K. Tojo
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COVARIANT DERIVATIVES ON KÄHLER C-SPACES

KOJI TOJO

0. Introduction

Let (M, g) be a Kähler C-space. R and ∇ denote the curvature tensor and the Levi-Civita connection of (M, g), respectively.

In [6], Takagi have proved that there exists an integer n such that

$$\hat{\nabla}^{n-1} R \neq 0, \ \hat{\nabla}^{n} R \neq 0,$$

where $\hat{\nabla}$ denotes the covariant derivative of (1,0)-type induced from ∇ (see Section 3 for the defintion). Moreover, Takagi classified Kähler C-spaces with $n = 2$ (Hermitian symmetric spaces of compact type are characterized as Kähler C-spaces with $n = 1$).

However, there is a mistake in deduction to lead a certain formula. The purpose of this paper is to correct the mistake and to classify Kähler C-spaces with $n = 2$. Moreover, in Section 5, we shall classify Kähler C-spaces with $n = 3$.

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1. Preliminaries

Let G be a Lie group and K a closed subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Suppose that $\mathrm{Ad}(K)$ is compact. Then there exist an $\mathrm{Ad}(K)$-invariant decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of $\mathfrak{g}$ and an $\mathrm{Ad}(K)$-invariant scalar product $\langle , \rangle$ on $\mathfrak{p}$. Then

$$[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \tag{1.1}$$

$$\langle [u, x], y \rangle + \langle [u, y], x \rangle = 0 \ (u \in \mathfrak{k}, \ x, y \in \mathfrak{p}). \tag{1.2}$$

Moreover, under the canonical identification of $\mathfrak{p}$ with the tangent space $T_o(G/K)$ $(o = \{K\})$ of homogeneous space G/K, the scalar product $\langle , \rangle$ can be extended to

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a G-invariant metric on G/K.

Let Λ be the connection function of $(G/K, \langle , \rangle)$ (cf.[5]). Then for $x, y \in \mathfrak{p}$,

$$\Lambda(x)(y) = \frac{1}{2}[x, y]_{\mathfrak{p}} + U(x, y) \tag{1.3}$$

where

$$\langle U(x, y), z\rangle = \frac{1}{2}\{\langle [z, x]_{\mathfrak{p}}, y\rangle + \langle [z, y]_{\mathfrak{p}}, x\rangle\} \quad (z \in \mathfrak{p}). \tag{1.4}$$

Furthermore the curvature tensor R is given by

$$\begin{aligned} R(x, y)z = {} & [\Lambda(x), \Lambda(y)]z - [[x, y]_{\mathfrak{k}}, z] \\ & - \Lambda([x, y]_{\mathfrak{p}})z. \end{aligned} \tag{1.5}$$

In the remaining part of this section we describe irreducible Kähler C-spaces and recall some properties with respect to the connection functions (see [3] for example).

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbf{C}$ with $\mathrm{rk}(\mathfrak{g}) = l$, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, Δ denotes the set of non-zero roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. For some lexicographic order we denote by $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ the fundamental root system of Δ. Moreover let Δ^+ be the set of positive roots of Δ with respect to the order. Since $\mathfrak{g}$ is simple, we can define $H_\alpha \in \mathfrak{h}$ $(\alpha \in \Delta)$ by

$$B(H, H_\alpha) = \alpha(H) \quad (H \in \mathfrak{h})$$

where B is the Killing form of $\mathfrak{g}$. We choose root vectors $\{E_\alpha\}$ $(\alpha \in \Delta)$ so that for $\alpha, \beta \in \Delta$

$$\begin{aligned} & B(E_\alpha, E_{-\alpha}) = 1, \\ & [E_\alpha, E_\beta] = N_{\alpha,\beta}E_{\alpha+\beta}, \; N_{\alpha,\beta} = -N_{-\alpha,-\beta} \in \mathbf{R}. \end{aligned} \tag{1.6}$$

Then $[E_\alpha, E_{-\alpha}] = H_\alpha$. Moreover the following hold (cf. [2]).

$$N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha} \text{ if } \alpha + \beta + \gamma = 0 \tag{1.7}$$

$$N_{\alpha,\beta}N_{\gamma,\delta} + N_{\beta,\gamma}N_{\alpha,\gamma} + N_{\gamma,\alpha}N_{\beta,\delta} = 0, \tag{1.8}$$

if $\alpha + \beta + \gamma + \delta = 0$ (no two of which have sum 0). Let $\{\beta + n\alpha; p \leq n \leq q\}$ be the α-series containing β. Then

$$(N_{\alpha,\beta})^2 = \frac{q(1-p)}{2}\alpha(H_\alpha), \; \frac{2\alpha(H_\beta)}{\alpha(H_\alpha)} = -(p+q). \tag{1.9}$$

As is well-known, the subalgebra $\mathfrak{g}_u$ of $\mathfrak{g}$ defined in the following is a compact real form of $\mathfrak{g}$:

$$\mathfrak{g}_u = \sum_{\alpha\in\Delta^+} \mathbf{R}\sqrt{-1}\, H_\alpha + \sum_{\alpha\in\Delta^+} (\mathbf{R}A_\alpha + \mathbf{R}B_\alpha),$$

where $A_\alpha = E_\alpha - E_{-\alpha}$ and $B_\alpha = \sqrt{-1}(E_\alpha + E_{-\alpha})$.

Consider a non-empty subset $\Psi = \{\alpha_{i_1}, \ldots, \alpha_{i_r}\}$ of Π. Set

$$\Delta^+(\Psi) = \left\{\alpha = \sum_{j=1}^{l} n_j\alpha_j \in \Delta^+;\, n_{i_k} > 0 \text{ for some } \alpha_{i_k} \in \Psi\right\}. \tag{1.10}$$

Then we define a subalgebra $\mathfrak{k}_\Psi$ as follows:

$$\mathfrak{k}_\Psi = \sum_{\alpha\in\Delta^+} \mathbf{R}\sqrt{-1}H_\alpha + \sum_{\alpha\in\Delta^+-\Delta^+(\Psi)} (\mathbf{R}A_\alpha + \mathbf{R}B_\alpha).$$

Let G_u and K_Ψ be a simply connected Lie group and its connected closed subgroup which correspond to $\mathfrak{g}_u$ and $\mathfrak{k}_\Psi$ respectively. Then G_u/K_Ψ is an irreducible C-space.

Put

$$\mathfrak{p} = \sum_{\alpha\in\Delta^+(\Psi)} (\mathbf{R}A_\alpha + \mathbf{R}B_\alpha).$$

Then $\mathfrak{g}_u = \mathfrak{k}_\Psi + \mathfrak{p}$ (direct sum) and the tangent space $T_o(G_u/K_\Psi)$ of G_u/K_Ψ at $o = \{K_\Psi\}$ is identified with $\mathfrak{p}$. Then a complex structure I is given at o by

$$I(A_\alpha) = B_\alpha,\ I(B_\alpha) = -A_\alpha\ (\alpha \in \Delta^+(\Psi)). \tag{1.11}$$

We set

$$\mathfrak{p}^\pm = \sum_{\alpha\in\Delta^+(\Psi)} \mathbf{C}E_{\pm\alpha}. \tag{1.12}$$

Then we have $\mathfrak{p}^\pm = \{X \in \mathfrak{p}^{\mathbf{C}};\, I(X) = \pm\sqrt{-1}X\}$. An element of $\mathfrak{p}^+$ is said to be of (1,0)-type.

Define a mapping $p : \Delta^+(\Psi) \to \mathbf{Z}^r$ as follows:

$$p(\alpha) = (n_{i_1}(\alpha), \ldots, n_{i_r}(\alpha)) \text{ for } \alpha = \sum_{i=1}^{l} n_i(\alpha)\alpha_i \in \Delta^+(\Psi).$$

Let ω^α and $\bar\omega^\alpha$ be the dual forms of E_α and $E_{-\alpha}$, respectively. Then any G_u-invariant Kähler metric g is given at o by

$$g = -2\sum_{\alpha\in\Delta^+(\Psi)} (c\cdot p(\alpha))\,\omega^\alpha\cdot\bar\omega^\alpha \tag{1.13}$$

where $c = (c_1, \ldots, c_r)$ $(c_j > 0)$ and $c \cdot p(\alpha) = \sum_{j=1}^{r} c_j n_{i_j}(\alpha)$. Conversely, any bilinear form $-2\sum_\alpha (c \cdot p(\alpha))\omega^\alpha \cdot \bar{\omega}^\alpha$ on $\mathfrak{p}^{\mathbf{C}} \times \mathfrak{p}^{\mathbf{C}}$ can be extended to a G_u-invariant metric on G_u/K_Ψ.

In the following we regard the metrics, connections and tensors as ones extended naturally over $\mathbf{C}$.

In [3] the connection functions of Kähler spaces are determined.

For $\alpha, \beta \in \Delta$ we write $p(\alpha) > p(\beta)$ if $n_{i_k}(\alpha) \geq n_{i_k}(\beta)$ $(k = 1, \ldots, r)$ and $n_{i_j}(\alpha) > n_{i_j}(\beta)$ for some j. Then

LEMMA 1.1. *For $\alpha \in \Delta^+(\Psi)$, identify α with E_α and $\bar{\alpha}$ with $E_{-\alpha}$. Then*

$$\Lambda(\alpha)(\beta) = \frac{c \cdot p(\beta)}{c \cdot p(\alpha + \beta)}[\alpha, \beta]$$

$$\Lambda(\bar{\alpha})(\beta) = \begin{cases} [\bar{\alpha}, \beta] & p(\alpha) < p(\beta) \\ 0 & \textit{otherwise} \end{cases}$$

$$\Lambda(\alpha)(\bar{\beta}) = \begin{cases} [\alpha, \bar{\beta}] & p(\alpha) < p(\beta) \\ 0 & \textit{otherwise} \end{cases}$$

$$\Lambda(\bar{\alpha})(\bar{\beta}) = \frac{c \cdot p(\beta)}{c \cdot p(\alpha + \beta)}[\bar{\alpha}, \bar{\beta}].$$

2. Covariant derivatives on homogeneous spaces

In this section we shall write the Levi-Civita connections of Riemannian homogeneous spaces in terms of the Lie algebras.

Let (M, g) be an n-dimensional Riemannian manifold and ∇ the Levi-Civita connection of (M, g). Let $\{e_1, \ldots, e_n\}$ be local orthonormal frame fields and $\{\omega^1, \ldots, \omega^n\}$ their dual 1-forms. Associated with $\{e_1, \ldots, e_n\}$, there uniquely exist local 1-forms $\{\omega_i^j\}$ $(i, j = 1, \ldots, n)$, which are called the connection forms, such that

$$\omega_i^j + \omega_j^i = 0 \tag{2.1}$$

$$d\omega^i + \sum_{j=1}^{n} \omega_j^i \wedge \omega^j = 0. \tag{2.2}$$

Then the following holds.

$$\nabla_{e_i} e_j = \sum_{k=1}^{n} \omega_j^k(e_i) e_k \tag{2.3}$$

(see [4]).

