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ON AN OPTIMAL CONTROL PROBLEM FOR A PARABOLIC INCLUSION

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Let H, U be two real Hilbert spaces and let g be a proper lower semi-continuous convex function from $L^2(0, T; H)$ into R^+. For each t in $[0, T]$, let $\varphi(t,.)$ be a proper l.s.c. convex function from H into R^+ with effective domain $D(\varphi(t,.))$ and let h be a l.s.c. convex function from a closed convex subset $\mathcal{U}$ of U into $L^2(0, T; H)$ with

$$h(u) \geq \gamma \| u \|_U^2 + C$$

for all u in $\mathcal{U}$. The constants γ and C are positive.

The main purpose of this paper is to establish the existence of a solution of the optimal control problem

$$\text{(0.1)} \qquad \inf\{g(y) + h(u) : y' \in -\partial\varphi(t, y) + \mathcal{F}(t, y) + Bu,\ 0 \leq t \leq T,\ y(0) = y_0;\ u \in \mathcal{U},\ y \in L^2(0, T; H)\}$$

where B is a bounded linear mapping of U into $L^2(0, T; H)$ and $\mathcal{F}$ is an upper semi-continuous set-valued mapping of $L^2(0, T; H)$ into the closed convex subsets of $L^2(0, T; H)$ with at most a linear growth in y. The existence is shown by using an approximation scheme introduced recently by Barbu and Tiba [4], Barbu and Neittaanmaki and Niemisto [5] for elliptic variational problems. Optimal control problems for differential inclusions of parabolic type involving continuous convex multi-valued mappings have been considered by Makhmudov and Pshemichnyi [9].

Notations, the basic assumptions and the main result of the paper are given in Section 1. The following differential inclusion is studied in Section 2

$$\text{(0.2)} \qquad y' \in -\partial\varphi(t, y) + \mathcal{F}(t, y) + Bu \text{ on } (0, T),\ y(0) = y_0.$$

The proof of the main result of the paper is carried out in Section 3. Extremality relations for an approximating problem are considered in Section 4.

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Applications to control problems for differential inclusions of parabolic type in non-cylindrical domains are given in Section 5.

§1. Notations, assumptions and statement of the main result

For each t in $[0, T]$, let $\varphi(t,.)$ be a proper lower semi-continuous (l.s.c.) convex function from H into R^+ with effective domain

$$D(\varphi(t,.)) = \{y : y \in H,\ 0 \leq \varphi(t, y) < \infty\}$$

and with $0 \in D(\varphi(t,.))$.

The subdifferential $\partial\varphi(t, x)$ of $\varphi(t, x)$ at x shall be written as $\mathscr{A}(t, x)$. It is known that $\mathscr{A}(t, x)$ is a maximal monotone set-valued mapping of H into H and that $D(\mathscr{A}(t, x))$ is dense in $D(\varphi(t,.))$. Since $\mathscr{A}(t,.)$ is maximal monotone in H, the mapping $I + \lambda\mathscr{A}(t,.)$ is 1-1 and onto for each positive λ and hence the Yosida approximants $J_\lambda^t = (I + \lambda\mathscr{A}(t,.))^{-1}$ is well-defined.

The following results are known and can be found in Brezis [6] or in Watanabe [10].

LEMMA 1.1. *For each t in $[0, T]$, let $\varphi(t,.)$ be a proper l.s.c. convex function from H into R^+ with effective domain $D(\varphi(t,.))$ in H. Then*

1. *For each t and each positive λ, the Yosida approximant J_λ^t is a single-valued non-expansive mapping of H into H.*
2. *For each t, $A_\lambda^t(.) = \lambda^{-1}(I - J_\lambda^t)$ is a single-valued maximal monotone, Lipschitzean mapping of H into H with constant λ^{-1}.*
3. *For each t and each x, $A_\lambda^t x \in \mathscr{A}(t, J_\lambda^t x)$.*
4. *For all x in $D(\mathscr{A}(t,.))$:*
   $J_\lambda^t x \to x$ in H ; $A_\lambda^t x \to m[\mathscr{A}(t, x)]$ in H where $m[\mathscr{A}(t, x)]$ is the element of $\mathscr{A}(t, x)$ with minimal H-norm.
5. *Let $\varphi_\lambda(t, x) = \inf_{y \in D(\varphi)}\left\{\varphi(t, y) + \dfrac{1}{2\lambda}\| x - y \|^2\right\}$, then: $\varphi_\lambda(t,.)$ is Frechet differentiable and $\partial\varphi_\lambda(t, x) = A_\lambda^t x$.*

We shall assume some continuity hypotheses on φ.

ASSUMPTION I.1. *Let $r > 0$ and t_0 be in $[0, T]$. Then for each $y_0 \in D(\varphi(t_0,.)$ with $\| y_0 \| \leq r$, we assume that there exists $y(t)$ in $D(\varphi(t,.))$ such that*

1. $\| y(t) - y_0 \|^2 \leq | k_r(t) - k_r(t_0) |^2 (K_r + \varphi(t_0, y_0))$.
2. $0 \leq \varphi(t, y(t)) \leq \varphi(t_0, y_0) + | l_r(t) - l_r(t_0) | (K_r + \varphi(t_0, y_0))$,

where K_r is a non-negative constant and k_r, l_r are two absolutely continuous functions on $[0, T]$ with k'_r, l'_r in $L^2(0, T)$.

Using Assumption I.1, Yamada [11] has proved the following result.

LEMMA 1.2. *Let φ be as in Lemma 1.1 and suppose that Assumption I.1 is satisfied. Let $y(t)$ be an absolutely continuous function from $[0, T]$ into H. Then for each positive λ, $\varphi_\lambda(t, y(t))$ is absolutely continuous on $[0, T]$ and*

$$| \left(\frac{d}{dt} \varphi_\lambda(t, y(t)\right) - \left(A^t_\lambda y(t), \frac{d}{dt} y(t))\right) | \leq | l'_r(t) | (K_r + \varphi_r(t, y))^{\frac{1}{2}}$$

where K_r is as in Assumption I.1 and $r = \sup\{\| J^t_\lambda y(s) \|: 0 < \lambda \leq 1 ; 0 \leq s, t \leq T\}$.

A compactness assumption is needed in the paper.

ASSUMPTION I.2. *For each t in $[0, T]$ and each positive c, the set*

$$X_c(t) = \{y : y \in H, 0 \leq \varphi(t, y) \leq c\}$$

is compact in H.

We shall consider set-valued mappings $\mathcal{F}(t, x)$ of $L^2(0, T ; H)$ into the subsets of $L^2(0, T ; H)$ satisfying the following assumption.

ASSUMPTION I.3.

1. *$\mathcal{F}$ is an upper semi-continuous (u.s.c) set-valued mapping of $L^2(0, T; H)$ into the subsets of $L^2(0, T; H)$.*
2. *For each and each x, $\mathcal{F}(t, x)$ is a closed convex subset of $L^2(0, T; H)$.*
3. *There exists C such that*

$$\sup\{\| f(t, x) \|^2 : f(t, x)) \in \mathcal{F}(t, x)\} \leq C(1 + \| x \|^2)$$

for all x in H and almost all t in $[0, T]$.

