J. DIFFERENTIAL GEOMETRY
100 (2015) 491-506

L-SPACE SURGERIES, GENUS BOUNDS, AND THE CABLING CONJECTURE

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Abstract

We establish a tight inequality relating the knot genus $g(K)$ and the surgery slope p under the assumption that p-framed Dehn surgery along K is an L-space that bounds a sharp 4-manifold. This inequality applies in particular when the surgered manifold is a lens space or a connected sum thereof. Combined with work of Gordon-Luecke, Hoffman, and Matignon-Sayari, it follows that if surgery along a knot produces a connected sum of lens spaces, then the knot is either a torus knot or a cable thereof, confirming the cabling conjecture in this case.

1. Introduction

1.1. Lens space surgeries. Denote by K a knot in S^3, $p/q \in \mathbb{Q} \cup \{1/0\}$ a slope, and $K(p/q)$ the result of p/q–Dehn surgery along K. By definition, the lens space $L(p,q)$ is the oriented manifold $-U(p/q)$, where U denotes the unknot and $|p| > 1$.

When can surgery along a non-trivial knot K produce a lens space? This question remains unanswered forty years since Moser first raised it [**23**], although work by several researchers has led to significant progress on it. For example, the cyclic surgery theorem of Culler-Gordon-Luecke-Shalen asserts that K is the unknot or a torus knot or else the surgery slope is an integer [**5**], and a conjecturally complete construction due to Berge accounts for all the known examples [**3**]. Furthermore, the complete list of lens spaces obtained by integer surgery along a knot is determined in [**14**].

On the basis of Berge's construction, Goda-Teragaito conjectured an inequality relating the surgery slope that produces a lens space and the knot genus $g = g(K)$ [**9**]. As the case of torus knots is completely settled [**23**], we assume that the surgery slope is an integer p. Reflect K if necessary in order to assume that $p > 0$. The Goda-Teragaito

This work was supported by an NSF Post-doctoral Fellowship, NSF grant DMS-1207812, and by Boston College, a changemaker campus [**36**].
Received 11/04/2013.

conjecture asserts that if K is a hyperbolic knot and $K(p)$ is a lens space, then

$$\frac{p-1}{2} \leq 2g-1 \leq p-9. \tag{1}$$

Note that $2g-1$ equals minus the maximum Euler characteristic of a Seifert surface for K.

Both conjectured bounds in (1) are now close to settled. Rasmussen established the lower bound

$$\frac{p-5}{2} \leq 2g-1$$

for any (not necessarily hyperbolic) knot K for which $K(p)$ is a lens space, observing that it is attained for $p = 4k+3$ and K the $(2, 2k+1)$-torus knot [**32**, Theorem 1]. His argument involves an application of Heegaard Floer homology, and in particular a use of the *correction terms* in the theory that we discuss below. Kronheimer-Mrowka-Ozsváth-Szabó established the upper bound

$$2g-1 \leq p \tag{2}$$

by an application of monopole Floer homology [**21**, Corollary 8.5]. Their argument utilizes the fact that the Floer homology of a lens space is as simple as possible: $\mathrm{rk}\, \widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|$. A space with this property is called an *L-space*, and a knot with a positive L-space surgery is called an *L-space knot*. The proof of (2) extends to show that the set of positive rational slopes for which surgery along K results in an L-space is either empty or consists of all rational values greater than or equal to $2g-1$. This fact holds in the setting of Heegaard Floer homology as well [**25**], the framework in place for the remainder of this paper.

As remarked in [**21**, pp. 537-8], the bound (2) can often be improved for the case of a lens space surgery. Indeed, a closer examination of the Berge knots suggests the bound

$$2g-1 \leq p - 2\sqrt{(4p+1)/5} \tag{3}$$

whenever $K(p)$ is a lens space, with the exception of K the right-hand trefoil and $p = 5$ (cf. [**34**]). This bound is attained by an infinite sequence of type VIII Berge knots K and slopes $p \to \infty$. Rasmussen, utilizing extensive computer calculations of the rational genera of knots in lens spaces, established the bound (3) for all $p \leq 100,000$ [**33**].

