t = U_{xx} = (C + 1)U_x$ in $\bar{\Omega}$. Thus Γ and Γ^* are straight lines and, since they are periodic in t , they are vertical. This contradiction implies that $t(s)$ does not approach $+\infty$. A similar argument shows that $t(s)$ does not approach $-\infty$. \square

REMARK. It does not follow from the τ -periodicity of U , U_x , U_t , etc. that maximal connected sets of the form $\gamma = \{-bU_x + aU_t = 0\}$ are τ -periodic. There are no τ -periodic connected sets $\gamma = \{-bU_x + aU_t = 0\}$, since $-bU + aU_x$ would be monotonic and τ -periodic on γ .

LEMMA 16. *Suppose the hypotheses of Lemma 15 are satisfied. Then γ must terminate at a point $p_0^* \in \Gamma^*$ with the same normal vector (to Γ^*) $\nu = a\vec{i} + b\vec{j}$.*

Proof. Suppose $p_0 = (x_0, t_0) \in \Gamma$ is a ν -minimum and γ begins at p_0 . Let $(x(s), t(s))$ be an arclength parametrization of γ and let $f = -bU + aU_x$. Now γ remains in a bounded subset of Ω . Further, once it leaves a neighborhood of p_0 , γ remains bounded away from Γ . Moreover, it can be shown using Lemmas 10, 11, and 14 that γ has no accumulation points in Ω . Since γ does not end at a point of Ω , there must be a terminal point or an accumulation point $p_0^* = (x_0^*, t_0^*) \in \Gamma^*$ of γ . Since ∇U is continuous on $\bar{\Omega}$ and $U_\lambda = 0$ at each point of γ , $U_\lambda(p_0^*) = 0$; thus the normal to Γ^* at p_0^* is $\nu = a\vec{i} + b\vec{j}$ (recall $\lambda = -b\vec{i} + a\vec{j}$). Since $t(s)$ is monotonic (at least for $s \geq s_0$), $t(s) \rightarrow t_0^*$ as $s \rightarrow |\gamma|$ ($|\gamma|$ may be ∞). If $x(s)$ does not have a limit as $s \rightarrow \infty$, then there exists $\delta \rightarrow 0$ such that each point of $[x_0^*, x_0^* + \delta] \times \{t_0^*\}$ is an accumulation point of γ and hence $U_\lambda = 0$ on this interval, a horizontal line segment in Γ^* . This is not possible, since $\Gamma^* = \{(x, t) : x = X^*(t)\}$, and so $x(s) \rightarrow x_0^*$ as $s \rightarrow |\gamma|$. \square

REMARK. Everything above carries over when $p_0 \in \Gamma$ is a ν -maximum except that $-bU + aU_x$ is decreasing on γ .

LEMMA 17. *Suppose Γ^* is not a vertical line. Suppose $a > 0$, $b \geq 0$, and $p_0 = (x_0, t_0) \in \Gamma$ is a ν -minimum of Γ , where $\nu = a\vec{i} + b\vec{j}$. Let γ be the curve of constant gradient direction beginning at p_0 as in Lemma 16 and let γ terminate at $p_1 = (x_1, t_1) \in \Gamma^*$. Suppose that $x \geq x_1$ for every point $(x, t) \in \gamma$. Then $\nu \cdot (p_0 - p_1) > a$.*

Proof. Let us consider the path independent integral

$$I = \int_{p_0}^{p_1} (-bU + aU_x) dx + (-bU_x + aU_t) dt.$$

If we integrate along γ from p_0 to p_1 and integrate by parts, we see

that

$$I = \int_{p_0}^{p_1} (-bU + aU_x) dx$$

and so

$$I = (x - x_1)(-bU + aU_x)|_{p_0}^{p_1} - \int_{p_0}^{p_1} (x - x_1)d(-bU + aU_x).$$

Since $-bU + aU_x$ is increasing along γ and $x - x_1 \geq 0$, we see that $I < a(x_0 - x_1)$.