In Section 3, we shall consider the optimal control problem

$$(1.1) \quad \inf\Big\{g(y) + h(u) + \eta \| v \|^2_{L^2(0,T;H)} + \varepsilon^{-1}\int_0^T [\varphi(t, y) + \varphi^*(t, v) - (y, v)]dt :$$
$$y' \in \mathcal{F}(t, y) + Bu - v, 0 \le t \le T, y(0) = y_0, \varepsilon \le \eta ; u \in \mathcal{U}, y, v \in L^2(0, T; H)\Big\}$$

where φ^*, the conjugate function of φ, is given by

$$\varphi^*(t, v) = \sup_{x \in D(\varphi)}[(x, v) - \varphi(t, x)].$$

It will be shown that the set of solutions $\{y^\eta_\varepsilon, u^\eta_\varepsilon, v^\eta_\varepsilon\}$ of (1.1) is compact in $L^2(0, T; H) \times U_{\text{weak}} \times (L^2(0, T; H))_{\text{weak}}$ for each η and ε. The limit of $\{y^\eta_\varepsilon, u^\eta_\varepsilon, v^\eta_\varepsilon\}$ as $\varepsilon \to 0$, is a solution of the problem

$$(1.2) \quad \inf\{g(y) + h(u) + \eta \| v \|^2_{L^2(0,T;H)} :$$
$$y' \in -\mathcal{A}(t, y) + \mathcal{F}(t, y) + Bu, 0 \le t \le T, y(0) = y_0;$$
$$u \in \mathcal{U}, y \in L^2(0, T; H), v \in \mathcal{A}(t, y)\}.$$

The main result of the paper is the following theorem.

THEOREM 1.1. *Let $\varphi(t,.)$ be a proper l.s.c. convex function from its effective domain $D(\varphi(t,.)) \subset H$ into R^+, satisfying Assumption* I.1 *and suppose that*

$$\varphi(t, y) \ge c \| y \|^2$$

for all y in $D(\varphi(t,.))$ and all $t \in [0, T]$.

Let $\mathcal{F}$ be an u.s.c. set-valued mapping of $L^2(0, T; H)$ into the closed convex subsets of $L^2(0, T; H)$ verifying Assumption I.3. *Suppose that Assumption* I.2 *is satisfied and let y_0 be in $D(\varphi(t,.))$. Then*

1. *For each η and ε, the set $\{y^\eta_\varepsilon, u^\eta_\varepsilon, v^\eta_\varepsilon\}$ of solutions of* (1.1) *is compact in* $L^2(0, T; H) \times U_{\text{weak}} \times (L^2(0, T; H))_{\text{weak}}$.
2. *Every limit point $\{y^\eta, u^\eta, v^\eta\}$ of the set $\{y^\eta_\varepsilon, u^\eta_\varepsilon, v^\eta_\varepsilon\}$ as $\varepsilon \to 0$, is a solution of* (1.2).
3. *The set $\{y^\eta, u^\eta\}$ is compact in $L^2(0, T; H) \times U_{\text{weak}}$ and every limit point of the set as $\eta \downarrow 0$, is a solution of* (0.1).

§2. The differential inclusion

$$y' \in -\mathcal{A}(t, y) + \mathcal{F}(t, y) + Bu \text{ on } (0, T),\ y(0) = y_0. \tag{2.1}$$

Under slightly different hypotheses, the existence of a solution of the differential inclusion (2.1) has been established in [8] using Attouch and Damlamian [1], Yamada [11] results together with the Schauder fixed point theorem.

First we shall consider the initial-value problem

$$y' + \mathcal{A}_\lambda^t y \in \mathcal{F}(t, y) + Bu \text{ on } (0, T),\ y(0) = y_0. \tag{2.2}$$

LEMMA 2.1. *Let φ be as in Theorem* 1.1 *and suppose that Assumptions* I.1-I.2 *are verified. Let f be a continuous single-valued mapping of $L^2(0, T; H)$ into $L^2(0, T; H)$ with*

$$\| f(t, x) \|^2 \leq C(1 + \| x \|^2)$$

for all x in H and for almost all t in $(0, T)$. Then for any y_0 in $D(\varphi(0,.))$ and any u in $\mathcal{U}$, there exists y_λ in $L^2(0, T; H)$ with $y' \in L^2(0, T; H)$ such that

$$y_\lambda' + A_\lambda^t y_\lambda = f(t, y_\lambda) + Bu \text{ on } (0, T),\ y_\lambda(0) = y_0. \tag{2.3}$$

Moreover

$$\| y_\lambda' \|^2_{L^2(0,T;H)} + \| A_\lambda^t y_\lambda \|^2_{L^2(0,T;H)} + \sup_{0 \leq t \leq T} \varphi_\lambda(t, y_\lambda) \leq M(1 + \| u \|_U^2 + \varphi(0, y_0)) \tag{2.4}$$

where M is independent of λ, y_0, u.

Proof. Since A_λ^t is Lipschitzean with constant λ^{-1}, the existence of a solution of (2.3) is a consequence of Peano's theorem.

From (2.3) we get

$$\frac{d}{dt} \| y_\lambda \|^2 \leq C(\| y_\lambda(t) \|^2 + \| Bu \|^2 + 1)$$

and

$$(y_\lambda', A_\lambda^t y_\lambda) + \| A_\lambda^t y_\lambda \|^2 \leq \frac{1}{2} \| A_\lambda^t y_\lambda \|^2 + \frac{3}{4} \| f(t, y_\lambda) \|^2 + \frac{3}{4} \| Bu \|^2.$$

Applying Lemma 1.2 and we have

$$\frac{d}{dt} \varphi_\lambda(t, y_\lambda) + \frac{1}{2} \| A_\lambda^t y_\lambda \|^2 \leq$$

$$\frac{1}{2}\| f(t, y_\lambda) \|^2 + \frac{1}{4}\| A_\lambda^t y_\lambda \|^2 + \| Bu \|^2 + (K_r + \varphi_\lambda(t, y_\lambda))\Big(| l_r'(t) | + \frac{3}{4}| k_r'(t) |^2\Big).$$

Since

$$c\| J_\lambda^t x \|^2 \leq \varphi(t, J_\lambda^t x) \leq \varphi_\lambda(t, x)$$

we obtain by taking into account our hypotheses on f

$$\frac{d}{dt}\varphi_\lambda(t, y_\lambda) + \frac{d}{dt}\| y_\lambda \|^2 + \frac{1}{4}\| A_\lambda^t y_\lambda \|^2$$
$$\leq \| Bu \|^2 + (K_r + \varphi_\lambda(t, y_\lambda) + 1)(| l_r'(t) | + | k_r'(t) |^2) + C(1 + \varphi_\lambda(t, y_\lambda).$$

Thus,

$$\varphi_\lambda(t, y_\lambda(t)) + \| y_\lambda(t) \| + \frac{1}{4}\int_0^t \| A_\lambda^s y_\lambda \|^2 ds \leq \tag{2.5}$$
$$\varphi_\lambda(0, y_0) + \| Bu \|^2 + M + M\int_0^t \varphi_\lambda(s, y_\lambda(s))ds,$$

where M is independent of λ, y_0 and of u.

The Gronwall lemma gives

$$\begin{aligned}&\varphi_\lambda(t, y_\lambda(t))\\ &\quad\leq \varphi_\lambda(0, y_0) + \| Bu \|^2 + M_2.\\ &\quad\leq \varphi(0, y_0) + \| Bu \|^2 + M_2.\end{aligned}$$

where M is again a positive constant independent of λ, y_0 and of u, All the other estimates are now a consequence of (2.5)–(2.6).