Next, let $(G/K, \langle, \rangle)$ be a homogeneous space with a G-invariant metric $\langle, \rangle$ as stated in Section 1.

Let $\pi : G \rightarrow G/K$ be the canonical projection and W an open subset in $\mathfrak{p}$ such that $0 \in W$ and the mapping

$$\pi \circ \exp : W \rightarrow \pi(\exp W)$$

is diffeomorphic. Let $\{e_\alpha\}_{\alpha \in A}$ be a basis of $\mathfrak{k}$ and $\{e_i\}_{i \in I}$ an orthonormal basis of $(\mathfrak{p}, \langle, \rangle)$. In this section we use the following convention on the range of indices, unless otherwise stated:

$$i, j, k, \ldots \in I, \ \alpha, \beta, \gamma, \ldots \in A,$$
$$p, q, r, \ldots \in I \cup A.$$

Let $\{X_\alpha\}$ and $\{X_i\}$ be the left invariant vector fields on G such that $(X_\alpha)_e = e_\alpha$ and $(X_i)_e = e_i$ (e is the identity of G). Furthermore we define an orthonormal frame field $\{E_i\}$ on $\pi(\exp W)$ and the mapping $\mu : \pi(\exp W) \rightarrow \exp W$ as follows:

$$(E_i)_{\pi(\exp x)} = \tau(\exp x)_*(e_i)$$
$$\mu(\pi(\exp x)) = \exp x \quad (x \in W),$$

where $\tau(g)$ $(g \in G)$ denotes the left transformation of G/K. Then since $\pi_*(X_i) = E_i$, $\pi_*(X_\alpha) = 0$ and $\pi_*\mu_* = \mathrm{id}$, we can put

$$\mu_*(E_i) = X_i + \sum_\alpha \eta_{\alpha i} X_\alpha. \tag{2.4}$$

Let $\{\omega^\alpha\}$, $\{\omega^i\}$ and $\{\theta^i\}$ be the dual 1-forms of $\{X_\alpha\}$, $\{X_i\}$ and $\{E_i\}$, respectively. Then it is easy to see

$$\mu^*(\omega^i) = \theta^i. \tag{2.5}$$

Set $[X_p, X_q] = \sum_r c_{pq}{}^r X_r$. Then the following is known as the equation of Maurer-Cartan (cf. [4]).

$$d\omega^p = -\frac{1}{2} \sum_{q,r} c_{qr}{}^p \omega^q \wedge \omega^r. \tag{2.6}$$

For the sake of completeness we show the following well-known fact.

LEMMA 2.1 *Let $\{\theta_j{}^i\}$ be the connection forms of $(G/K, \langle, \rangle)$ associated with $\{E_i\}$. Then*

$$\theta_j^{\ i} = -\mu^*\{\sum_\alpha c_{j\alpha}^{\ \ i}\omega^\alpha + \frac{1}{2}\sum_k (c_{jk}^{\ \ i} - c_{ik}^{\ \ j} - c_{ij}^{\ \ k})\omega^k\}.$$

Proof. It follows from (1.1) and (1.2) that

$$c_{j\alpha}^{\ \ \beta} = 0, \quad c_{i\alpha}^{\ \ j} + c_{i\alpha}^{\ \ j} = 0. \tag{2.7}$$

Moreover since $\mathfrak{k}$ is subalgebra of $\mathfrak{g}$, we get

$$c_{\alpha\beta}^{\ \ i} = 0. \tag{2.8}$$

From equations (2.5), (2.6), (2.7) and (2.8) it follows that

$$\begin{aligned} d\theta^i &= \mu^* d\omega^i \\ &= -\sum_j \mu^*\{\sum_\alpha c_{j\alpha}^{\ \ i}\omega^j \wedge \omega^\alpha + \frac{1}{2}\sum_k (c_{jk}^{\ \ i} - c_{ik}^{\ \ j} - c_{ij}^{\ \ k})\omega^j \wedge \omega^k\} \\ &= \sum_j \mu^*\{\sum_\alpha c_{j\alpha}^{\ \ i}\omega^\alpha + \frac{1}{2}\sum_k (c_{jk}^{\ \ i} - c_{ik}^{\ \ j} - c_{ij}^{\ \ k})\omega^k\} \wedge \theta^j \end{aligned}$$

(note that $\sum_{j,k}(c_{ij}^{\ \ k} + c_{ik}^{\ \ j})\omega^j \wedge \omega^k = 0$).

Put $\theta_j^{\ i} = -\mu^*\{\sum_\alpha c_{j\alpha}^{\ \ i}\omega^\alpha + (1/2)\sum_k(c_{jk}^{\ \ i} - c_{ik}^{\ \ j} - c_{ij}^{\ \ k})\omega^k\}$. Then it is easy to see $\theta_j^{\ i} + \theta_i^{\ j} = 0$.

Consequently, by (2.1) and (2.2), the connection forms coincide with $\{\theta_j^{\ i}\}$. □

By (2.3), (2.4) and the above lemma, we have the following.

PROPOSITION 2.2.

$$\nabla_{E_i}E_j = \sum_k \{\sum_\alpha c_{\alpha j}^{\ \ k}\eta_{\alpha i} + \frac{1}{2}(c_{ij}^{\ \ k} - c_{ik}^{\ \ j} - c_{jk}^{\ \ i})\}E_k.$$

Next we shall rewrite Proposition 2.2 in terms of the bracket operation [,] of $\mathfrak{g}$.

For $x \in W$, we define $z_x^{\ i}(t) \in W$ and $h_x^{\ i}(t) \in K$ ($t \in \mathbf{R}$, $|t|$: small enough) by the following:

$$\exp x \cdot \exp te_i = \exp z_x^{\ i}(t) \cdot h_x^{\ i}(t) \tag{2.9}$$

with $z_x^{\ i}(0) = x$ and $h_x^{\ i}(0) = e$. Then

$$\begin{aligned} \mu_*(E_i)_{\pi(\exp x)} &= \frac{d}{dt}\Big|_0 \mu(\pi(\exp x \cdot \exp te_i)) \\ &= \frac{d}{dt}\Big|_0 \mu(\pi(\exp z_x^{\ i}(t))) \end{aligned}$$

$$= (\exp_*)_x\Big(\frac{d}{dt}\big|_0 z_x^{\,i}(t)\Big).$$

Here, the differential map $\exp_*$ of $\exp$ has the following form (see [2]).

LEMMA 2.3. *Let x, $y \in \mathfrak{g}$. Then*

$$(\exp_*)_x(y) = (L_{\exp x})_* \circ \Phi_x(y),$$

where $\Phi_x(y) = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(n+1)!} (\mathrm{ad}x)^n(y)$.

Thus we have

$$\mu_*(E_i)_{\pi(\exp x)} = (L_{\exp x})_* \circ \Phi_x\Big(\frac{d}{dt}\big|_0 z_x^{\,i}(t)\Big). \tag{2.10}$$

On the other hand, (2.9) and Lemma 2.3 give

$$\begin{aligned}(L_{\exp x})_* \circ \Phi_x\Big(\frac{d}{dt}\big|_0 z_x^{\,i}(t)\Big) &= (L_{\exp x})_*(e_i) \\ &\quad - (L_{\exp x})_*\Big(\frac{d}{dt}\big|_0 h_x^{\,i}(t)\Big).\end{aligned} \tag{2.11}$$

Considering (2.4), (2.10) and (2.11), we obtain

$$\frac{d}{dt}\big|_0 h_x^{\,i}(t) = -\sum_{\alpha} \eta_{\alpha i}(\exp x)\, e_\alpha. \tag{2.12}$$

Therefore, by (2.12) and Proposition 2.2, we have

$$(\nabla_{E_i} E_j)_{\pi(\exp x)} = \tau(\exp x)_*\Big\{\Lambda(e_i)(e_j) - \Big[\frac{d}{dt}\big|_0 h_x^{\,i}(t),\, e_j\Big]\Big\}. \tag{2.13}$$

Remark. For $x \in \mathfrak{p}$($|x|$: small), the mapping

$$p_{\mathfrak{p}} \circ \Phi_x : \mathfrak{p} \longrightarrow \mathfrak{p}$$

is an isomorphism ($p_{\mathfrak{p}} : \mathfrak{g} \longrightarrow \mathfrak{p}$ denotes the canonical projection.). So we can assume that for each $x \in W$ the mapping $p_{\mathfrak{p}} \circ \Phi_x$ is an isomorphism. Therefore we can regard the equation (2.11) as a characterization of $\dfrac{d}{dt}\big|_0 z_x^{\,i}(t)$ ($\in \mathfrak{p}$) and $\dfrac{d}{dt}\big|_0 h_x^{\,i}(t)$ ($\in \mathfrak{k}$).

For $X \in \mathfrak{p}$, we denote by X_* the vector field on $\pi(\exp W)$ defined by

$$(X_*)_{\pi(\exp x)} = \tau(\exp x)_*(X).$$

Then the following theorem is easily derived from the above arguments.

THEOREM 2.4. *Let $x \in W$ and $X, Y \in \mathfrak{p}$, Then*

$$(\nabla_{X*}Y_*)_{\pi(\exp x)} = \tau(\exp x)_*\{\Lambda(X)(Y) - [h_x(X), Y]\}.$$

Here $h_x(X) = -p_{\mathfrak{k}} \circ \Phi_x \circ (p_{\mathfrak{p}} \circ \Phi_x)^{-1}(X)$ ($p_{\mathfrak{k}}: \mathfrak{g} \to \mathfrak{k}$ denotes the canonical projection).

3. Covariant derivatives on Kähler C-spaces

In this section we shall write higher covariant derivatives of (1,0)-type on Kähler C-spaces in terms of the connection functions.

Let $(G_u/K_\Psi, \langle , \rangle)$ be a Kählerian C-space as stated in Section 1. For $\alpha \in \Delta^+(\Psi)$, since $\alpha = (1/2)(A_\alpha - \sqrt{-1}B_\alpha)$ (under the identification E_α with a), we have

$$\alpha_* = \frac{1}{2}(A_{\alpha*} - \sqrt{-1}B_{\alpha*}).$$

At first we calculate the value of $\nabla^n(X_*; \alpha_1^*, \ldots, \alpha_n^*)$ at o ($X \in \mathfrak{p}^{\mathbf{C}}$, $\alpha_i \in \Delta^+(\Psi)$).