On the other hand, $I = a \int_{p_0}^{p_1} U_x dx + U_t dt - b \int_{p_0}^{p_1} U dx + U_x dt$, and the first integral equals a . If we integrate the second integral first along Γ from p_0 to $q_0 \equiv (x_3, t_1)$ and then along the horizontal line from q_0 to p_1 , we see that $I > a + b(t_1 - t_0)$. If we combine the two inequalities for I , we obtain $\nu \cdot (p_0 - p_1) > a$. \square

LEMMA 18. *Suppose σ is any curve from a point $p_4 = (x_4, t_4) \in \Omega$ to a point $p_1 = (x_1, t_1) \in \Gamma^*$ with $x_4 > x_1$ along which $U_t = 0$ and $U_t > 0$ locally to the right of σ . Then $x > x_1$ for all points $(x, t) \in \sigma$.*

Proof. Suppose first that the curve σ stays to the right of the vertical line $x = x_1$ near p_1 . Suppose σ crosses the line $x = x_1$ at a point $p_3 = (x_1, t_3)$ and stays to the right of the vertical line between p_3 and p_1 . Let $p_2 = (x_2, t_2) \in \sigma$ be the furthest point to the right on σ between p_3 and p_1 . From the monotonicity of $U - x$ on σ , we see that for some number $D > 0$, $U_x < -D$ on σ between p_3 and p_2 and $U_x > -D$ on σ between p_2 and p_1 . Then

$$U(p_3) - U(p_1) = \int_{p_1}^{p_3} U_x dx + U_t dt = \int_{p_1}^{p_3} U_x dx > 0.$$

Thus $U(p_3) > U(p_1) = 1$ in violation of the maximum principle.

Suppose next that σ lies to the left of the line $x = x_1$ near p_1 . Let us assume that in a neighborhood of p_1 , σ stays above $t = t_1$ and between Γ^* and $x = x_1$. Then for some $\epsilon > 0$, $U_t(X^*(t), t) < 0$ for $t_1 < t < t_1 + \epsilon$. Since $U_t > 0$ to the right of σ , there must be a curve $\tilde{\sigma}$ which begins at p_1 and lies between Γ^* and σ along which $U_t = 0$ and U_x increases as points move away from p_1 . This curve $\tilde{\sigma}$ must either terminate at a point p_5 of Γ^* or cross the line $x = x_1$ at a point p_6 . If we argue in a similar manner to the first paragraph, we obtain either $U(p_5) > U(p_1)$ or $U(p_6) > U(p_1)$; in either case, a contradiction results.

Finally, if neither case holds, then σ intersects the line $x = x_1$ infinitely often. An argument similar to the argument of the first paragraph implies that there is a sequence (q_n) (of “every other point of intersection of σ and $x = x_1$ ”) such that q_n converges to p_1 as $n \rightarrow \infty$ and $U(q_n) < U(q_{n+1})$ for each n . Since U is continuous, this contradicts the maximum principle. \square

REMARK. Using a similar (but more complicated) proof, it can be shown that a curve σ as above with $U_\lambda = 0$ on σ and $U_\lambda > 0$ locally to the right of σ cannot cross the tangent line $a(x - x_1) + b(t - t_1) = 0$.

THEOREM 4. (a) *If Γ has ν -minima (ν -maxima) in one period (i.e. in $\Gamma \cap (\mathcal{R} \times [t_0, t_0 + \tau])$), then Γ^* has at least n ν -minima (ν -maxima) in one period.*

(b) *The total curvature of Γ in a single period cannot exceed the total curvature of Γ^* in one period.*

(c) *The x -variation of Γ in one period (i.e. $\int_0^\tau |x'(t)| dt$, where $(X(t), t)$ is a parametrization of Γ) cannot exceed that of Γ^* in one period.*