LEMMA 2.2. *Suppose all the hypotheses of Theorem* 1.1 *are satisfied. Then there exists y_λ in $L^2(0, T; H)$ with y_λ' in $L^2(0, T; H)$, solution of* (2.2). *Moreover*

$$\| y_\lambda' \|^2_{L^2(0,T;H)} + \| A_\lambda^t y_\lambda \|^2_{L^2(0,T;H)} + \sup_{0\leq t\leq T}\varphi_\lambda(t, y_\lambda(t)) \leq M(1 + \varphi(0, y_0) + \| u \|_U^2),$$

where M is a positive costant independent of λ, y_0, u.

Proof. Since $\mathscr{F}$ is u.s.c. from $L^2(0, T; H)$ into the closed convex subsets of $L^2(0, T; H)$, it follows from the approximate selection theorem that there exists $\{f_n\}$ of single-valued continuous mappings of $L^2(0, T; H)$ into $L^2(0, T; H)$ such that

1. Graph $f_n \subset$ Graph $\mathscr{F} + \dfrac{1}{n}$ (unit ball about the graph of $\mathscr{F}$),

2. Range $f_n \subset \text{co}(\text{Range } \mathscr{F})$.

Lemma 2.1 yields the existence of a solution of the intial value problem

$$y_n' + A_\lambda^t y_n = f_n(t, y_n) + B_u \text{ on } (0, T), \; y_n(0) = y_0. \tag{2.6}$$

Furthermore

$$\| y_n' \|^2_{L^2(0,T;H)} + \| A_\lambda^t y_\lambda \|^2_{L^2(0,T;H)} + \sup_{0 \leq t \leq T} \varphi_\lambda(t, y_n) \leq \\ M(1 + \| u \|_U^2 + \varphi(0, y_0), \tag{2.7}$$

where M is a positive constant independent of n, u, y_0, λ.

We obtain by taking subsequences (denoted again by n): $\{y_n, y_n'\} \to \{y, y'\}$ weakly in $L^2(0, T; H) \times L^2(0, T; H)$. Taking into account the lower semi-continuity of φ_λ, we get

$$\| y' \|^2_{L^2(0,T;H)} + \sup_{0 \leq t \leq T} \varphi_\lambda(t, y(t)) \leq M(1 + \| u \|_U^2 + \varphi(0, y_0)).$$

It follows from Assumption I.2 that: $y_n \to y$ in $L^2(0, T; H)$. From the definition of A_λ^t we obtain

$$A_\lambda^t y_n = \lambda^{-1}(y_n - J_\lambda^t y_n) \to A_\lambda^t y$$

weakly in $L^2(0, T; H)$. So:

$$\| A_\lambda^t y \|^2_{L^2(0,T;H)} \leq M(1 + \| u \|_U^2 + \varphi(0, y_0)).$$

We know that

$$\| w_n - f_n(., y_n) \|_{L^2(0,T;H)} \leq n^{-1}; \, w_n \in \mathscr{F}(t, y_n).$$

With our hypotheses on $\mathscr{F}$, we get: $\| w_n \|_{L^2(0,T;H)} \leq M$. Taking subsequences, we have: $w_n \to w$ weakly in $L^2(0, T; H)$. Since $\mathscr{F}$ is u.s.c. we get: $w \in \mathscr{F}(t, y)$.

The lemma is proved.

THEOREM 2.1. *Suppose all the hypotheses of Theorem 1.1 are satisfied. Then for any given y_0 in $D(\varphi(0,.))$ and any $u \in \mathscr{U}$, there exists y in $L^2(0, T; H)$ with y' in $L^2(0, T; H)$, solution of (2.1). Moreover*

$$\| y' \|^2_{L^2(0,T;H)} + \sup_{A \in \mathscr{A}} \| A(., y) \|^2_{L^2(0,T;H)} + \\ \sup_{0 \leq t \leq T} \varphi(t, y(t)) \leq M(1 + \| u \|_U^2 + \varphi(0, y_0)),$$

where M is a postitive constant independent of u, y_0.

Proof. Let y_λ be a solution of (2.2) given by Lemma 2.2. From the estimates of the lemma, we obtain by taking subsequences: $\{y_\lambda, y'_\lambda\} \to \{y, y'\}$ weakly in $L^2(0, T; H)$. Since

$$\sup_{0 \le t \le T} \varphi(t, y_\lambda) \le M,$$

it follows from the estimate and from Assumption I.2 that $y_\lambda \to y$ in $L^2(0, T; H)$ as $\lambda \to 0$. The u.s.c. of $\mathscr{F}$ gives: $f(t, y_\lambda) \to f(t, y)$ weakly in $L^2(0, T; H)$ for any $f \in \mathscr{F}$.

From Lemma 2.2 we know that: $A^t_\lambda(y_\lambda) \to z$ weakly in $L^2(0, T; H)$. Since $A^t_\lambda(y_\lambda) \in \mathscr{A}(t, y_\lambda)$, we have

$$0 \le \int_0^T (A^t_\lambda(y_\lambda) - A(t, x), y_\lambda - x)dt$$

for any $A \in \mathscr{A}$ and all x in $D(\mathscr{A}) \cap L^2(0, T; H)$.

Therefore

$$0 \le \int_0^T (z - A(t, x), y - x)dt$$

for any $A \in \mathscr{A}$ and all x in $D(\mathscr{A}) \cap L^2(0, T; H)$.

Now a standard argument yields $z \in \mathscr{A}(t, y)$.

The estimates of the theorem are now an immediate consequence of those of Lemma 2.2.

§3. The optimal control problem (0.1)

First let us consider the problem (1.1).

LEMMA 3.1. *Suppose all the hypotheses of Theorem* 1.1 *are satisfied. Then for each* $\varepsilon < \eta$, *there exists at least one solution* $\{y^\eta_\varepsilon, u^\eta_\varepsilon, v^\eta_\varepsilon\}$ *in* $L^2(0, T; H) \times \mathscr{U} \times L^2(0, T; H)$ *of the optimal control problem* (1.1).

Proof. 1) It is clear that the admissible set is non-empty as it contains $\{y_0, 0, v\}$ with any v in $\mathscr{F}(t, y_0) + B(0)$.

Let d^η_ε which we shall write as d_ε be given by

$$d_\varepsilon = \inf\{g(y) + h(u) + \eta \| v \|^2_{L^2(0,T;H)} + \varepsilon^{-1} \int_0^T [\varphi(t, y) + \varphi^*(t, v) - (y, v)]dt :$$

$$y' \in \mathcal{F}(t, y) + Bu - v,\ 0 \leq t \leq T,$$
$$y(0) = y_0 ; x \in \mathcal{U} ; y,\ v \in L^2(0,\ T; H))\}.$$

It is clear that: $0 \leq d_\varepsilon$.

Let $\{y_n,\ u_n,\ v_n\}$ be a minimizing sequence of (1.1) with

$$d_\varepsilon \leq g(y_n) + h(u_n) + \eta \| v_n \|^2_{L^2(0,T;H)} + \varepsilon^{-1} \int_0^T [\varphi(t,\ y_n) + \varphi^*(t,\ v_n) - (v_n,\ y_n)] dt \leq d_\varepsilon + n^{-1}. \tag{3.1}$$

With h as in the paper and $g(.) \geq 0$, we get

$$\eta \| v_n \|^2_{L^2(0,T;H)} + \| u_n \|_U \leq C(\varepsilon).$$

$C(\varepsilon)$ is independent of n, η.