Let X_i $(i = 1, \ldots, n)$ be one of $\{A_i, B_i\}$ $(A_i = A_{\alpha_i}, B_i = B_{\alpha_i})$. For $s_1, \ldots, s_n \in \mathbf{R}$ ($|s_i|$: small enough), we define $z^i(s_1, \ldots, s_i) \in W$ $(1 \leq i \leq n)$ inductively as follows:

(3.1)
$$z^1(s_1) = s_1X_1$$
$$\pi(\exp z^i(s_1, \ldots, s_i)) = \pi(\exp z^{i-1}(s_1, \ldots, s_{i-1})\exp s_iX_i).$$

Then

(3.2)
$$z^i(s_1, \ldots, s_{i-1}, 0) = z^{i-1}(s_1, \ldots, s_{i-1}).$$

Then it follows Lemma 2.3, (3.1) and (3.2) that

(3.3)
$$X_i = p_{\mathfrak{p}} \circ \Phi_{z^{i-1}(s_1,\ldots,s_{i-1})}\Big(\frac{\partial}{\partial s_i}\Big|_0 z^i(s_1, \ldots, s_i)\Big).$$

From Theorem 2.4 we have

(3.4)
$$(\nabla_{X_{n*}}X_*)_{\pi(\exp z^n(s_1,\ldots,s_{n-1},0))}$$
$$= \tau(\exp z^{n-1}(s_1, \ldots, s_{n-1}))_*\{\Lambda(X_n)(X) - [h_{n-1}(s_1, \ldots, s_{n-1}), X]\}$$

where

$$h_{n-1}(s_1,\dots,s_{n-1}) = -p_{\mathfrak{k}}\circ \Phi_{z^{n-1}(s_1,\dots,s_{n-1})}(V_{n-1}(s_1,\dots,s_{n-1}))$$
$$X_n = p_{\mathfrak{p}}\circ \Phi_{z^{n-1}(s_1,\dots,s_{n-1})}(V_{n-1}(s_1,\dots,s_{n-1})).$$

Thus, by (3.3) we get

$$V_{n-1} = \frac{\partial}{\partial s_n}\Big|_0 z^n. \tag{3.5}$$

Similarly, we have by (3.4) and Theorem 2.4

$$\begin{aligned}(3.6)\quad &(\nabla_{X_{n-1*}}\nabla_{X_{n*}}X_*)_{\pi(\exp z^{n-2}(s_1,\dots,s_{n-2}))}\\ &= \tau(\exp z^{n-2}(s_1,\dots,s_{n-2}))_*\{\Lambda(X_{n-1})\Lambda(X_n)(X)\\ &\quad - \Lambda(X_{n-1})([h_{n-1}(s_1,\dots,s_{n-2},0),X] - \frac{\partial}{\partial s_{n-1}}\Big|_0[h_{n-1}(s_1,\dots,s_{n-1}),X])\\ &\quad - [h_{n-2}(s_1,\dots,s_{n-2}),\Lambda(X_n)(X) - [h_{n-1}(s_1,\dots,s_{n-2},0),X]]\}\end{aligned}$$

where

$$h_{n-2}(s_1,\dots,s_{n-2}) = -p_{\mathfrak{k}}\circ \Phi_{z^{n-2}(s_1,\dots,s_{n-2})}\Big(\frac{\partial}{\partial s_{n-1}}\Big|_0 z^{n-1}\Big)$$
$$X_{n-1} = p_{\mathfrak{k}}\circ \Phi_{z^{n-2}(s_1,\dots,s_{n-2})}\Big(\frac{\partial}{\partial s_{n-1}}\Big|_0 z^{n-1}\Big).$$

Therefore, by induction, we can see

$$\begin{aligned}(3.7)\quad &(\nabla_{X_{1*}}\cdots\nabla_{X_{n*}}X_*)_o\\ &= \Lambda(X_1)\cdots\Lambda(X_n)(X)\\ &\quad + \Big\{\text{terms containing } \frac{\partial^r}{\partial s_{i_1}\cdots\partial s_{i_r}}\Big|_{s_1=\cdots=s_{k-1}=0} h_{k-1}(s_1,\dots,s_{k-1})\\ &\quad \text{for some } k,\ r\Big\}.\end{aligned}$$

Here

$$h_{k-1}(s_1,\dots,s_{k-1}) = -p_{\mathfrak{k}}\circ \Phi_{z^{k-1}(s_1,\dots,s_{k-1})}\Big(\frac{\partial}{\partial s_k}\Big|_0 z^k\Big) \tag{3.8}$$

$$X_k = p_{\mathfrak{p}}\circ \Phi_{z^{k-1}(s_1,\dots,s_{k-1})}\Big(\frac{\partial}{\partial s_k}\Big|_0 z^k\Big). \tag{3.9}$$

LEMMA 3.1. *Expand* $z^n(s_1, \dots, s_n)$ *as*

$$z^n(s_1, \dots, s_n) = \sum_{i_1, \dots, i_k} s_{i_1} \cdots s_{i_k} a_{i_1, \dots, i_k}.$$

Then there exists a multi-linear function

$$F_{i_1, \dots, i_k} : (\mathfrak{p}^{\mathbf{C}})^k \to \mathfrak{p}^{\mathbf{C}}$$

such that

$$a_{i_1, \dots, i_k} = F_{i_1, \dots, i_k}(X_{i_1}, \dots, X_{i_k}).$$

Proof. At first we note that $z^n(0, \dots, 0) = 0$ and

$$z^n(s_1, \dots, s_i, 0, \dots, 0) = z^i(s_1, \dots, s_i),$$
$$z^n(0, \dots, 0, s_i, 0, \dots, 0) = s_i X_i.$$

We prove the lemma by induction.

Assume that for any r-tuple $(i_1, \dots, i_r)(1 \leq r \leq k, i_1 < \cdots < i_r)$ there exists r-linear function $F_{i_1, \dots, i_r}$, such that

$$a_{i_1, \dots, i_r} = F_{i_1, \dots, i_r}(X_{i_1}, \dots, X_{i_r}).$$

Then for any $(k+1)$-tuple $(j_1, \dots, j_k, j_{k+1})$ $(j_1 < \cdots < j_{k+1})$ it follows from (3.9) that

$$X_{j_{k+1}} = p_{\mathfrak{p}} \circ \Phi_{z^{j_{k+1}}(s_1, \dots, s_{j_{k+1}-1}, 0)}\left(\frac{\partial}{\partial s_{j_{k+1}}}\Big|_0 z^{j_{k+1}}\right).$$

Considering the $(s_{j_1} \cdots s_{j_k})$-term of the above equation, we have

$$0 = a_{j_1, \dots, j_{k+1}} + \sum_{l=1}^{k} \frac{(-1)^l}{(l+1)!} \sum_{J_1, \dots, J_{l+1}} [a_{J_1}, [\cdots [a_{J_l}, a_{J_{l+1}}] \cdots]_{\mathfrak{p}}.$$

Here, each J_p, $1 \leq p \leq l+1$, is a subset of $\{j_1, \dots, j_{k+1}\}$ such that $J_p \cap J_q = \emptyset$ $(p \neq q)$, $J_p \subset \{j_1, \dots, j_k\}$ for $1 \leq p \leq l$ and

$$J_1 \cup \cdots \cup J_l \cup J_{l+1} = \{j_1, \dots, j_{k+1}\}.$$

Therefore, by the inductive assumption, the $(s_{j_1} \cdots s_{j_{k+1}})$-term of z^n is written as in the lemma. This completes the proof of the lemma. □

Let $h^r_{j_1, \dots, j_k}$ be the $(s_{j_1} \cdots s_{j_k})$-term of $h_r(s_1, \dots, s_r)$. Then, by (3.8) and the proof of Lemma 3.1, we have

$$(3.10) \qquad h^r_{j_1,\ldots,j_k} = -\sum_{l=1}^{k}\sum_{J_1,\ldots,J_{l+1}} \frac{(-1)^l}{(l+1)!}[a_{J_1}, [\cdots, [a_{J_l}, a_{J_{l+1}}]\cdots]_{\mathfrak{k}}.$$

Thus, by Lemma 3.1 and (3.10), there exists k-linear map

$$H^r_{j_1,\ldots,j_k} : (\mathfrak{p}^{\mathbf{C}})^k \to \mathfrak{k}^{\mathbf{C}}$$

such that

$$h^r_{j_1,\ldots,j_k} = H^r_{j_1,\ldots,J_k}(X_{j_1},\ldots, X_{j_k}).$$

Therefore (3.7) gives

$$\begin{aligned}
&(\nabla_{\alpha_{1*}}\cdots\nabla_{\alpha_{n*}}X_*)_o \\
&= \Lambda(\alpha_1)\cdots\Lambda(\alpha_n)(X) \\
&\quad + \{\text{terms containing } H^r_{j_1,\ldots,j_k}(\alpha_{j_1},\ldots,\alpha_{j_k})\}.
\end{aligned}$$

For α, $\beta\in\Delta^+(\Psi)$, it is obvious that $\alpha+\beta\in\Delta^+(\Psi)$ if $\alpha+\beta\in\Delta$. Considering the form of $H^r_{j_1,\ldots,j_k}$, it is easy to see that

$$H^r_{j_1,\ldots,j_k}(\alpha_{j_1},\ldots,\alpha_{j_k})\in\mathfrak{p}^+.$$

We have thus the following.

Proposition 3.2. *Let α_i $(i=1,\ldots,n)$ be in $\Delta^+(\Psi)$ and $X\in\mathfrak{p}^{\mathbf{C}}$. Then*

$$(\nabla_{\alpha_{1*}}\cdots\nabla_{\alpha_{n*}}X_*)_o = \Lambda(\alpha_1)\cdots\Lambda(\alpha_n)(X).$$

Remark 3.3. By similar argument as in the above, we can prove that

$$(\nabla_{\nabla_{\alpha*}\beta_*}\cdots)_o = \Lambda(\Lambda(\alpha_1)(\beta))(\cdots)$$

for $\alpha,\beta,\cdots\in\Delta^+(\Psi)$.

Now, we define $\Lambda^n R$ inductively as follows.

$$\begin{aligned}
&(\Lambda R)(X, Y, Z\,; T) \\
&= \Lambda(T)(R(X, Y)Z) - R(\Lambda(T)(X), Y)Z - R(X, \Lambda(T)(X))Z \\
&\quad - R(X, Y)\Lambda(T)(Z), \\
&\quad (\Lambda^n R)(X, Y, Z\,; T_1,\ldots, T_n) \\
&= \Lambda(T_n)((\Lambda^{n-1}R)(X, Y, Z\,; T_1,\ldots, T_{n-1})) \\
&\quad - (\Lambda^{n-1}R)(\Lambda(T_n)(X), Y, Z\,; T_1,\ldots, T_{n-1}) - (\Lambda^{n-1}R)(X, \Lambda(T_n)(Y), \\
&\quad Z\,; T_1,\ldots, T_{n-1}) - (\Lambda^{n-1}R)(X, Y, \Lambda(T_n)(Z)\,; T_1,\ldots, T_{n-1})
\end{aligned}$$

$$-\sum_{i=1}^{n-1}(\Lambda^{n-1}R)(X, Y, Z; T_1, \dots, \Lambda(T_n)(T_i), \dots, T_{n-1}).$$

Here $X, \dots, T_n \in \mathfrak{p}^{\mathbf{C}}$.