The current work addresses an improvement on the bound (2) in the direction of (3). We begin with the method introduced and carried out in [**21**, **25**], which uses a version of Theorem 2.4 below. That theorem uses the correction terms of a lens space $L(p,q)$ to place a restriction on the genus of a knot K with $K(p) \approx L(p,q)$. However, the formulae for these correction terms often prove unwieldy towards the end of extracting explicit bounds on the knot genus. The key advance presented here stems from the observation that a lens space bounds a *sharp* four-manifold (Definition 2.1), whose existence enables us to distill the desired information. In this more general set-up, we obtain the following result.

Theorem 1.1. *Let K denote an L-space knot and suppose that $K(p)$ bounds a smooth, negative definite 4-manifold X with $H_1(X;\mathbb{Z})$ torsion-free. Then the knot genus is bounded above by*

$$2g-1 \le p-\sqrt{p}-1. \tag{4}$$

If X can be chosen sharp, then we obtain the improved bound

$$2g-1 \le p-\sqrt{3p+1}. \tag{5}$$

Furthermore, there exists an infinite family of pairs (K_n, p_n) that attain equality in (5)*, where K_n denotes an n-fold iterated cable of the unknot, and $p_n \to \infty$.*

We do not know as much concerning the tightness of inequality (4). It does, however, lead to an improvement over [**24**, Proposition 1.3] for $p \ge 9$, which under the same assumptions establishes that $2g-1 \le p-4$ for the specific case of a torus knot K.

In fact, for the case of a lens space surgery, we completely establish the bound (3) in [**14**, Theorem 1.4]. We do so by an involved combinatorial argument that develops out of the approach of the present paper, and specifically from Theorem 3.3 below. For comparison between (3) and (5), note that $2\sqrt{4/5} \approx 1.79$ and $\sqrt{3} \approx 1.73$.

Thus, both (3) and (5) dramatically improve on Goda-Teragaito's second conjectured bound (1) for $p \gg 0$. In Section 5.2 we indicate how that bound follows for all except two values $p \in \{14, 19\}$. In addition, Baker-Grigsby-Hedden [**2**] and Rasmussen [**33**] have proposed programs to prove the completeness of Berge's construction using Floer homology. One step involved in both approaches is to argue the non-existence of a non-trivial knot K for which $K(2g-1)$ is a lens space. This fact follows immediately from Theorem 1.1 or [**14**, Theorem 1.4].

1.2. Reducible surgeries. Given a knot $\kappa \subset S^3$, let $C(q,r) \circ \kappa \subset S^3$ denote the knot obtained by taking a curve of slope q/r on $\partial\nu(\kappa)$. A knot K is a *cable knot* if $K \simeq C(q,r) \circ \kappa$ for some knot κ and slope q/r, where $|r| \ge 2$ and either $\kappa \not\simeq U$ or $|q| \ge 2$.

When can surgery along a knot K produce a reducible 3-manifold? The cabling conjecture of Gonzalez-Acuña – Short asserts that this can only occur when the knot is a cable knot with surgery slope equal to the boundary of the cabling annulus [**10**, Conjecture A], [**20**, Problem 1.79]. The cabling slope is given by the value $qr \in \mathbb{Z}$, and $K(qr) \approx$

$\kappa(q/r)\#U(r/q)$. Thus, the cabling conjecture predicts that if Dehn surgery along a knot is reducible, then the surgery slope is an integer and the reducible manifold has two prime summands, one of which is a lens space.

Progress has been made on these implications. Gordon-Luecke established that the surgery slope is an integer [**12**] (which we may, once again, assume is positive upon reflecting the knot) and also that the surgered manifold contains a lens space summand [**13**, Theorem 3]. Both of these results rely on a detailed combinatorial analysis of the pattern of intersection between a pair of carefully chosen surfaces in the knot complement. Further work of Howie, Sayari, and Valdez Sánchez, using graph-theoretic and group-theoretic arguments, implies that the surgered manifold has at most three prime summands, and if it has three, then two are lens spaces of coprime orders and the third is a homology sphere [**19**, **35**, **37**].

Apparently unknown to practitioners of Floer homology, Matignon-Sayari established a bound strikingly opposite to (2) in the context of the cabling conjecture.