But

$$y_n' \in \mathcal{F}(t,\ y_n) + Bu_n - v_n \text{ on } (0,\ T),\ y_n(0) = y_0. \tag{3.2}$$

Thus,

$$\| y_n(t) \|^2 \leq \| y_0 \|^2 + C_2 \int_0^t [\| y_n(s) \|^2 + 1 + \| u_n \|^2 + \| v_n \|^2] ds. \tag{3.3}$$

The Gronwall lemma gives

$$\| y_n \|_{L^\infty(0,T;H)} \leq C_3(\varepsilon,\ \eta). \tag{3.4}$$

The constants C are all independent of n.

From the definition of the conjugate function, we get

$$\begin{aligned} &\int_0^T \varphi(t,\ y_n) dt \\ &\leq \varepsilon(1 + d_\varepsilon) + \int_0^T [\varphi(t,\ 0) + (y_n,\ v_n)] dt \\ &\leq (1 + d_\varepsilon) + \int_0^T [\varphi(t,\ 0) + \| y_n \| \| v_n \|] dt \\ &\leq C_4(\varepsilon,\ \eta). \end{aligned}$$

From the equation (3.2) and from (3.4), we have

$$\| y_n' \|_{L^2(0,T;H)} \leq C_5(\varepsilon,\ \eta). \tag{3.5}$$

2) Let $n \to \infty$ to obtain by taking subsequences (again denoted by n): $u_o \to u$ weakly in U, $\{y_n,\ y_n',\ v_n\} \to \{y,\ y',\ v\}$ weakly in $(L^2(0,\ T; H))^3$. Since $u_n \in \mathcal{U}$

and $\mathcal{U}$ is closed, u is also in $\mathcal{U}$. In view of (3.4)-(3.5), Assumption I.2 gives: $y_n \to y$ in $L^2(0, T; H)$. The lower semi-continuity of both φ and of its conjugate yield

$$\int_0^T [\varphi(t, y) + \varphi^*(t, v)]\,dt \leq \liminf_{n\to\infty} \int_0^T \varphi(t, y_n)\,dt + \liminf_{n\to\infty} \int_0^T \varphi^*(t, v_n)\,dt.$$

Clearly

$$\int_0^T (v, y)\,dt \leq \lim_{n\to\infty} \int_0^T (v_n, y_n)\,dt.$$

We now have

$$d_\varepsilon = g(y) + h(u) + \eta \| v \|^2_{L^2(0,T;H)} + \varepsilon^{-1} \int_0^T [\varphi(t, y) + \varphi^*(t, v) - (y, v)]\,dt.$$

It remains to show that

$$y' \in \mathcal{F}(t, y) + Bu - v \text{ on } (0, T),\ y(0) = y_0.$$

Since $\mathcal{F}$ is u.s.c. from $L^2(0, T; H)$ into the subsets of $L^2(0, T; H)$ and since $y_n \to y$ in $L^2(0, T; H)$, we get: $f(t, y_n) \to f(t, y)$ weakly in $L^2(0, T; H)$ for any $f \in \mathcal{F}$.

The lemma is proved.

LEMMA 3.2. *Suppose all the hypotheses of Theorem* 1.1 *are satisfied. Then for any* η, *there exists at least one solution* $\{y^\eta, u^\eta\}$ *in* $L^2(0, T; H) \times U$ *of the optimal problem* (1.2).

Proof. Let

$$\begin{aligned} 0 \leq d^\eta = \inf\{ & g(y) + h(u) + \eta \| v \|^2_{L^2(0,T;H)} : \\ & y' \in -\mathcal{A}(t, y) + \mathcal{F}(t, y) + Bu,\ 0 \leq t \leq T,\ y(0) = y_0; \\ & u \in \mathcal{U},\ y \in L^2(0, T; H),\ v \in \mathcal{A}(t, y)\}. \end{aligned}$$

From Theorem 2.1 we know that the admissible set

$$\{y : y' \in \mathcal{A}(t, y) + \mathcal{F}(t, y) + Bu \text{ on } (0, T),\ y(0) = y_0\}$$

is non-empty.

Let $\{y_n, u_n, v_n\}$ be a minimizing sequence such that

$$d^\eta \leq g(y_n) + h(u_n) + \eta \| v_n \|^2_{L^2(0,T;H)} \leq d^\eta + n^{-1},$$

and

$$y_n' \in \mathscr{A}(t, y_n) + \mathscr{F}(t, y_n) + Bu_n \text{ on } (0, T), \ y_n(0) = y_0.$$

Then it follows from Theorem 2.1 that

$$\begin{aligned}&\| y_n' \|^2_{L^2(0,T;H)} + \sup_{A\in\mathscr{A}} \| A(., y_n) \|^2_{L^2(0,T;H)} + \sup_{0\le t\le T} \varphi(t, y_n)\\ &\quad \le M(1 + \| u_n \|^2_U + \varphi(0, y_0))\\ &\quad \le C_2(\eta).\end{aligned}$$

We get by taking subsequences: $y_n \to y$ weakly in $L^2(0, T; H)$, $y_n' \to y'$ weakly in $L^2(0, T; H)$, $u_n \to u$ weakly in U, $\{A(t, y_n), v_n\} \to \{z, v\}$ weakly in $(L^2(0, T; H))^2$. From Assumption I.2, we obtain: $y_n \to y$ in $L^2(0, T; H)$.

The u.s.c. of $\mathscr{F}$ yields: $f(t, y_n) \to f(t, y)$ weakly in $L^2(0, T; H)$ for any $f \in \mathscr{F}$. The maximal monotonicity of $\mathscr{A}$ gives: $z \in \mathscr{A}(t, y)$.

Therefore:

$$y' \in - \mathscr{A}(t, y) + \mathscr{F}(t, y) + Bu \text{ on } (0, T), \ y(0) = y_0.$$

It is clear that

$$d^\eta = g(y) + h(u) + \eta \| v \|^2_{L^2(0,T;H)}$$

with $v \in \mathscr{A}(t, y)$.

The lemma is proved.

LEMMA 3.3. *Suppose all the hypotheses of Theorem* 1.1 *are satisfied. Then the set* $\{y_\varepsilon^\eta, u_\varepsilon^\eta, v_\varepsilon^\eta\}$ *of solutions of* (1.1) *given by Lemma* 3.1, *is compact in* $L^2(0, T; H) \times U_{\text{weak}} \times (L^2(0, T; H))_{\text{weak}}$ *for each fixed* η. *Every limit point in* $L^2(0, T; H) \times U_{\text{weak}} \times (L^2(0, T; H))_{\text{weak}}$ *of that set as* $\varepsilon \to 0$, *is a solution of the optimal control problem* (1.2).

Proof. 1) Let $\{y^*, u^*, v^*\}$ be a solution of the optimal control problem (1.2) given by Lemma 3.2.

Taking $y = y^*$, $u = u^*$, $v = v^* = A(t, y^*)$ for some $A \in \mathscr{A}$, i.e. $v^* \in \partial\varphi(t, y^*)$ in (1.1): we get:

$$\begin{aligned}&g(y_\varepsilon) + h(u_\varepsilon) + \eta \| v_\varepsilon \|^2_{L^2(0,T;H)} +\\ &\qquad \varepsilon^{-1} \int_0^T [\varphi(t, y_\varepsilon) + \varphi^*(t, v_\varepsilon) - (y_\varepsilon, v_\varepsilon)]dt\\ &\qquad \le g(y^*) + h(u^*) + \eta \| v^* \|^2_{L^2(0,T;H)},\end{aligned}$$

since

$$\varphi(t, y) + \varphi^*(t, v) - (y, v) = 0 \text{ for all } v = A(t, y) \in \partial\varphi(t, y).$$

It follows that

$$\| u_\varepsilon \|_U + \eta \| v_\varepsilon \|^2_{L^2(0,T;H)} \leq M, \tag{3.6}$$

where M is a positive constant independent of both ε and of η.