Since

$$R(\alpha_*, \beta_*)\gamma_* = (R(\alpha, \beta)\gamma)_*,$$

Proposition 3.2 and Remark 3.3 give the following Theorem which is the correction of (2.11) and (3.11) of [6].

THEOREM 3.4. *Let $X, Y, Z \in \mathfrak{p}^{\mathbf{C}}$ and $\delta_1, \dots, \delta_n \in \Delta^+(\Psi)$. Then*

$$(\nabla^n R)(X, Y, Z; \delta_1, \dots, \delta_n) = (\Lambda^n R)(X, Y, Z; \delta_1, \dots, \delta_n).$$

COROLLARY 3.5. *Let α, β, and γ be in Δ such that E_α, E_β and E_γ are elements of $\mathfrak{p}^{\mathbf{C}}$. Moreover, let $\delta_1, \dots, \delta_n$ be in $\Delta^+(\Psi)$. Then*

$$(\nabla^n R)(\alpha, \beta, \gamma; \delta_1, \dots, \delta_n) \in \mathbf{C}E_{\alpha+\beta+\gamma+\delta_1+\dots+\delta_n}.$$

We denote by $\hat{\nabla}$ the covariant derivative in the direction of $\mathfrak{p}^+$. Then, from Corollary 3.5, there is a number n such that $\hat{\nabla}^n R = 0$ and $\hat{\nabla}^{n-1}R \neq 0$. We call the integer n the degree of $(G_u/K_\Psi, \langle, \rangle)$. It is known that Hermitian symmetric spaces of compact type are characterized as Kähler C-spaces with degree one.

4. Degree two

In this section, using a similar method as in [6], we shall determine the class of Kählerian C-spaces with degree two.

Let α, β, γ, δ and λ be elements of $\Delta^+(\Psi)$. From Theorem 3.4, we have

$$\begin{aligned}(4.1)\quad &(\nabla^2 R)(\alpha, \bar{\lambda}, \beta; \gamma, \delta)\\ &= \Lambda(\delta)\Lambda(\gamma)R(\alpha, \bar{\lambda})\beta - \Lambda(\Lambda(\delta)\gamma)R(\alpha, \bar{\lambda})\beta - \Lambda(\gamma)R(\Lambda(\delta)\alpha, \bar{\lambda})\beta\\ &- \Lambda(\gamma)R(\alpha, \Lambda(\delta)\bar{\lambda})\beta - \Lambda(\gamma)R(\alpha, \bar{\lambda})\Lambda(\delta)\beta - \Lambda(\delta)R(\Lambda(\gamma)\alpha, \bar{\lambda})\beta\\ &+ R(\Lambda(\Lambda(\delta)\gamma)\alpha, \bar{\lambda})\beta + R(\Lambda(\gamma)\Lambda(\delta)\alpha, \bar{\lambda})\beta + R(\Lambda(\gamma)\alpha, \Lambda(\delta)\bar{\lambda})\beta\\ &+ R(\Lambda(\gamma)\alpha, \bar{\lambda})\Lambda(\delta)\beta - \Lambda(\delta)R(\alpha, \Lambda(\gamma)\bar{\lambda})\beta + R(\Lambda(\delta)\alpha, \Lambda(\gamma)\bar{\lambda})\beta\\ &+ R(\alpha, \Lambda(\Lambda(\delta)\gamma)\bar{\lambda})\beta + R(\alpha, \Lambda(\gamma)\Lambda(\delta)\bar{\lambda})\beta + R(\alpha, \Lambda(\gamma)\bar{\lambda})\Lambda(\delta)\beta\\ &- \Lambda(\delta)R(\alpha, \bar{\lambda})\Lambda(\gamma)\beta + R(\Lambda(\delta)\alpha, \bar{\lambda})\Lambda(\gamma)\beta + R(\alpha, \Lambda(\delta)\bar{\lambda})\Lambda(\gamma)\beta\\ &+ R(\alpha, \bar{\lambda})\Lambda(\Lambda(\delta)\gamma)\beta + R(\alpha, \bar{\lambda})\Lambda(\gamma)\Lambda(\delta)\beta.\end{aligned}$$

LEMMA 4.1. *Suppose that* $\alpha, \beta\ (\in \Delta^{+}(\Psi))\ (\alpha \neq \beta)$ *satisfy the following conditions*:

(1) $\alpha+\beta \in \Delta$, (2) $\alpha-\beta \notin \Delta$, (3) $2\alpha+\beta \notin \Delta$, (4) $\alpha+2\beta \notin \Delta$.

Then $(\nabla^2 R)(\alpha, \overline{\alpha+\beta}, \beta\,; \alpha, \beta) \neq 0$.

Proof. From (4.1) and the conditions in the lemma, we have

$$\begin{aligned}
&(\nabla^2 R)(\alpha, \overline{\alpha+\beta}, \beta\,; \alpha, \beta) \\
&= -\Lambda(\alpha) R(\Lambda(\beta)\alpha, \overline{\alpha+\beta})\beta - \Lambda(\alpha) R(\alpha, \Lambda(\beta)\overline{\alpha+\beta})\beta \\
&\quad - \Lambda(\beta) R(\alpha, \Lambda(\alpha)\overline{\alpha+\beta})\beta + R(\Lambda(\beta)\alpha, \Lambda(\alpha)\overline{\alpha+\beta})\beta - \Lambda(\beta) R(\alpha, \overline{\alpha+\beta})\Lambda(\alpha)\beta \\
&\quad + R(\Lambda(\beta)\alpha, \overline{\alpha+\beta})\Lambda(\alpha)\beta + R(\alpha, \Lambda(\beta)\overline{\alpha+\beta})\Lambda(\alpha)\beta \\
&= \Lambda(\alpha)\,[[\Lambda(\beta)\alpha, \overline{\alpha+\beta}], \beta] + \Lambda(\alpha)\,\{\Lambda(\Lambda(\beta)\overline{\alpha+\beta})\Lambda(\alpha)\beta + [[\alpha, \Lambda(\beta)\overline{\alpha+\beta}], \beta]\} \\
&\quad + \Lambda(\beta)\Lambda(\Lambda(\alpha)\overline{\alpha+\beta})\Lambda(\alpha)\beta - \Lambda([\Lambda(\beta)\alpha, \Lambda(\alpha)\overline{\alpha+\beta}])\beta \\
&\quad + \Lambda(\beta)\Lambda([\alpha, \overline{\alpha+\beta}])\Lambda(\alpha)B - [[\Lambda(\beta)\alpha, \overline{\alpha+\beta}], \Lambda(\alpha)\beta] \\
&\quad + \Lambda(\alpha)\Lambda(\Lambda(\beta)\overline{\alpha+\beta})\Lambda(\alpha)\beta - [[\alpha, \Lambda(\beta)\overline{\alpha+\beta}], \Lambda(\alpha)\beta].
\end{aligned}$$

It follows from (1.6) and Lemma 1.1 that

$$\begin{aligned}
&(\nabla^2 R)(\alpha, \overline{\alpha+\beta}, \beta\,; \alpha, \beta) \\
&= -\frac{(c\cdot p(\alpha))(c\cdot p(\beta))}{(c\cdot p(\alpha+\beta))^2}(N_{\alpha,\beta})^2 \beta(H_{\alpha+\beta})\cdot(\alpha+\beta) \\
&\quad + 2\frac{(c\cdot p(\beta))^2}{(c\cdot p(\alpha+\beta))^2}(N_{\alpha,\beta})^2 N_{\beta,-(\alpha+\beta)} N_{-\alpha,\alpha+\beta}\cdot(\alpha+\beta) \\
&+ \frac{c\cdot p(\beta)}{c\cdot p(\alpha+\beta)} N_{\alpha,\beta} N_{\beta,-(\alpha+\beta)} \beta(H_\alpha)\cdot(\alpha+\beta) \\
&- 3\frac{(c\cdot p(\alpha))(c\cdot p(\beta))}{(c\cdot p(\alpha+\beta))^2}(N_{\alpha,\beta})^2 N_{\alpha,-(\alpha+\beta)} N_{-\beta,\alpha+\beta}\cdot(\alpha+\beta) \\
&+ \frac{(c\cdot p(\alpha))(c\cdot p(\beta))}{(c\cdot p(\alpha+\beta))^2}(N_{\alpha,\beta})^2 (\alpha+\beta)(H_{\alpha+\beta})\cdot(\alpha+\beta) \\
&- \frac{c\cdot p(\beta)}{c\cdot p(\alpha+\beta)} N_{\alpha,\beta} N_{\beta,-(\alpha+\beta)} \alpha(H_{\alpha+\beta})\cdot(\alpha+\beta).
\end{aligned}$$

It follows from (1.7) that

$$N_{\beta,-(\alpha+\beta)} = -N_{\alpha,-(\alpha+\beta)} = N_{\alpha,\beta},$$

form which we have

$$(4.2)\qquad (\nabla^2 R)(\alpha, \overline{\alpha+\beta}, \beta\,; \alpha, \beta)$$
$$= \frac{c\cdot p(\beta)}{(c\cdot p(\alpha+\beta))}(N_{\alpha,\beta})^2\Big\{-\frac{c\cdot p(\alpha)}{(c\cdot p(\alpha+\beta))}\beta(H_{\alpha+\beta})$$
$$+ 2\frac{(c\cdot p(\beta)}{(c\cdot p(\alpha+\beta))}(N_{\alpha,\beta})^2 + \beta(H_\alpha) - 3\frac{c\cdot p(\alpha)}{(c\cdot p(\alpha+\beta))}(N_{\alpha,\beta})^2$$
$$+ \frac{c\cdot p(\alpha)}{(c\cdot p(\alpha+\beta))}(\alpha+\beta)(H_{\alpha+\beta}) - \alpha(H_{\alpha+\beta})\Big\}\cdot(\alpha+\beta).$$

From the conditions of Lemma 4.1, the α-series containing β is given by $\{\beta, \beta+\alpha\}$. Hence, by (1.9) we have

$$\alpha(H_\beta) = -\frac{e}{2},\ (N_{\alpha,\beta})^2 = \frac{e}{2},$$

where $e = \alpha(H_\alpha) = \beta(H_\beta)$. Therefore we have from (4.2)

$$(4.3)\qquad (\nabla^2 R)(\alpha, \overline{\alpha+\beta}, \beta\,; \alpha, \beta) = -\frac{e^2(c\cdot p(\alpha))(c\cdot p(\beta))}{(c\cdot p(\alpha+\beta))^2}\cdot(\alpha+\beta).$$

We have thus proved the lemma. □

Now, we prove the following theorem.

THEOREM 4.2. *The only Kählerian C-spaces of which degrees are at most two are Hermitian symmetric spaces of compact type.*

In the following we denote by $M(\mathfrak{g}, \Psi, g)$ the Kählerian C-space corresponding to Ψ. We show the theorem by case by case check.