Theorem 1.2 (Theorem 1.1, [**22**]). *If $K(p)$ is reducible, then either K is a cable knot with cabling slope p, or else*

$$p \leq 2g - 1. \tag{6}$$

The proof involves studying graphs of surface intersections, and it is a swift application of Hoffman's work [**17**]. In light of the bounds (2) and (6), in order to establish the cabling conjecture under the additional assumption that the surgered manifold is an L-space, it suffices to show that $K(2g-1)$ is never a reducible L-space for any knot K. Indeed, the cabling conjecture predicts that this is never the case, due to the following observation.

Proposition 1.3. *If K is a cable knot with cabling slope p and $K(p)$ is an L-space, then $2g - 1 < p$.*

Note that without the L-space condition, there is no relation in general between the genus of a cable knot and its cabling slope.

Proof. Express $K = C(q, r) \circ \kappa$ according to the definition of a cable knot. Then $p = qr$ and $K(p) \approx \kappa(q/r)\#U(r/q)$. In order for $K(p)$ to be an L-space, $\kappa(q/r)$ must be as well, so (2) implies that

$$2g(\kappa) - 1 < q/r;$$

the inequality is strict, since the left side is an integer while the right side is not. On the other hand, an elementary calculation shows that K bounds a surface F with

$$2g(F) - 1 = qr + r(2g(\kappa) - 1) - q.$$

Thus, $2g - 1 < qr = p$, as desired. q.e.d.

We do not know whether $K(2g - 1)$ is ever a reducible L-space, but we can show that this is not the case under the additional hypotheses stated at the beginning of Theorem 1.1.

Corollary 1.4. *Suppose that $K(p)$ is a reducible L-space that bounds a smooth, negative definite 4-manifold X with $H_1(X;\mathbb{Z})$ torsion-free. Then K is a cable knot with cabling slope p.*

Proof. This follows directly from Theorems 1.1 and Theorem 1.2.

q.e.d.

Corollary 1.4 applies to a connected sum of lens spaces, a natural case of interest in view of the fact that any reducible surgery has a lens space summand. Accordingly, the cabling conjecture follows in this case.

Theorem 1.5. *Suppose that surgery along a knot $K \subset S^3$ produces a connected sum of lens spaces. Then K is either a (q, r)-torus knot or a (q, r)-cable of an (s, t)-torus knot with $q = rst \pm 1$, and the surgery slope is qr. The surgered manifold is $U(q/r)\#U(r/q)$ or $U(q/rt^2)\#U(\pm r)$, respectively.*

Proof. Corollary 1.4 implies that K is a cable knot $C(q, r) \circ \kappa$ and the surgery slope is the cabling slope qr. The remainder of the argument is essentially a redaction of [**12**, §3]. Since $K(qr) \approx \kappa(q/r)\#U(r/q)$, it follows that $\kappa(q/r)$ is a lens space. Since $|r| \geq 2$, the slope q/r is not integral. It follows by the cyclic surgery theorem that κ is either an unknot or a torus knot [**5**]. If κ is an unknot, then $K \simeq T(q, r)$, and $K(qr) \approx U(q/r)\#U(r/q)$. If $\kappa \simeq T(s, t)$ is a torus knot, then [**23**] implies that $q = rst \pm 1$ and $\kappa(q/r) \approx U(q/rt^2)$, so $K(qr) \approx U(q/rt^2)\#U(r/q) \approx U(q/rt^2)\#U(\pm r)$. q.e.d.

1.3. Organization. Section 2 reviews the necessary background about Heegaard Floer homology and intersection pairings on 4-manifolds. Section 3 uses this material in conjunction with combinatorial arguments in order to establish the genus bounds stated in Theorem 1.1. Section 4 constructs the knots that attain the bound in Theorem 1.1. Section 5 concludes with a discussion about the sharpness of the bound in Theorem 1.1 and the Goda-Teragaito conjecture.

1.4. Conventions. We use homology groups with integer coefficients throughout. All 4-manifolds are assumed smooth. For a compact 4-manifold X, regard $H_2(X)$ as a $\mathbb{Z}$-module equipped with the intersection pairing on X. Let $W_{\pm p}(K)$ denote the 4-manifold obtained by attaching a $\pm p$-framed 2-handle to D^4 along the knot $K \subset S^3 = \partial D^4$.

Acknowledgments. Thanks to John Baldwin for his advice on this paper. Thanks to Cameron Gordon and John Luecke for enjoyable discussions and for drawing my attention to the references [**17**, **22**]. Thanks to the referees for suggesting improvements to the exposition.