Proof. Suppose $p_k^0 = (x_k^0, t_k^0) \in \Gamma$, $k = 1, \dots, n$, with $t_1^0 \leq t_2^0 \leq \dots \leq t_n^0 < t_1^0 + \tau$ such that p_k^0 is a ν -minimum of Γ , $k = 1, \dots, n$, and, for some $t \in (t_k, t_{k+1})$, $U_\lambda(X(t), t) \neq 0$, for each $1 \leq k \leq n - 1$. For each $k = 1, \dots, n$, there is a curve γ_k starting at p_k^0 and ending at a point $p_k^* \in \Gamma^*$ such that $U_\lambda = 0$ on γ_k and $U_\lambda > 0$ locally to the right of γ_k . Then ν is a normal to Γ^* at p_k^* , $k = 1, \dots, n$, and the curves γ_k do not intersect in $\bar{\Omega}$. To see this, let $q \in \Gamma$ lie between p_k and p_{k+1} such that q is a ν^* -minimum of Γ , where ν^* is not parallel to ν . Let σ_1 be the curve of constant gradient direction beginning at q and ending at a point $q_1 \in \Gamma^*$. If σ_1 intersects γ_k , for example, at p , then $\nabla U(p) = \vec{0}$. From Lemma 2 and the maximum principle, we see that $U_x \neq 0$ in $\bar{\Omega}$ and so $\sigma_1 \cap \gamma_k = \emptyset$. Since σ_1 separates γ_k and γ_{k+1} near Γ , the γ_k cannot intersect in $\bar{\Omega}$. Notice then that $p_k^* \neq p_{k+1}^*$, $1 \leq k \leq n - 1$.

Next, let us fix $k \in \{1, \dots, n\}$. Let $q_1 \in \Gamma$ with q_1 between p_k and p_{k+1} and $q_2 \in \Gamma$ with q_2 between p_{k-1} and p_k such that $\nu_1 = a\vec{i} + b_1\vec{j}$ and $\nu_2 = a\vec{i} + b_2\vec{j}$ are (exterior) normals to Γ at q_1 and q_2 respectively with $b_1 < b < b_2$ and q_1 and q_2 are strict local minima of Γ with respect to their normals ν_k (where $p_{n+1} \equiv p_1 + (0, \tau)$ and $p_0 \equiv p_n - (0, \tau)$). Let $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ be curves of constant gradient direction beginning at q_1 and q_2 respectively. Then $\tilde{\gamma}_1$ and

$\tilde{\gamma}_2$ terminate at points q_1^* and q_2^* of Γ^* respectively and ν_1 and ν_2 are (interior) normals to Γ^* at q_1^* and q_2^* respectively. Since Γ^* is C^1 and p_k^* lies between q_1^* and q_2^* , there must be a point $p_k^{**} \in \Gamma^*$ at which Γ^* has a ν -minimum (in the sense mentioned at the beginning of this section). The proof of (a) and (b) follows.

Suppose $p_0 = (x_0, t_0) \in \Gamma$ is a $(1, 0)$ -minimum of Γ and $p_2 = (x_2, t_2) \in \Gamma$ is the next $(1, 0)$ -maximum of Γ . Suppose that γ and $\tilde{\gamma}$ are curves of constant gradient direction starting from p_0 and p_2 respectively as in Lemma 16 and let these curves terminate at points $p_1 = (x_1, t_1)$ and $p_3 = (x_3, t_3)$ of Γ^* respectively. Notice that $U_t > 0$ locally to the left of $\tilde{\gamma}$. Let $a = 1$ and $b = 0$. From Lemmas 17 and 18, we see that $x_0 - x_1 > 1$. If we apply the first part of the proof of Lemma 17 to $\tilde{\gamma}$, we see that $I > x_2 - x_3$. (An argument similar to that of Lemma 18 shows that $\tilde{\gamma}$ cannot cross the line $x = x_3$.) Since $I = a = 1$, we obtain $x_2 - x_3 < 1$. Thus the x -variation of Γ between p_0 and p_2 is less than the x -variation of Γ^* between p_1 and p_3 . The last part follows from this. \square

REMARK. Results similar, for example, to Theorem 4 of [4] and Theorem 5 of [6] for this problem follow from our methods.

REMARK. The results in §§2 and 3 were obtained by the first author (Acker) while he was a guest of SFB 123, University of Heidelberg, in summer, 1987. Section 4 represents joint work.