The case where $\mathfrak{g}$ *is of type* A_l $(l \geq 2)$.

We identify Δ with

$$\{e_i - e_j\,; 1 \leq i \neq j \leq l+1\}$$

(for example, see [2]), where $\{e_1, \ldots, e_{l+1}\}$ is an orthonormal basis. Moreover, set $\alpha_i = e_i - e_{i+1}$. Then $M\{\mathfrak{g}, \{\alpha_i\}, g)$ $(i = 1, \ldots, l)$ are Hermitian symmetric spaces.

Suppose that Ψ contains α_i and α_j $(i < j)$. Then $\alpha = \alpha_1 + \cdots + \alpha_i$ and $\beta = a_{i+1} + \cdots + \alpha_l$ are contained in $\Delta^+(\Psi)$. Furthermore, it is easy to see that α and

β satisfy the conditions (1), (2), (3) and (4) of Lemma 4.1. Thus the degree of $M(\mathfrak{g}, \Psi, g)$ is not equal to two.

The case where $\mathfrak{g}$ *is of type* B_l $(l \geq 3)$.

$$\Delta = \{\pm e_i, \pm e_i \pm e_j ; 1 \leq i \neq j \leq l\}.$$

Set

$$\alpha_i = e_i - e_{i+1} \ (1 \leq i \leq l-1), \ \alpha_l = e_l.$$

In this case Hermitian symmetric spaces are $M(\mathfrak{g}, \{\alpha_i\}, g)$ $(i = 1, l)$.

Put

$$\alpha = e_1 - e_l = \alpha_1 + \cdots + \alpha_{l-1}, \beta = e_2 + e_{l-1} = \alpha_2 + \cdots + \alpha_{l-1} + 2\alpha_l.$$

Then we can easily see that α and β satisfy the conditions of Lemma 4.1. Then Kählerian C-spaces of which degrees are at most two are only Hermitian symmetric spaces. In fact, if Ψ contains some α_i $(2 \leq i \leq l-1)$, then $\alpha, \beta \in \Delta^+(\Psi)$. Moreover, $\alpha, \beta \in \Delta^+(\{\alpha_1, \alpha_l\})$.

The case where $\mathfrak{g}$ *is of type* C_l $(l \geq 3)$.

$$\Delta = \{\pm 2e_i, \pm e_i \pm e_j ; 1 \leq i \neq j \leq l\}.$$

Set

$$\alpha_i = e_i - e_{i+1} \ (1 \leq i \leq l-1), \ \alpha_l = 2e_l.$$

In this case Hermitian symmetric spaces are $M(\mathfrak{g}, \{\alpha_i\}, g)$ $(i = 1, l)$.

If $\alpha_i \in \Psi$ for some i $(2 \leq i \leq l-1)$, then

$$\alpha = e_1 + e_l = \alpha_1 + \cdots + \alpha_l, \beta = e_i - e_l = \alpha_i + \cdots + \alpha_{l-1}$$

are elements of $\Delta^+(\Psi)$ and satisfy the conditions of Lemma 4.1. Therefore the degree of $M(\mathfrak{g}, \Psi, g)$ is not equal to two.

Let $\Psi = \{\alpha_1, \alpha_l\}$. Then set $\alpha = \alpha_1$ and $\beta = \alpha_2 + \cdots + \alpha_l$. As above, we see that the degree of $M(\mathfrak{g}, \Psi, g)$ is not equal to two.

The case where $\mathfrak{g}$ *is of type* $D_l (l \geq 4)$.

$$\Delta = \{\pm e_i \pm e_j ; 1 \leq i \neq j \leq l\}.$$
$$\alpha_i = e_i - e_{i+1} \ (i = 1, \ldots, l-1), \ \alpha_l = e_{l-1} + e_l.$$

In this case Hermitian symmetric spaces are $M(\mathfrak{g}, \{\alpha_i\}, g)$ $(i = 1, l-1, l)$.

If $\alpha_i \in \Psi$ for some i $(2 \leq i \leq l-2)$, then

$$\alpha = e_1 - e_l = \alpha_1 + \cdots + \alpha_{l-1},\ \beta = e_i + e_l = \alpha_i + \cdots + \alpha_l$$

are in $\Delta^+(\Psi)$ and satisfy the conditions of Lemma 4.1.

Next we check $M(\mathfrak{g}, \{\alpha_1, \alpha_l\}, g)$ and $M(\mathfrak{g}, \{\alpha_{l-1}, \alpha_l\}, g)$.

Set

$$\alpha = \alpha_1 + \cdots + \alpha_{l-1},\ \beta = \alpha_2 + \cdots + \alpha_{l-2} + \alpha_l.$$

Then α and β satisfy the conditions of Lemma 4.1 and are elements of $\Delta^+(\Psi)$, regardless of whether $\Psi = \{\alpha_1, \alpha_l\}$ or $\Psi = \{\alpha_{l-1}, \alpha_l\}$.

The case where $\mathfrak{g}$ *is of type* E_8.

In this case Δ consists of the following.

$$\pm e_i \pm e_j \ (1 \leq i \neq j \leq 8),\ \frac{1}{2} \sum_{i=1}^{8} \nu(i) e_i \ (\sum \nu(i) : \text{even}).$$

Set

$$\alpha_1 = \frac{1}{2}(e_1 + e_8 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7)$$
$$\alpha_2 = e_1 + e_2,\ \alpha_i = e_{i-1} - e_{i-2}\ (3 \leq i \leq 8).$$

We denote a root $\alpha = \sum_{i=1}^{8} n_i \alpha_i$ by

$$\begin{pmatrix} n_8 & n_7 & n_6 & n_5 & n_4 & n_3 & n_1 \\ & & & & n_2 & & \end{pmatrix}$$

Then there is no $M(\mathfrak{g}, \Psi, g)$ with degree two. In fact, the following α, β satisfy the conditions (1)~(4) of Lemma 4.1 (cf. [1]).

$$\alpha = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & & & & 1 & & \end{pmatrix},\quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 3 & 1 \\ & & & & 2 & & \end{pmatrix}$$

The case where $\mathfrak{g}$ *is of type* E_7.

We use the same notation as in the case E_8. Then $\{\alpha_1, \ldots, \alpha_7\}$ is a fundamental root system and Δ consists of the following.

$$\pm e_i \pm e_j \ (1 \leq i \neq j \leq 6),\ \pm(e_7 - e_8)$$
$$\pm \frac{1}{2}\left(e_7 - e_8 + \sum_{i=1}^{6} \nu(i) e_i\right) \left(\sum_{i=1}^{6} \nu(i) : \text{odd}\right).$$

In this case Hermitian symmetric space is only $M(\mathfrak{g}, \{\alpha_7\}, g)$. We denote a root $\alpha = \sum_{i=1}^{7} n_i\alpha_i$ by

$$\begin{pmatrix} n_7 & n_6 & n_5 & n_4 & n_3 & n_1 \\ & & & n_2 & & \end{pmatrix}$$

Then

$$\alpha = \begin{pmatrix} 0 & 1 & 1 & 2 & 1 & 1 \\ & & & 1 & & \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 1 \\ & & & 1 & & \end{pmatrix}$$

satisfy (1)~(4) of Lemma 4.1.

The case where $\mathfrak{g}$ *is of type* E_6.

Δ consists of

$$\pm e_i \pm e_j \ (1 \leq i \neq j \leq 5)$$
$$\pm \frac{1}{2}\left(e_8 - e_7 - e_6 + \sum_{i=1}^{5} \nu(i) e_i\right) \left(\sum_{i=1}^{5} \nu(i) : \text{even}\right).$$

In this case Hermitian symmetric spaces are $M(\mathfrak{g}, \{\alpha_i\}, g)$ $(i = 1,6)$. We identify $\alpha = \sum_{i=1}^{6} n_i\alpha_i$ with

$$\begin{pmatrix} n_6 & n_5 & n_4 & n_3 & n_1 \\ & & n_2 & & \end{pmatrix}.$$

Then

$$\alpha = \begin{pmatrix} 0 & 1 & 2 & 1 & 1 \\ & & 1 & & \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ & & 1 & & \end{pmatrix}$$

satisfy (1)~(4) of Lemma 4.1.

The case where $\mathfrak{g}$ *is of type* F_4.

$$\Delta = \left\{\pm e_i, \ \pm e_i \pm e_j \ (1 \leq i \neq j \leq 4), \ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\right\}$$

$$\alpha_1 = e_2 - e_3, \ \alpha_2 = e_3 - e_4, \ \alpha_3 = e_4, \ \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4).$$

We identify $\alpha = \sum_{i=1}^{4} n_i\alpha_i$ with (n_1, n_2, n_3, n_4).

If Ψ contains α_i for some i $(1 \leq i \leq 3)$, then

$$\alpha = (1, 1, 2, 2) \text{ and } \beta = (1, 2, 2, 0)$$

are elements of $\Delta^+(\Psi)$ and satisfy (1)~(4) of Lemma 4.1.

Let $\Psi=\{\alpha_4\}$, $\alpha=(0,0,0,1)$ and $\beta=(1,2,3,1)$. Then the degree of $M(\mathfrak{g}, \{\alpha_4\}, g)$ is not equal to two.

The case where $\mathfrak{g}$ *is of type* G_2.

Δ consists of the following.

$$\pm(e_2-e_3),\ \pm(e_3-e_1),\ \pm(e_1-e_2)$$
$$\pm(2e_1-e_2-e_3),\ \pm(2e_2-e_1-e_3),\ \pm(2e_3-e_1-e_2).$$

Let $\alpha_1=e_1-e_2$ and $\alpha_2=-2e_1+e_2+e_3$. Then $M(\mathfrak{g}, \{\alpha_1\}, g)$ is a Hermitian symmetric space.

Suppose that $\alpha_2\in\Psi$. Then $\alpha=3\alpha_1+\alpha_2$ and $\beta=\alpha_2$ is contained in $\Delta^+(\Psi)$ and satisfy (1)~(4).

Finally we check $M(B_2, \{\alpha, \beta\}, g)$ $(\alpha=e_1-e_2, \beta=e_2)$.