On the other hand, using the definition of the conjugate function we have

$$\int_0^t [\varphi(s, y_\varepsilon) - \varphi(s, x) + (x - y_\varepsilon, v_\varepsilon)]ds \leq \varepsilon M \tag{3.7}$$

for all x in $D(\varphi) \cap L^2(0, T; H)$.

Since we assume that $x = 0$ is in $D(\varphi(t,.))$, we get

$$\int_0^t \varphi(s, y_\varepsilon)ds \leq C_2\eta^{-1} + \int_0^t \| y_\varepsilon \|^2 ds. \tag{3.8}$$

But

$$y'_\varepsilon \in \mathscr{F}(t, y_\varepsilon) + Bu_\varepsilon - v_\varepsilon \text{ on } (0, T)\,;\ y_\varepsilon(0) = y_0.$$

So:

$$\| y_\varepsilon \|^2 \leq \| y_0 \|^2 + C_3 \int_0^t \| y_\varepsilon(s) \|^2 ds + C_4\eta^{-1}. \tag{3.9}$$

The different constants C are all independent of ε, η.

The Gronwall lemma applied to (3.10) yields

$$\| y_\varepsilon \|_{L^\infty(0,T;H)} \leq C_5\eta^{-1}. \tag{3.10}$$

It now follows from (3.9) that

$$\int_0^t \varphi(s, y_\varepsilon)ds \leq C_6\eta^{-1}, \tag{3.11}$$

and

$$\| y'_\varepsilon \|_{L^2(0,T;H)} \leq M(\eta).$$

The set $\{y_\varepsilon, u_\varepsilon, v_\varepsilon\}$ is now compact in $L^2(0, T; H) \times U_{\text{weak}} \times (L^2(0, T; H))_{\text{weak}}$.

2) Let $\varepsilon \to 0$ and we have, by taking subsequences: $\{u_\varepsilon, v_\varepsilon\} \to \{u, v\}$ weakly in $U \times L^2(0, T; H)$. Since $u_\varepsilon \in \mathscr{U}$ and $\mathscr{U}$ is closed, u is in $\mathscr{U}$. Assumption I.2 gives: $y_\varepsilon \to y$ in $L^2(0, T; H)$. Clearly $y'_\varepsilon \to y'$ weakly in $L^2(0, T; H)$. Since

$$
\begin{aligned}
&g(y_\varepsilon) + h(u_\varepsilon) + \eta \| v_\varepsilon \|^2_{L^2(0,T;H)} \\
&\quad \le g(y_\varepsilon) + h(u_\varepsilon) + \eta \| v_\varepsilon \|^2_{L^2(0,T;H)} + \varepsilon^{-1} \int_0^T [\varphi(t, y_\varepsilon) + \varphi^*(t, v_\varepsilon) - (y_\varepsilon, v_\varepsilon)] dt \\
&\quad \le g(y^*) + h(u^*) + \eta \| v^* \|^2_{L^2(0,T;H)},
\end{aligned}
$$

we obtain

$$
(3.12) \qquad g(y) + h(u) + \eta \| v \|^2_{L^2(0,T;H)} \le g(y^*) + h(u^*) + \eta \| v^* \|^2_{L^2(0,T;H)}.
$$

3) We now show that

$$
y' \in \mathscr{A}(t, y) + \mathscr{F}(t, y) + Bu \text{ on } (0, T), \; y(0) = y_0.
$$

The upper semi continuity of $\mathscr{F}$ gives: $f(t, y_n) \to f(t, y)$ weakly in $L^2(0, T; H)$ for any $f \in \mathscr{F}$. In (3.7), we get by replacing v_ε with $f(t, y_\varepsilon) + Bu_\varepsilon - y'_\varepsilon$

$$
\int_0^T [\varphi(t, y_\varepsilon) - \varphi(t, x) + (x - y_\varepsilon, f(t, y_\varepsilon) + Bu_\varepsilon - y'_\varepsilon)] dt \le \varepsilon M.
$$

So:

$$
\begin{aligned}
\frac{1}{2} \| y_\varepsilon(T) \|^2 + \int_0^T [\varphi(t, y_\varepsilon) - \varphi(t, x) + (x - y_\varepsilon, f(t, y_\varepsilon + Bu_\varepsilon)] dt \\
- \int_0^T (x, y'_\varepsilon) dt \le \varepsilon M + \frac{1}{2} \| y_0 \|^2.
\end{aligned}
$$

Hence, letting $\varepsilon \to 0$ yields

$$
\int_0^T [\varphi(t, y) - \varphi(t, x) + (x - y, f(t, y) + Bu - y')] dt \le 0.
$$

It follows that

$$
\int_0^T (y' - Bu - f(t, y), x - y) dt \le \int_0^T [\varphi(t, x) - \varphi(t, y)] dt
$$

for all x in $D(\varphi) \cap L^2(0, T; H)$ and for some $f \in \mathscr{F}$.

Thus, $y' - Bu - f(t, y) \in \partial\varphi(t, y)$. It is now clear that $v \in \partial\varphi(t, y)$. Since $\{y^*, u^*, v^*\}$ is a solution of (1.2) any y, u are in the admissible set of the problem (1.2); in view of (3.13) the lemma is proved.

Proof of Theorem 1.1. In view of Lemma 3.3 it remains only to show that there exists at least a solution of (0.1) and that the set of solutions $\{y^\eta, u^\eta, v^\eta\}$ of (1.2) is relatively compact in $L^2(0, T; H) \times U_{\text{weak}} \times (L^2(0, T; H))_{\text{weak}}$ and that the limit in $L^2(0, T; H) \times U_{\text{weak}}$ of any $\{y^\eta, u^\eta, v^\eta\}$ is a solution of (0.1).

The existence of a solution of (0.1) can be established as in the proof of Lemma 3.3 by using a minimizing sequence. We shall not reproduce the proof here.

Let $\{y^*, u^*\}$ be a solution of (0.1) and let $v^* = m[\mathscr{A}[t, y^*)]$, i.e. the element of the convex set $\mathscr{A}(., y^*)$ with minimal $L^2(0, T; H)$-norm. Since y^η, u^η, v^η is a solution of (1.1), we have

$$\begin{aligned} & g(y^\eta) + h(u^\eta) + \eta \| v^\eta \|^2_{L^2(0,T;H)} \\ & \quad \le g(y^*) + h(u^*) + \eta \| v^* \|^2_{L^2(0,T;H)} \\ & \quad \le M . \end{aligned}$$

Thus,

$$\| u^\eta \|_U \le M ,$$

where M is independent of η.

On the other hand

$$(y^\eta)' \in \mathscr{A}(t, y^\eta) + \mathscr{F}(t, y^\eta) + Bu^\eta, \; y^\eta(0) = y_0.$$

Theorem 2.1 gives

$$\| (y^\eta)' \|^2_{L^2(0,T;H)} + \sup\nolimits_{A \in \mathscr{A}} \| A(., u^\eta) \|^2_{L^2(0,T;H)} + \sup \varphi(t, y^\eta) \le C(1 + \varphi(0, y_0) + \| u^\eta \|^2_U).$$

Thus, as before the set $\{y^\eta\}$ is relatively compact in $L^2(0, T; H)$.