REFERENCES

- [1] A. Acker, *Free boundary optimization—a constructive iterated method*, J. Appl. Math. Phys. (ZAMP), **30** (1979), 885–900.
- [2] —, *How to approximate the solutions of certain free-boundary problems for the Laplace equation by using the contraction principle*, J. Appl. Math. Phys. (ZAMP), **32** (1981), 22–33.
- [3] —, *On the geometric form of free boundaries satisfying a Bernoulli condition*, Math. Methods Appl. Sci., **6** (1984), 449–456.
- [4] —, *On the geometric form of free boundaries satisfying a Bernoulli condition, II*, Math. Methods Appl. Sci., (1986), 387–404.
- [5] —, *Convergence results for an analytical trial free-boundary method*, IMA J. Numer. Anal., **8** (1988), 357–364.
- [6] A. Acker and K. Lancaster, *The geometry of curves of constant contact angle for doubly-connected minimal surfaces*, Comm. Partial Differential Equations, **14** (1989), 375–390.
- [7] G. Alessandrini, *An identification problem for an elliptic equation in two variables*, (manuscript).
- [8] S. Angenent, *The zero set of a solution of a parabolic equation*, J. Reine Angew. Math., **390** (1988), 79–96.

- [9] C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities*, Wiley, Chichester, 1984.
- [10] G. Birkhoff and E. Zarantonello, *Jets, Wakes, and Cavities*, Academic Press, New York, 1957.
- [11] J. Cannon, *The One-Dimensional Heat Equation*, Addison-Wesley, Menlo Park, 1984.
- [12] M. Chipot, *Variational Inequalities and Flow in Porous Media*, Springer, New York, 1984.
- [13] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. 1, Springer, New York, 1970.
- [14] J. Crank, *Free and Moving Boundary Problems*, Clarendon Press, Oxford, 1984.
- [15] A. Crowley and C. Elliott (editors). *Proceedings of the Conference on Crystal Growth*, IMA J. Appl. Math., (1985), 115–264.
- [16] C. Cryer, *Numerical Methods for Free and Moving Boundary Problems*, (presented at State of the Art in Numerical Analysis, Birmingham, April, 1986).
- [17] R. Finn, *Equilibrium Capillary Surfaces*, Springer, New York, 1986.
- [18] A. Friedman, *Variational Principles and Free Boundary Problems*, Wiley, New York, 1982.
- [19] A. Friedman and R. Jensen, *Convexity of the free boundary in the Stefan problem and in the dam problem*, Arch. Rational Mech. Anal., **67** (1978), 1–24.
- [20] A. Friedman and T. Vogel, *Cavitational flow in a channel with oscillatory wall*, Nonlinear Analysis TMA, **7** (1983), 1175–1192.
- [21] E. Holmgren, *Sur une application de l'equation integrale de M. Volterra*, Ark. für Matematik, Astronomi och Fysik, **3** (1907), 1–4.
- [22] D. Kinderlehrer and Nirenberg, *Regularity in free boundary problems*, Ann. Scuola Norm. Sup. Pisa, ser. IV, **4** (1977), 373–391.
- [23] M. Lavrentiev, *Variational Methods*, Groningen, 1963.
- [24] M. Lavrentiev and B. Schabat, *Methoden der komplexen Funktionentheorie*, Berlin, 1967.
- [25] K. Nickel, *Einige Eigenschaften von Lösungen der Praditschen Grenzschicht-Differentialgleichungen*, Arch. Rational Mech. Anal., **2** (1958), 1–31.
- [26] —, *Gestaltaussagen über Lösungen parabolischer Differentialgleichungen*, J. Reine Angew. Math., **211** (1962), 78–94.
- [27] P. Panagiotopoulos, *Inequality Problems in Mechanics and Applications*, Birkhäuser, Boston, 1985.
- [28] L. Payne, *On two conjectures in the fixed membrane eigenvalue problem*, J. Appl. Math. Phys. (ZAMP), **24** (1973), 721–729.
- [29] M. Protter and H. Weinberger, *Maximum Principles in Partial Differential Equations*. Springer-Verlag, 1970.
- [30] L. Rubiňstein, *The Stefan Problem*, Transl. Math. Monographs, vol. 27, Amer. Math. Soc., Providence, R.I., 1971.
- [31] A. Schatz, *Free boundary problems of Stephen type with prescribed flux*, J. Math. Anal. Appl., **28** (1969), 569–580.
- [32] W. Walter, *Differential and Integral Inequalities*, Springer-Verlag, New York, 1970.

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