We compute $(\nabla^2R)(\alpha, \overline{\alpha+\beta}, \beta; \alpha, \beta)$. Since

$$\alpha+\beta,\ \alpha+2\beta\in\Delta \quad\text{and}\quad \alpha-\beta,\ 2\alpha+\beta\notin\Delta, \tag{4.4}$$

we have

$$\begin{aligned}
&(\nabla^2R)(\alpha, \overline{\alpha+\beta}, \beta; \alpha, \beta)\\
&=-\Lambda(\alpha)R(\Lambda(\beta)\alpha, \overline{\alpha+\beta})\beta-\Lambda(\alpha)R(\alpha, \Lambda(\beta)\overline{\alpha+\beta})\beta-\Lambda(\beta)R(\alpha, \Lambda(\alpha)\overline{\alpha+\beta})\beta\\
&\quad+R(\Lambda(\beta)\alpha, \Lambda(\alpha)\overline{\alpha+\beta})\beta-\Lambda(\beta)R(\alpha, \overline{\alpha+\beta})\Lambda(\alpha)\beta+R(\Lambda(\beta)\alpha, \overline{\alpha+\beta})\Lambda(\alpha)\beta\\
&\quad+R(\alpha, \Lambda(\beta)\overline{\alpha+\beta})\Lambda(\alpha)\beta+R(\alpha, \overline{\alpha+\beta})\Lambda(\Lambda(\beta)\alpha)\beta.
\end{aligned}$$

Comparing the above equation with the right hand side of (4.2), we get

$$\begin{aligned}
&(\nabla^2R)(\alpha, \overline{\alpha+\beta}, \beta; \alpha, \beta)\\
&=R(\alpha, \overline{\alpha+\beta})\Lambda(\Lambda(\beta)\alpha)\beta+\Lambda(\alpha)\Lambda(\overline{\alpha+\beta})\Lambda(\Lambda(\beta)\alpha)\beta-\Lambda(\Lambda(\alpha)\overline{\alpha+\beta})\Lambda(\Lambda(\beta)\alpha)\beta\\
&\quad+\text{the right hand side of (4.2)}.
\end{aligned}$$

Thus

$$\begin{aligned}
&(\nabla^2R)(\alpha, \overline{\alpha+\beta}, \beta; \alpha, \beta)\\
&=-2\frac{(c\cdot p(\alpha))(c\cdot p(\beta))^2}{(c\cdot p(\alpha+\beta))^2(c\cdot p(2\beta+\alpha))}(N_{\alpha,\beta})^2(N_{\beta,\alpha+\beta})^2\cdot(\alpha+\beta)\\
&\quad+\text{the right hand side of (4.2)}.
\end{aligned}$$

From (4.4), we have

$$(N_{\alpha,\beta})^2 = (N_{\beta,\alpha+\beta})^2 = e,\ \alpha(H_\beta) = -e,$$

where $e = \beta(H_\beta) = (1/2)\alpha(H_\alpha)$. Therefore

$$\begin{aligned}
&(\nabla^2 R)(\alpha, \overline{\alpha+\beta}, \beta\,;\alpha, \beta)\\
&= -\frac{c\cdot p(\beta)}{c\cdot p(\alpha+\beta)}(N_{\alpha,\beta})^2\Big\{-\frac{2e(c\cdot p(\alpha))(c\cdot p(\beta))}{(c\cdot p(\alpha+\beta))(c\cdot p(\alpha+2\beta))} + \frac{2e(c\cdot p(\beta))}{c\cdot p(\alpha+\beta)}\\
&\quad - 2e - \frac{3e(c\cdot p(\alpha))}{c\cdot p(\alpha+\beta)} + \frac{e(c\cdot p(\alpha))}{c\cdot p(\alpha+\beta)}\Big\}\cdot(\alpha+\beta)\\
&= -2e^2\frac{(c\cdot p(\beta))(c\cdot p(\beta))}{(c\cdot p(\alpha+\beta))^2(c\cdot p(2\beta+\alpha))}(c\cdot p(\alpha)+4c\cdot p(\beta))\cdot(\alpha+\beta).
\end{aligned}$$

Therefore the degree of $M(B_2, \{\alpha, \beta\}, g)$ is not equal to two.

We have thus proved the theorem.

5. Degree three

For $\alpha_i \in \Pi$, set $\Delta_i^+(k) = \{\alpha = \sum_j n_j\alpha_j \in \Delta^+\,;\, n_i = k\}$.

We devote this section to proving the following theorem.

THEOREM 5.1. *Let α_i, α_q and α_r be elements of Π such that $\Delta_i^+(k) = \emptyset$, $\Delta_g^+(m) = \emptyset$ and $\Delta_r^+(n) = \emptyset$ for $k \geq 3$, $m, n \geq 2$. Then Kähler C-space with degree three is one of $M(\mathfrak{g}, \{\alpha_i\}, g)$ and $M(\mathfrak{g}, \{\alpha_q, \alpha_r\}, g)$*

At first we show that the degrees of $M(\mathfrak{g}, \{\alpha_i\}, g)$ and $M(\mathfrak{g}, \{\alpha_q, \alpha_r\}, g)$ are at most three.

In the following we suppose that α, β, γ, δ, ω and λ are elements of $\Delta^+(\Psi)$.

Suppose $\Psi = \{\alpha_i\}$. Since

$$\Lambda(\mathfrak{p}^{\mathbf{C}})\mathfrak{p}^\pm \subset \mathfrak{p}^\pm,\quad R(\mathfrak{p}^{\mathbf{C}}, \mathfrak{p}^{\mathbf{C}})\mathfrak{p}^\pm \subset \mathfrak{p}^\pm,$$

we can see

$$(\nabla^3 R)(\alpha, \bar\lambda, \beta\,;\gamma, \delta, \omega) \in \mathfrak{p}^+$$
$$(\nabla^3 R)(\bar\alpha, \lambda, \bar\beta\,;\gamma, \delta, \omega) \in \mathfrak{p}^-.$$

Therefore, If $(\nabla^3 R)(\alpha, \bar\lambda, \beta\,;\gamma, \delta, \omega) \neq 0$, then $\alpha+\beta+\gamma+\delta+\omega-\lambda$ must be in $\Delta^+(\Psi)$. Similarly, if $(\nabla^3 R)(\bar\alpha, \lambda, \bar\beta\,;\gamma, \delta, \omega) \neq 0$, then $\alpha+\beta-\gamma-\delta-\omega-\lambda$ must be in $\Delta^+(\Psi)$.

Each $\alpha \in \Delta^{+}(\Psi)$ has $1 \leq p(\alpha) \leq 2$ so that

$$p(\alpha + \beta + \gamma + \delta + \omega - \lambda) \geq 1 + 1 + 1 + 1 + 1 - 2 = 3.$$

However, this is impossible, since $\Delta_i^{+}(k) = \emptyset$ for $k \geq 3$. Similarly we have

$$p(\alpha + \beta - \gamma - \delta - \omega - \lambda) \leq 2 + 2 - 1 - 1 - 1 - 1 = 0.$$

Thus the degree of $M(\mathfrak{g}, \{\alpha_1\}, g)$ is not more than three.

Next, suppose $\Psi = \{\alpha_q, \alpha_r\}$ $(q < r)$. Since $\Delta_q^{+}(m) = \emptyset$ and $\Delta_r^{+}(n) = \emptyset$ for $m, n \geq 2$, it is easy to see that the possibilities of $p(\alpha)$ are only (1,0),(0,1) and (1,1). Therefore

$$p(\alpha + \beta + \gamma + \delta + \omega - \lambda) \neq (1,0), (0,1), (1,1)$$
$$p(\alpha + \beta - \gamma - \delta - \omega - \lambda) \neq (1,0), (0,1), (1,1).$$

Thus the degree of $M(\mathfrak{g}, \{\alpha_q, \alpha_r\}, g)$ is not more than three.

Next, we prove that Hermitian symmetric spaces, $M(\mathfrak{g}, \{\alpha_t\}, g)$ and $M(\mathfrak{g}, \{\alpha_q, \alpha_r\}, g)$ are only Kähler C-spaces of which degrees are at most three.

As in Section 4, we shall prove the following lemmas.

LEMMA 5.2. *Suppose that there are $\alpha, \beta, \gamma \in \Delta^{+}(\Psi)$ $(\alpha \neq \beta, \beta \neq \gamma, \gamma \neq \alpha)$ satisfying the following*:

(1) $\alpha + \beta \in \Delta$, (2) $\alpha + \gamma \in \Delta$, (3) $\alpha + \beta + \gamma \in \Delta$,
(4) $\alpha - \beta \notin \Delta$, (5) $\beta + \gamma \notin \Delta$, (6) $\beta - \gamma \notin \Delta$, (7) $2\alpha + \beta \notin \Delta$
(8) $2\beta + \alpha \notin \Delta$, (9) $2\alpha + \gamma \notin \Delta$, (10) $\alpha + \gamma - \beta \notin \Delta$
(11) $2\alpha + \beta + \gamma \notin \Delta$, (12) $2\beta + \alpha + \gamma \notin \Delta$, (13) $2\alpha + 2\beta + \gamma \notin \Delta$
(14) $\alpha - \gamma \notin \Delta$, (15) $2\gamma + \alpha \notin \Delta$.

Then the degree of $M(\mathfrak{g}, \Psi, g)$ is more than three.

LEMMA 5.3. *Let α and β be in $\Delta^{+}(\Psi)$ $(\alpha \neq \beta)$. If the following conditions are satisfied, then the degree of $M(\mathfrak{g}, \Psi, g)$ is more than three*:

(1) $\alpha + \beta \in \Delta$, (2) $\alpha - \beta \notin \Delta$, (3) $2\alpha + \beta \notin \Delta$
(4) $2\beta + \alpha \in \Delta$, (5) $3\beta + \alpha \notin \Delta$.

Proof of Lemma 5.2. We shall show

$$(\nabla^3 R)(\alpha, \bar{\lambda}, \beta\,; \alpha, \beta, \gamma) \neq 0 \quad (\lambda = \alpha + \beta + \gamma).$$

By Theorem 3.4 and (10) of Lemma 5.2, we have

$$
\begin{aligned}
&(\nabla^3 R)(\alpha, \bar{\lambda}, \beta\,; \alpha, \beta, \gamma) \\
&= -(\Lambda^2 R)(\Lambda(\gamma)\alpha, \bar{\lambda}, \beta\,; \alpha, \beta) \\
&\quad -(\Lambda^2 R)(\alpha, \Lambda(\gamma)\bar{\lambda}, \beta\,; \alpha, \beta) \\
&\quad -(\Lambda^2 R)(\alpha, \bar{\lambda}, \beta\,; \Lambda(\gamma)\alpha, \beta).
\end{aligned}
$$