Let $\{y, u\}$ be the limit in $L^2(0, T; H) \times U_{\text{weak}}$ of $\{y^\eta, u^\eta\}$, then a proof as before gives

$$y' \in -\mathscr{A}(t, y) + \mathscr{F}(t, y) + Bu \text{ on } (0, T), \; y(0) = y_0$$

and

$$g(y) + h(u) \le g(h^*) + h(u^*).$$

Since $\{y, u)$ is now in the admissible set of the problem and since $\{y^*, u^*\}$ is a solution of (0.1), the theorem is then an immediate consequence of the above inequality.

§4. Extremality relations for (1.1)

The first order necessary conditions of optimality for the differential inclusion (1.1) are derived in this section.

Let $K(u)$ be a closed convex subset of $L^2(0, T; H)$ defined by

$$(4.1)\quad K(u) = \left\{y : y \in L^2(0, T; H),\ 0 \leq \int_0^T \varphi(t, y)dt \leq c(1 + \|u\|_U^2 + \varphi(0, y_0))\right\}$$

and let $I_{K(u}(x)$ be its indicator function.

LEMMA 4.1. *Let $\mathcal{F}$ be a set-valued mapping of $L^2(0, T; H)$ into the subsets of $L^2(0, T; H)$ verifying Assumption* I.3. *Suppose further that:*

1. *For each y, $\mathcal{F}(y)$ is a compact subset of $L^2(0, T; H)$.*
2. *$\mathcal{F}$ is convex, i.e. the graph of $\mathcal{F}$ is a convex subset of $L^2(0, T; H) \times L^2(0, T; H)$.*

Then for each fixed x in $L^2(0, T; H)$, the function

$$(4.2)\qquad F(y\,; x) = \inf_{f \in \mathcal{F}(y)} \int_0^T (f, x)dt$$

is convex and l.s.c. from $L^2(0, T; H)$ into R.

Proof. For each fixed x in $L^2(0, T; H)$, $F(.\,; x)$ is a mapping of $L^2(0, T; H)$ into R and its lower semi-continuity is an immediate consequence of a known result. (Cf. [2] p.67). We now show that it is convex, i.e.

$$F(\lambda y_1 + (1-\lambda)y_2\,; x) \leq \lambda F(y_1\,; x) + (1-\lambda)F(y_2\,; x).$$

for any pair y_1, y_2 in $L^2(0, T; H)$ and any $0 \leq \lambda \leq 1$. Let $\{y_j, f(y_j)\}$ be in Graph $\mathcal{F}(y_i)$, then $\{\lambda y_1 + (1-\lambda)y_2, \lambda f(y_1) + (1-\lambda)f(y_2)\} \in$ Graph $\mathcal{F}(\lambda y_1 + (1-\lambda)y_2)$ for any $0 \leq \lambda \leq 1$. We have

$$\begin{aligned}
&F(\lambda y_1 + (1-\lambda)y_2\,; x)\\
&\quad = \inf_{f \in \mathcal{F}(\lambda y_1 + (1-\lambda)y_2)} \int_0^T (f, x)dt\\
&\quad \leq \inf_{g \in \lambda\mathcal{F}(y_1)} \int_0^T (g, x)dt + \inf_{g \in (1-\lambda)\mathcal{F}(y_2)} \int_0^T (g, x)dt\\
&\quad \leq \lambda \inf_{h \in \mathcal{F}(y_1)} \int_0^T (h, x)dt + (1-\lambda) \inf_{h \in \mathcal{F}(y_2)} \int_0^T (h, x)dt\\
&\quad \leq \lambda F(y_1\,; x) + (1-\lambda)F(y_2\,; x).
\end{aligned}$$

The lemma is proved.

Let

$$\gamma(y_1\, u,\, v;\, p) =$$
$$F(y\,;\, p) - \int_0^T (v - Bu,\, p)dt + \int_0^T (y,\, p')dt$$
$$- (y(T),\, p(T)) + (y_0,\, p(0))$$

for any $\{y, u, v\}$ in $W^{1,2}(0, T; H) \times U \times L^2(0, T; H)$ and $p \in W^{1,2}(0, T; H)$.

The strategy set S is given by

(4.3) $\quad S = \{\{y, u, v\} : \{y, u, v\} \in W^{1,2}(0, T; H) \times U \times L^2(0, T; H),\, \gamma(y, u, v\,; p) \leq 0$

for all p in $W^{1,2}(0, T; H)\}$.

Let Γ be the mapping of $L^2(0, T; H) \times U \times L^2(0, T; H)$ into R^+ defined by

(4.4)
$$\Gamma(y,\, u,\, v) = g(y) + h(u) + \eta \| v \|^2_{L^2(0,T;H)} + I_{K(u)}(y) +$$
$$\varepsilon^{-1} \int_0^T [\varphi(t,\, y) + \varphi^*(t,\, v) - (v,\, y)]dt.$$

Now problem (1.1) may be rephrased as

(1.1′)
$$\inf_{\{y,u,v\} \in S} \Gamma(y,\, u,\, v).$$

The Lagrangian of the problem (1.1′) is

(4.5)
$$L(y,\, u,\, v;\, p) = \Gamma(y,\, u,\, v) + \gamma(y,\, u,\, v;\, p).$$

It is defined for $\{y, u, v\}$ in S and p in $W^{1,2}(0, T; H)$.

The Lagrange multipliers p_* are given by

$$\inf\{L(y,\, u,\, v;\, p_*) : \{y,\, u,\, v\} \in L^2(0,\, T;\, H) \times U \times L^2(0,\, T;\, H)\}$$
$$= \inf\{ \sup_{p \in W^{1,2}(0,T;H)} L(y,\, u,\, v;\, p) : \{y,\, u,\, v\} \in L^2(0,\, T;\, H) \times U \times L^2(0,\, T;\, H)\}.$$

It is known that $\{y_*, u_*, p_*\}$ is an optimal solution of (1.1) iff:

(i) $\{y_*, u_*, v_*\}$ minimizes $L(y, u, v; p_*)$ on $L^2(0, T; H) \times U \times L^2(0, T; H)$ and

(ii) $\gamma\{y_*, u_*, v_*; p_*\} = 0$.

Thus, from (4.4), (4.6) we get

(4.6)
$$p' + \partial_y F(y_*, p) + \partial g(y_*) + \partial I_{K(u_*)}(y_*) + \varepsilon^{-1} \int_0^T (\partial\varphi(t, y_*) - v_*)\, dt \ni 0\,;$$
$$p(T) = y_*(T).$$

and

$$\partial h(u_*) + B^* p \ni 0 \tag{4.7}$$

with

$$2\eta v_* + \varepsilon^{-1}\partial\varphi^*(t, v_*) - \varepsilon^{-1} y_* - p \ni 0. \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$\begin{aligned}&p' + \partial_y F(y_*, p) + \partial g(y_*) + \partial\sigma^*_{K(u_*)}(y_*)\\ &- (2\varepsilon\eta)^{-1}\int_0^T p\,dt + \varepsilon^{-1}\int_0^T [\partial\varphi(t, y_*) - (2\eta)^{-1} y_* + (2\varepsilon\eta)^{-1}\partial\varphi^*(t, v_*)]\,dt \ni 0,\\ &p(T) = y_*(T).\end{aligned} \tag{4.9}$$

Let $\mathscr{P}(y_*; p)$ be given by

$$\mathscr{P}(y_*; p) = \partial_y F(y_*; p) - (2\varepsilon\eta)^{-1}\int_0^T p\,dt.$$

Then $\mathscr{P}(y_*; p)$ is a mapping of $L^2(0, T; H)$ into $L^2(0, T; H)$ and is linear in p. Since h is convex and l.s.c. from U to R^+, its sub-differential $\partial h(x)$ is a closed convex subset of U. Let

$$\mathscr{K}(t) = \{p(t) : p \in L^2(0, T; H),\ B^* p \in -\partial h(u_*(t))\}$$

for almost all t in $(0, T)$.