By (4.1) and the conditions of the lemma, we have

$$
\begin{aligned}
&(\nabla^2 R)(\Lambda(\gamma)\alpha, \bar{\lambda}, \beta\,; \alpha, \beta) \\
&= -\Lambda(\alpha)R(\Lambda(\beta)\Lambda(\gamma)\alpha, \bar{\lambda})\beta - \Lambda(\alpha)R(\Lambda(\gamma)\alpha, \Lambda(\beta)\bar{\lambda})\beta + R(\Lambda(\gamma)\alpha, \Lambda(\Lambda(\beta)\alpha)\bar{\lambda})\beta \\
&\quad + R(\Lambda(\gamma)\alpha, \Lambda(\alpha)\Lambda(\beta)\bar{\lambda})\beta - \Lambda(\beta)R(\Lambda(\gamma)\alpha, \bar{\lambda})\Lambda(\alpha)\beta \\
&\quad + R(\Lambda(\beta)\Lambda(\gamma)\alpha, \bar{\lambda})\Lambda(\alpha)\beta + R(\Lambda(\gamma)\alpha, \Lambda(\beta)\bar{\lambda})\Lambda(\alpha)\beta \\
&= \Lambda(\alpha)\,[[\Lambda(\beta)\Lambda(\gamma)\alpha, \bar{\lambda}], \beta] \\
&\quad + \Lambda(\alpha)\,\{\Lambda(\Lambda(\beta)\bar{\lambda})\Lambda(\Lambda(\gamma)\alpha)\beta + [[\Lambda(\gamma)\alpha, \Lambda(\beta)\bar{\lambda}]\} \\
&\quad - \{\Lambda(\Lambda(\Lambda(\beta)\alpha)\bar{\lambda})\Lambda(\Lambda(\gamma)\alpha)\beta + \Lambda([\Lambda(\gamma)\alpha, \Lambda(\Lambda(\beta)\alpha)\bar{\lambda}]\beta\} \\
&\quad - \{\Lambda(\Lambda(\alpha)\Lambda(\beta)\bar{\lambda})\Lambda(\Lambda(\gamma)\alpha)\beta + \Lambda([\Lambda(\gamma)\alpha, \Lambda(\alpha)\Lambda(\beta)\bar{\lambda}]\beta\} \\
&\quad + \Lambda(\beta)\Lambda([\Lambda(\gamma)\alpha, \bar{\lambda}])\Lambda(\alpha)\beta - [[\Lambda(\beta)\Lambda(\gamma)\alpha, \bar{\lambda}], \Lambda(\alpha)\beta] \\
&\quad - [[\Lambda(\gamma)\alpha, \Lambda(\beta)\bar{\lambda}], \Lambda(\alpha)\beta].
\end{aligned}
$$

Now, put $c_\alpha = c\cdot p(\alpha)$ $(\alpha \in \Delta^+(\Psi))$. Then, by Lemma 1.1 and (1.7), we have

(5.1)

$$
\begin{aligned}
&(\nabla^2 R)(\Lambda(\gamma)\alpha, \bar{\lambda}, \beta\,; \alpha, \beta) \\
&= -\frac{c_\alpha c_\beta c_{\alpha+\gamma}}{c_{\alpha+\beta}c_{\alpha+\gamma}c_\lambda} N_{\gamma,\alpha}N_{\beta,-\lambda}\beta(H_\lambda)\cdot[\alpha, \beta] \\
&\quad + \frac{c_\alpha c_\beta}{c_{\alpha+\beta}c_{\alpha+\gamma}} N_{\gamma,\alpha}N_{\beta,-\lambda}\left\{\frac{c_\beta}{c_\lambda}(N_{\beta,-\lambda})^2 + \beta(H_{\gamma+\alpha})\right\}[\alpha, \beta] \\
&\quad - \frac{(c_\alpha)^2 c_\beta}{c_{\alpha+\beta}c_{\alpha+\gamma}}\left\{\frac{1}{c_{\alpha+\beta}}(N_{\gamma,\alpha})^2 N_{\beta,\alpha}N_{\gamma,-\lambda} - \frac{1}{c_\lambda}N_{\gamma,\alpha}N_{\beta,-\lambda}(N_{\gamma,-\lambda})^2\right\}[\alpha, \beta] \\
&\quad + \frac{c_\alpha c_\beta}{c_{\alpha+\gamma}}\left\{\frac{1}{c_\gamma}(N_{\gamma,\alpha}N_{\beta,-\gamma})^2 N_{\gamma,-\lambda}\cdot(\alpha+\beta) + \frac{1}{c_{\alpha+\beta}}(N_{\gamma,\alpha})^3 N_{\beta,-\lambda}\cdot[\alpha, \beta]\right\} \\
&\quad - \frac{(c_\alpha)^2 c_\beta}{(c_{\alpha+\beta})^2 c_{\alpha+\gamma}} N_{\gamma,\alpha}N_{\beta,-\lambda}(N_{\alpha,\beta})^2\cdot[\alpha, \beta] + \frac{c_\alpha c_\beta c_{\alpha+\gamma}}{c_{\alpha+\beta}c_{\alpha+\gamma}c_\lambda} N_{\gamma,\alpha}N_{\beta,-\lambda}\lambda(H_{\alpha+\beta})\cdot[\alpha, \beta] \\
&\quad - \frac{c_\alpha c_\beta}{c_{\alpha+\beta}c_{\alpha+\gamma}} N_{\gamma,\alpha}N_{\beta,-\gamma}(\alpha+\beta)(H_{\gamma+\alpha})\cdot[\alpha, \beta].
\end{aligned}
$$

For simplicity, put $e = \alpha(H_\alpha)$. Then, by (1.9) and the conditions of the lemma, we get the following.

$$\beta(H_\beta) = \gamma(H_\gamma) = e,\ \alpha(H_\beta) = \alpha(H_\gamma) = -\frac{e}{2}$$

$$\beta(H_\gamma) = 0,\ (N_{\alpha,\beta})^2 = (N_{\alpha,\gamma})^2 = \frac{e}{2}.$$

Moreover it follows from (1.8) that

$$N_{\alpha,\beta}N_{\gamma,-\lambda} + N_{\gamma,\alpha}N_{\beta,-\lambda} = 0.$$

Therefore (5.1) gives

$$(\Lambda^2 R)(\Lambda(\gamma)\alpha,\ \bar{\lambda},\ \beta\ ;\ \alpha,\ \beta) = \frac{e^2 N_{\gamma,-\lambda}(c_\alpha)^2 c_\beta}{2(c_{\alpha+\beta})^2 c_{\alpha+\gamma}} \cdot (\alpha+\beta). \tag{5.2}$$

Similarly, we have

$$(\Lambda^2 R)(\alpha,\ \bar{\lambda},\ \beta\ ;\ \Lambda(\gamma)\alpha,\ \beta) = \frac{e^2 c_\alpha c_\beta}{2(c_{\alpha+\beta})^2} N_{\gamma,-\lambda}\cdot(\alpha+\beta). \tag{5.3}$$

From (4.3) we get

$$\begin{aligned} &(\Lambda^2 R)(\alpha,\ \Lambda(\gamma)\bar{\lambda},\ \beta\ ;\ \alpha,\ \beta) \\ &= N_{\gamma,-\lambda}(\Lambda^2 R)(\alpha,\ \overline{\alpha+\beta},\ \beta\ ;\ \alpha,\ \beta) \\ &= -\frac{e^2 c_\alpha c_\beta}{(c_{\alpha+\beta})^2} N_{\gamma,-\lambda}\cdot(\alpha+\beta). \end{aligned} \tag{5.4}$$

Therefore it follows from (5.2), (5.3) and (5.4) that

$$\begin{aligned} &(\nabla^3 R)(\alpha,\ \bar{\lambda},\ \beta\ ;\ \alpha,\ \beta,\ \gamma) \\ &= \frac{e^2 c_\alpha c_\beta}{2(c_{\alpha+\beta})^2} N_{\gamma,-\lambda}\cdot\left\{\frac{c_\alpha}{c_{\alpha+\gamma}} + 1 - 2\right\}\cdot(\alpha+\beta) \\ &= -\frac{e^2 c_\alpha c_\beta c_\gamma}{2(c_{\alpha+\beta})^2 c_{\alpha+\gamma}} N_{\gamma,-\lambda}\cdot(\alpha+\beta). \end{aligned}$$

This completes the proof of Lemma 5.2. □

Proof of Lemma 5.3. We shall show that

$$(\Lambda^3 R)(\alpha,\ \bar{\lambda},\ \alpha\ ;\ \beta,\ \beta,\ \beta) \neq 0 \ \ (\lambda = 2\beta + \alpha).$$

In fact

$$\begin{aligned} &(\Lambda^3 R)(\alpha,\ \bar{\lambda},\ \alpha\ ;\ \beta,\ \beta,\ \beta) \\ &= \Lambda(\beta)(\Lambda^2 R)(\alpha,\ \bar{\lambda},\ \alpha\ ;\ \beta,\ \beta) \end{aligned}$$

$$
\begin{aligned}
&(\Lambda^2 R)(\Lambda(\beta)\alpha, \bar{\lambda}, \alpha; \beta, \beta) \\
&\quad - (\Lambda^2 R)(\alpha, \Lambda(\beta)\bar{\lambda}, \alpha; \beta, \beta) \\
&\quad - (\Lambda^2 R)(\alpha, \bar{\lambda}, \Lambda(\beta)\alpha; \beta, \beta) \\
&= 3\Lambda(\beta)\{R(\Lambda(\beta)\Lambda(\beta)\alpha, \bar{\lambda})\alpha + R(\alpha, \Lambda(\beta)\Lambda(\beta)\bar{\lambda})\alpha \\
&\quad + R(\alpha, \bar{\lambda})\Lambda(\beta)\Lambda(\beta)\alpha + 2R(\Lambda(\beta)\alpha, \Lambda(\beta)\bar{\lambda})\alpha \\
&\quad + 2R(\Lambda(\beta)\alpha, \bar{\lambda})\Lambda(\beta)\alpha + 2R(\alpha, \Lambda(\beta)\bar{\lambda})\Lambda(\beta)\alpha\} \\
&\quad - 3\{R(\Lambda(\beta)\Lambda(\beta)\alpha, \Lambda(\beta)\bar{\lambda})\alpha + R(\Lambda(\beta)\Lambda(\beta)\alpha, \bar{\lambda})\Lambda(\beta)\alpha \\
&\quad + R(\Lambda(\beta)\alpha, \Lambda(\beta)\Lambda(\beta)\bar{\lambda})\alpha + R(\alpha, \Lambda(\beta)\Lambda(\beta)\bar{\lambda})\Lambda(\beta)\alpha \\
&\quad + R(\Lambda(\beta)\alpha, \bar{\lambda})\Lambda(\beta)\Lambda(\beta)\alpha + R(\alpha, \Lambda(\beta)\bar{\lambda})\Lambda(\beta)\Lambda(\beta)\alpha\} \\
&\quad - 6R(\Lambda(\beta)\alpha, \Lambda(\beta)\bar{\lambda})\Lambda(\beta)\alpha.
\end{aligned}
$$

As before, we set $e = \alpha(H_\alpha)$. Then we obtain

$$\beta(H_\beta) = (N_{\alpha,\beta})^2 = (N_{\beta,-\lambda})^2 = \frac{e}{2}, \quad \alpha(H_\beta) = -\frac{e}{2}.$$

Thus, by a straightforward computation we have

$$(\Lambda^3 R)(\alpha, \bar{\lambda}, \alpha; \beta, \beta, \beta) = \frac{3e^2 c_\alpha (c_\beta)^2}{2(c_{\alpha+\beta})^3} N_{\beta,-\lambda} \cdot (\alpha + \beta).$$

We have thus proved the lemma. □

Suppose that $\mathfrak{g}$ is not of G_2 type. For Kähler C-spaces except for those stated in Theorem 5.1, we take examples of $\{\alpha, \beta, \gamma\}$ satisfying the conditions of Lemma 5.2 or of $\{\alpha, \beta\}$ satisfying the conditions of Lemma 5.3.

The case where $\mathfrak{g}$ *is of type* A_l $(l \geq 3)$.

Suppose that α_i, α_j and α_k are elements of Ψ $(i < j < k)$. Then set

$$\alpha = \alpha_1 + \cdots + \alpha_{j-1}, \beta = \alpha_j, \gamma = \alpha_{j+1} + \cdots + \alpha_l.$$

Then α, β and γ satisfy (1)~(15) of Lemma 5.2.

The case where $\mathfrak{g}$ *is of type* B_l $(l \geq 2)$.

We use the notation in Section 4.

Suppose that Ψ contains α_i and α_j $(i < j)$. Put

$$\alpha = \alpha_i = e_i - e_{i+1}, \beta = e_{i+1} = \alpha_{i+1} + \cdots + \alpha_l.$$

Then α and β satisfy (1)~(5) of Lemma 5.3.

The case where $\mathfrak{g}$ *is of type* C_l $(l \geq 3)$.

Suppose that Ψ contains α_i and α_j $(i < j)$. Put $\beta = \alpha_i + \cdots + \alpha_{j-1} = e_i - e_j$ and

$$\alpha = 2e_j = \begin{cases} \alpha_l & \text{if } j = l, \\ 2\alpha_j + \cdots + 2\alpha_{l-1} + \alpha_l & \text{if } j < l. \end{cases}$$

Then α and β satisfy (1)~(5) of Lemma 5.3.

The case where $\mathfrak{g}$ *is of type* D_l $(l \geq 4)$.

Suppose that Ψ contains $\{\alpha_i, \alpha_l\}$ $(2 \leq i \leq l-2)$. Then put

$$\alpha = \alpha_l = e_{l-1} + e_l, \beta = \alpha_2 + \cdots + \alpha_{l-1} = e_2 - e_l, \gamma = \alpha_1 + \cdots + \alpha_{l-2} = e_1 - e_{l-1}.$$

Then α, β and γ are contained in $\Delta^+(\Psi)$ and satisfy (1)~(15) in Lemma 5.2.

Next, we assume that Ψ cotains $\{\alpha_i, \alpha_j\}$ $(1 \leq i < j \leq l-2)$. Set

$$\alpha = \alpha_1 + \cdots + \alpha_{j-1}, \beta = \alpha_j + \cdots + \alpha_{l-2} + \alpha_{l-1}, \gamma = \alpha_j + \cdots + \alpha_{l-2} + \alpha_l.$$

Then α, β and γ are contained in $\Delta^+(\Psi)$ and satisfy (1)~(15) in Lemma 5.2.

The case where $\mathfrak{g}$ *is of type* E_8.

Set

$$\alpha = \begin{pmatrix} 0 & 1 & 1 & 2 & 2 & 1 & 1 \\ & & & & 1 & & \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 1 & 0 \\ & & & & 1 & & \end{pmatrix},$$
$$\gamma = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 2 & 1 \\ & & & & 1 & & \end{pmatrix}.$$

Then α, β and γ satisfy (1)~(15) in Lemma 5.2.

The case where $\mathfrak{g}$ *is of type* E_7.

Put

$$\alpha = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ & & & 0 & & \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ & & & 1 & & \end{pmatrix},$$
$$\gamma = \begin{pmatrix} 0 & 0 & 1 & 2 & 1 & 1 \\ & & & 1 & & \end{pmatrix}.$$

Then α, β and γ satisfy (1)~(15) in Lemma 5.2. Therefore, if Ψ contains α_i $(i = 3,4$ or $5)$, the degree of $M(\mathfrak{g}, \Psi, g)$ is more than three. Moreover, if Ψ contains $\{\alpha_1, \alpha_6\}$, $\{\alpha_1, \alpha_7\}$, $\{\alpha_2, \alpha_6\}$ or $\{\alpha_2, \alpha_7\}$, the degree of $M(\mathfrak{g}, \Psi, g)$ is more than three.

Next, set

$$\alpha=\begin{pmatrix}0&1&1&1&1&0\\&&&1&&\end{pmatrix},\quad \beta=\begin{pmatrix}1&1&1&1&1&1\\&&&0&&\end{pmatrix},$$
$$\gamma=\begin{pmatrix}0&0&1&2&1&1\\&&&1&&\end{pmatrix}.$$

Then the degree of $M(\mathfrak{g}, \{\alpha_1, \alpha_2\}, g)$ is more than three.

Finally, suppose that $\Psi=\{\alpha_6, \alpha_7\}$. Set

$$\alpha=\begin{pmatrix}1&0&0&0&0&0\\&&&0&&\end{pmatrix},\quad \beta=\begin{pmatrix}0&1&1&1&1&1\\&&&1&&\end{pmatrix},$$
$$\gamma=\begin{pmatrix}0&1&3&3&2&1\\&&&1&&\end{pmatrix}.$$

Then α, β and γ are contained in $\Delta^+(\Psi)$ and satisfy (1)~(15) in Lemma 5.2.

The case where $\mathfrak{g}$ *is of type* E_6.

Set

$$\alpha=\begin{pmatrix}1&1&1&0&0\\&&0&&\end{pmatrix},\quad \beta=\begin{pmatrix}0&0&1&1&1\\&&1&&\end{pmatrix},$$
$$\gamma=\begin{pmatrix}0&1&1&1&0\\&&1&&\end{pmatrix}.$$

Thus we can see that the degree of $M(\mathfrak{g}, \Psi, g)$ is more than three if Ψ contains one of the following:

$$\{\alpha_4\},\ \{\alpha_2, \alpha_5\},\ \{\alpha_2, \alpha_6\},\ \{\alpha_3, \alpha_5\},\ \{\alpha_3, \alpha_6\}.$$

Finally, we check the case where $\Psi=\{\alpha_5, \alpha_6\}$. Then the following roots α, β and γ are contained in $\Delta^+(\Psi)$ and satisfy the conditions in Lemma 5.2:

$$\alpha=\begin{pmatrix}0&1&0&0&0\\&&0&&\end{pmatrix},\quad \beta=\begin{pmatrix}0&1&1&1&1\\&&1&&\end{pmatrix},$$
$$\gamma=\begin{pmatrix}0&1&2&1&0\\&&1&&\end{pmatrix}.$$

The case where $\mathfrak{g}$ *is of type* F_4.

Set $\alpha=(1,1,2,2)$ and $\beta=(0,1,1,0)$. Then α and β satisfy (1)~(5) of Lemma 5.3. Thus, if $\alpha_i \in \Delta^+(\Psi)$ ($i=2$ or 3), than the degree of $M(\mathfrak{g}, \Psi, g)$ is more than three.

Next, let $\Psi = \{\alpha_1, \alpha_4\}$. Then put $\alpha = (1,1,0,0)$ and $\beta = (0,0,1,1)$. Then α and β satisfy (1)~(5) of Lemma 5.3.

Finally we shall prove that the degree of $M(G_2, \{\alpha_1, \alpha_2\}, g)$ is more than three. Set $\alpha = \alpha_2$ and $\beta = \alpha_1$. Then Δ^+ consists of the following:

$$\alpha,\ \beta,\ \alpha+\beta,\ \alpha+2\beta,\ \alpha+3\beta,\ 2\alpha+3\beta.$$

Therefore we have from (1.9)

$$(5.5)\qquad (N_{\alpha,\beta})^2 = \frac{3}{2}\beta(H_\beta),\ (H_{\alpha+\beta,\beta})^2 = 2\beta(H_\beta),$$

$$(N_{-\beta,\alpha+3\beta})^2 = \frac{3}{2}\beta(H_\beta),\ \alpha(H_\alpha) = 3\beta(H_\beta),\ \alpha(H_\beta) = -\frac{3}{2}\beta(H_\beta).$$

We show that

$$(\nabla^3 R)(\alpha, \overline{\alpha+3\beta}, \beta\,;\beta, \beta, \beta) \neq 0.$$

From Theorem 3.4 we have

$$\begin{aligned}
&(\nabla^3 R)(\alpha, \overline{\alpha+3\beta}, \beta\,;\beta, \beta, \beta)\\
&= -(\Lambda^2 R)(\Lambda(\beta)\alpha, \overline{\alpha+3\beta}, \beta\,;\beta, \beta)\\
&\quad -(\Lambda^2 R)(\alpha, \Lambda(\beta)\overline{\alpha+3\beta}, \beta\,;\beta, \beta)\\
&= -3\{R(\Lambda(\beta)\alpha, \Lambda(\beta)\Lambda(\beta)\overline{\alpha+3\beta})\beta + R(\Lambda(\beta)\Lambda(\beta)\alpha, \Lambda(\beta)\overline{\alpha+3\beta})\beta\}\\
&\quad - R(\alpha, \Lambda(\beta)\Lambda(\beta)\Lambda(\beta)\overline{\alpha+3\beta})\beta - R(\Lambda(\beta)\Lambda(\beta)\Lambda(\beta)\alpha, \overline{\alpha+3\beta})\beta\\
&= N_{\beta,\alpha}N_{-\beta,\alpha+3\beta}N_{\beta,\alpha+\beta}\Big\{3\frac{c_\alpha}{c_{\alpha+2\beta}}\Big(\frac{c_\beta}{c_{\alpha+2\beta}}(N_{\beta,\alpha+\beta})^2 + \beta(H_{\alpha+\beta})\Big)\\
&\quad -3\frac{c_\alpha}{c_{\alpha+2\beta}}\Big(\frac{c_\beta}{c_{\alpha+3\beta}}(N_{-\beta,\alpha+3\beta})^2 + \beta(H_{\alpha+2\beta})\Big)\\
&\quad -\Big(\frac{c_\beta}{c_{\alpha+\beta}}(N_{\alpha,\beta})^2 + \beta(H_\alpha)\Big) + \frac{c_\alpha}{c_{\alpha+3\beta}}\beta(H_{\alpha+3\beta})\Big\}\cdot\beta\\
&= \frac{12c_a(c_\beta)^2}{c_{\alpha+\beta}c_{\alpha+2\beta}c_{\alpha+3\beta}}N_{\beta,\alpha}N_{-\beta,\alpha+3\beta}N_{\beta,\alpha+\beta}\beta(H_\beta)\cdot\beta\\
&\neq 0.
\end{aligned}$$

Therefore the degree of $M(G_2, \{\alpha, \beta\}, g)$ is more than three.

We have thus proved Theorem 5.1.

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Graduate School of Science and Technology
Chiba University
Chiba-shi, 263 Japan