The problem (4.9) may be rewritten as

$$-p' \in \mathscr{P}(y_*; p) + \mathscr{Y}(u_*, v_*) \text{ on } (0, T), p \in \mathscr{K}(t) \text{ a.e. on } (0, T), p(T) = y_*(T) \tag{4.10}$$

with

$$\mathscr{Y}(y_*, v_*) = \partial g(y_*) + \partial I_{K(u_*)}(y_*) + \varepsilon^{-1}\int_0^T [\partial\varphi(y_*) - (2\eta)^{-1} y_* + (2\varepsilon\eta)^{-1}\partial\varphi^*(v_*)]\,dt.$$

THEOREM 4.1. *Suppose all the hypotheses of Theorem* 1.1 *and of Lemma* 4.1 *are satisfied. Let* $\{y_*, u_*, v_*\}$ *be an optimal solution of the problem* (1.1). *If* $u_*(T)$ *is in* H, *then we assume that* $B^* y_*(T) \in -\partial h(u_*(T))$. *Then there exists a unique* p *with* p *and* p' *in* $L^2(0, T; H)$, *solution of the inclusion*

$$-p' \in \partial I_{\mathscr{K}(t)}(p) + \mathscr{P}(y_*; p) + \mathscr{Y}(u_*, v_*) \text{ on } (0, T); p(T) = y_*(T).$$

Proof. Let $\phi(t; x) = \frac{1}{2}\|x\|^2 + I_{\mathscr{K}(t)}(x)$, then ϕ is a convex, l.s.c. function from H into R^+ for almost all t. Its subdifferential $\partial\phi(t; x) = x + \partial I_{\mathcal{K}(t)}(x)$ is a set-valued mapping of H into H for almost all t.

Consider the problem

$$(4.11) \quad -p' + \partial\phi(t, p) - (\mathscr{P}(y_*, p) + p) - \mathscr{Y}(u_*, v_*) \ni 0 \text{ on } (0, T)\,; \qquad p(T) = y_*(T).$$

Since $y_*(T)$ is in $D(\phi(T,.)$; it is known that (4.11) has a unique solution p with p and p' in $L^2(0, T; H)$. The theorem is proved.

§5. Applications

Let Ω_t be a bounded open set of R^n with boundary Γ_t and set $\Omega = \cup_{0\le t\le T}(\Omega_t \times \{t\})$, $\Gamma = \cup_{0\le t\le T}(L_t \times \{t\})$. We shall make the following assumptions on Ω.

Assumption V.

1. *There exist $k \in N$ and ε_0 such that for each t in $[0, T]$, Γ_t consists of closed hypersurfaces Γ_t^j of class C^3 and* dist $(\Gamma_t^j, \Gamma_t^i) \geq \varepsilon_0$ *for $j \neq i$.*
2. *Let $\Omega_s^t = \cup_{s\le r\le t}(\Omega_r \times \{r\})$. Then the domain Ω is covered by N slices $\Omega_{t_j}^{\delta_j+t_j}$, $\delta_j > 0$ and $j = 1, \dots, N$. For each j, $\Omega_{t_j}^{t_j+\delta_j}$ is mapped onto a cylindrical domain $\Omega_{t_j} \times (t_j, t_j + \delta_j)$ by a diffeomorphism of class C^4 up to the boundary, which preserves the time-variable.*

Let G be an open ball of R^n with $cl\Omega \subset G$ for all t in $[0, T]$.

1. A strongly nonlinear parabolic inclusion

Let $U = L^2(0, T; L^2(G))$ and let $\mathcal{U}$, the set of admissible controls be a closed convex subset of U, e.g.

$$\{u : u \in U, \alpha \le u(x, t) \le \beta \text{ a.e. in } (0, T) \times G, \int_\Omega u(x, t)\,dxdt = M\}$$

Let K be a closed convex subset of $L^2(G)$ with $0 \in K$, a typical example of K is

$$K = \{y : y(x) \in L^2(G),\ 0 \le y(x) \text{ a.e. in } G\}.$$

We shall take $\mathcal{K}(t)$ to be the set

$$\mathcal{K}(t) = \{y(x, y) : y \in L^2(G) \cap K \text{ a.e. in } (0, T), \, y = 0 \text{ on } G - \Omega_t\}.$$

It is easy to see that $\mathcal{K}(t)$ is a closed convex subset of $L^2(G)$.

Let

$$\varphi(t, y) = \tilde{\varphi}(t, y) + I_{\mathcal{K}(t)}(y). \tag{5.1}$$

where

$$\tilde{\varphi}(t, y) = r^{-1} \| \nabla y \|^r_{L^r(G)} \text{ if } y \in W_0^{1,r}(G)$$

and

$$\tilde{\varphi}(t, y) = +\infty \text{ otherwise.}$$

Then $\varphi(t,.)$ is a l.s.c. convex function of $L^2(G)$ into R^+ with $D(\varphi(t,.)) = \{y : y \in \mathcal{K}(t) \cap W_0^{1,r}(G)\}$. Since $\mathcal{K}(t)$ is a closed convex subset of $L^2(G)$, the indicator of the set is a l.s.c. convex function on $L^2(G)$ and for any y in $D(\partial\varphi(t,.))$, the subdifferential of $\varphi(t, y)$ is

$$\partial\varphi(t, y) = -\nabla(|\nabla y|^{r-2} \nabla y) + \partial I_{\mathcal{K}(t)}.$$

With φ as above, its conjugate is given by

$$\varphi^*(t, v) = \sup_{z \in K(t) \cap W_0^{1,r}(\Omega_t)} \left\{ \int_{\Omega_t} vz dx - r^{-1} \int_{\Omega_t} |\nabla z|^r dx \right\}. \tag{5.2}$$

It is known that there exists a unique solution z_v of the nonlinear elliptic boundary-value problem

$$-\nabla(|\nabla z|^{r-2} \nabla z) + \partial I_{K(t)}(z) \ni v \text{ in } \Omega_t, \, z = 0 \text{ on } \partial\Omega_t. \tag{5.3}$$

for any given v in $L^2(G)$. It is not difficult to check that: $\varphi^*(t, v) = (1 - r^{-1}) \int_{\Omega_t} z_v v$.

Consider the optimal control problem

$$\inf\Big\{ \int_\Omega | y(x, t) - q(x, t) |^2 dxdt +$$
$$\frac{1}{2} \int_\Omega | u(x, t) |^2 dxdt +$$
$$\eta \int_\Omega | v |^2 dxdt + \varepsilon^{-1} \int_\Omega (r^{-1} | \nabla y |^r + (1 - r^{-1}) z_v v - yv) dxdt$$

$: y' \in \mathscr{F}(t, y) + Bu - v,\ y(0) = y_0\}.$

Let $q(x, t)$ be in $L^2(0, T; L^2(G))$ and let

$$g(y) = \frac{1}{2}\int_\Omega |y(x, t) - q(x, t)|^2 dxdt;\ h(u) = \frac{1}{2}\|u\|_U^2.$$

We shall study the control problem

$$\inf\left\{\int_\Omega |y(x, t) - q(x, t)|^2 dxdt + \frac{1}{2}\int_\Omega |u(x, t)|^2 dxdt : y \in S(u)\right\} \tag{5.5}$$

where $S(u)$ is the set of solutions of the initial boundary-value problem

$$y' - \nabla(|\nabla y|^{r-2}\nabla y) \in \mathscr{F}(t, y) + Bu \text{ on } \Omega,\ y(x, t) = 0 \text{ on } \Gamma,\ y(0) = y_0. \tag{5.6}$$

THEOREM 5.1. *Let $\mathscr{F}$ be an u.s.c. set-valued mapping of $L^2(0, T; L^2(G))$ into the subsets of $L^2(0, T; L^2(G))$ satisfying Assumption* I.3. *Suppose that Assumption* V *is verified and let φ be as above with $y_0 \in K(0) \cap W_0^{1,r}(\Omega_0)$. Then the set of solutions $\{y_\varepsilon^\eta, u_\varepsilon^\eta, v_\varepsilon^\eta\}$ of* (5.4) *is compact in $L^2(0, T; L^2(G)) \times (L^2(0, T; L^2(G))_{\text{weak}} \times (L^2(0, T; L^2(G))_{\text{weak}}$. Let $\varepsilon \to 0$ and then let $\eta \to 0$, then the set of limit points $\{y, u, v\}$ of $\{y_\varepsilon^\eta, u_\varepsilon^\eta, v_\varepsilon^\eta\}$ are solutions of the optimal control problem* (5.5)-(5.6).

Proof. With φ as in the theorem and with Assumption V, it was shown by Yamada [11] that φ satisfies Assumption I.1. It is clear that Assumption I.2 is a direct consequence of the Sobolev imbedding theorem and of Aubin's theorem. The stated result is now an immediate consequence of Theorem 1.1.

2. Mixed boundary problems for evolution inclusions

Let Ω_t be as before and let G be a bounded, open simply connected subset of R^n with a smooth boundary. We assume that Ω_t is a subset of G for all t and that $\gamma_t = \partial G \cap \Gamma_t$ is a non-empty closed surface. Set: $\gamma = \cup_{0 \le t \le T} \gamma_t$ and let

$$H(G) = \{y : y \in W^{1,2}(G),\ y = 0 \text{ on } \partial G - \gamma\}.$$

Let j be a proper l.s.c. convex function from R to $[0, \infty]$ with $j(0) = 0$ and let $\beta = \partial j$. Consider the l.s.c. convex funtion φ^* of $L^2(G)$ into R^+ defined by

$$\tilde{\varphi}(y) = \frac{1}{2}\int_G |\nabla y|^2 dx + \int_\gamma j(y)d\sigma \text{ if } y \in H(G),\ j(y) \in L^1(\gamma)$$

and

$$\tilde{\varphi}(y) = \infty \text{ otherwise.}$$

Let $K(t) = \{y : y \in L^2(G),\ y = 0 \text{ a.e. in } G - \Omega_t\}$ and set $\varphi(t, y) = \tilde{\varphi}(y) + I_{K(t)}(y)$. It was shown in [8] that

$$D(\varphi(t,.)) = \Big\{y : y \in L^2(G), \Delta y \in L^2(G),\ y|_{\Omega_t} \in W^{1,2}(\Omega_t),$$
$$y = 0 \text{ on } G - \Omega_t,\ -\frac{\partial y}{\partial n} \in \beta(y) \text{ on } \gamma_t\Big\}$$

with $\partial\varphi(t, y) = -\Delta y$.

The conjugate of $\varphi(t, y)$ is

$$\varphi^*(t, v) = \sup_{z \in W^{1,2}(\Omega_t), z=0 \text{ on } \partial G - \gamma_t}\Big\{\int_{\Omega_t}\Big[zv - \frac{1}{2}|\nabla z|^2\Big]dx - \int_{\gamma_t} j(z)\,d\sigma\Big\}.$$

Consider the mixed boundary-value problem

$$-\Delta z = v \text{ in } \Omega_t,\ -\frac{\partial z}{\partial n} \in \beta(z) \text{ on } \gamma_t,\ z = 0 \text{ on } \partial\Omega_t - \gamma t. \tag{5.7}$$

It was shown in [8] that (5.7) has a unique solution z_v in $W^{1,2}(\Omega_t)$. Since z_v is in $W^{1,2}(\Omega_t)$ and Δz_v is in $L^2(\Omega_t)$, it is known that $\dfrac{\partial z_v}{\partial n}$ is in $L^2(\partial\Omega_t)$ and it is not difficult to check that: $\varphi^*(t, v) = \dfrac{1}{2}\displaystyle\int_{\Omega_t} z_v v\,dx - \dfrac{1}{2}\int_{\gamma_t}\frac{\partial z_v}{\partial n}v\,d\sigma.$

As before we now consider the optimal control problem

$$\begin{aligned}
&\inf\Big\{\int_\Omega |y(x, t) - q(x, t)|^2 dxdt + \frac{1}{2}\int_\Omega |u(x, t)|^2 dxdt + \eta\int_\Omega |v|^2 dxdt + \\
&\varepsilon^{-1}\Big[\frac{1}{2}\int_\Omega\Big\{|\nabla y|^2 + \frac{1}{2}z_v v\Big\}dxdt + \int_\gamma j(y)\,d\sigma - \frac{1}{2}\frac{\partial z_v}{\partial n}v - yv)\,dxdt\Big] : \\
&y' \in \mathcal{F}(t, y) \text{ on } (0, T),\ y(0) = y_0\Big\}.
\end{aligned} \tag{5.8}$$

The problem (5.8) will serve as the approximating system for

$$\inf\Big\{\int_\Omega |y(x, t) - q(x, t)|^2 dxdt + \int_\Omega |u(x, t)|^2 dxdt : y \in S(u)\Big\} \tag{5.9}$$

where $S(u)$ is the set of solutions of the initial boundary-value problem

$$y' - \Delta y \in \mathcal{F}(t, y) + u \text{ on } \Omega,\ -\frac{\partial y}{\partial n} \in \beta(y) \text{ on } \gamma_t, \tag{5.10}$$

$$y = 0 \text{ on } \partial\Omega - \gamma\,;\, y(0) = y_0.$$

THEOREM 5.2. *Let* φ *be as above and let* $\mathscr{F}$ *be a set-valued mapping of* $L^2(0, T; L^2(G))$ *into the subsets of* $L^2(0, T; L^2(G))$ *satisfying Assumption* I.3. *Suppose that Assumption* V *is verified and let* y_0 *be in* $K(0) \cap H(G)$ *with* $j(y_0) \in L^1(\gamma)$. *Then the set of solutions* $\{y_\varepsilon^\eta, u_\varepsilon^\eta, v_\varepsilon^\eta\}$ *of the optimal control problem* (5.8) *is compact in* $L^2(0, T; L^2(G)) \times (L^2(0, T; L^2(G)))_{\text{weak}} \times (L^2(0, T; L^2(G)))_{\text{weak}}$. *The set of limit points* $\{y, u, v\}$ *of the solution-set of* (5.8) *as* $\varepsilon \to 0$ *and then as* $\eta \to 0$ *is a solution-set of the optimal control problem* (5.9)-(5.10).

Proof. Again with φ as in the theorem and with Assumption V, one can show that Assumption I.1 is verified. (Cf. Yamada [11]). It is clear that Assumption I.2 is satisfied and the stated result is an immediate consequence of Theorem 1.1.

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