2. Input from Floer homology

Ozsváth-Szabó associated a numerical invariant $d(Y, \mathfrak{t}) \in \mathbb{Q}$ called a *correction term* to an oriented rational homology sphere Y equipped with a spin^c structure $\mathfrak{t}$ [**25**]. It is analogous to Frøyshov's h-invariant in monopole Floer homology [**8**]. They proved that this invariant obeys the relation $d(-Y, \mathfrak{t}) = -d(Y, \mathfrak{t})$, and that if Y is the boundary of a negative definite 4-manifold X, then

$$c_1(\mathfrak{s})^2 + b_2(X) \leq 4d(Y, \mathfrak{t}) \tag{7}$$

for every $\mathfrak{s} \in \mathrm{Spin}^c(X)$ which extends $\mathfrak{t} \in \mathrm{Spin}^c(Y)$ [**25**, Theorem 9.6].

Definition 2.1. A negative definite 4-manifold X is *sharp* if, for every $\mathfrak{t} \in \mathrm{Spin}^c(Y)$, there exists some extension $\mathfrak{s} \in \mathrm{Spin}^c(X)$ that attains equality in the bound (7).

The following result provides the examples of L-spaces and sharp 4-manifolds that we will need.

Proposition 2.2 (Proposition 3.3 and Theorem 3.4, [**30**])**.** *Let L denote a non-split alternating link. Then the branched double-cover $\Sigma(L)$ is an L-space, and there exists a sharp 4-manifold X with $\partial X = \Sigma(L)$ and $H_1(X) = 0$.*

Every lens space $L(p, q)$ arises as the branched double-cover of a 2-bridge link. In this case, the 4-manifold $X(p, q)$ implied by Proposition 2.2 admits the following description. Assume $p > q > 0$, and write $p/q = [a_1, \ldots, a_n]^-$ as a Hirzebruch-Jung continued fraction, with each $a_i \geq 2$. Then $X(p, q)$ denotes plumbing along a linear chain of disk bundles over S^2 with Euler numbers $-a_1, \ldots, -a_n$, in that order. From this perspective, the sharpness of $X(p, q)$ also follows from [**26**, Corollary 1.5]. In particular, $W_{-p}(U)$ is sharp, since it is diffeomorphic to the disk bundle of Euler number $-p$ over S^2.

In order to make use of (7), we must understand spin^c structures on $K(p)$. Given $\mathfrak{t} \in \mathrm{Spin}^c(K(p))$, it extends to some $\mathfrak{s} \in \mathrm{Spin}^c(W_p(K))$, since $H_1(W_p(K)) = 0$. The group $H_2(W_p(K))$ is generated by the class of a surface Σ obtained by smoothly gluing the core of the handle attachment to a copy of a Seifert surface for K with its interior pushed into *int*(D^4). The quantity $\langle c_1(\mathfrak{s}), [\Sigma]\rangle + p$ is an even value $2i$ whose residue class (mod $2p$) does not depend on the choice of extension $\mathfrak{s}$.

The assignment $\mathfrak{t} \mapsto i$ sets up a 1-1 correspondence

$$\mathrm{Spin^c}(K(p)) \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}. \tag{8}$$

Next, suppose that $K(p)$ bounds a smooth, negative definite 4-manifold X with $n := b_2(X)$. The manifold $W := -W_p(K) \approx W_{-p}(\overline{K})$ is negative definite and has boundary $\overline{K}_{-p} = -K(p)$, where $\overline{K}$ denotes the mirror image of K. Form the closed, smooth, oriented 4-manifold $Z := X \cup W$. Since $b_1(K(p)) = 0$, it follows that $b_2(Z) = b_2(X) + b_2(W) = n+1$; and, since $H_2(X) \oplus H_2(W) \hookrightarrow H_2(Z)$, it follows that Z is negative definite. In particular, the square of a class in $H^2(Z)$ equals the sum of the squares of its restrictions to $H^2(X)$ and $H^2(W)$.

Lemma 2.3. *Suppose that $K(p)$ bounds a smooth, negative definite 4-manifold X with $H_1(X)$ torsion-free, and form $Z = X \cup W$ as above. Then every $i \in \mathrm{Spin^c}(K(p))$ extends to some $\mathfrak{s} \in \mathrm{Spin^c}(Z)$, and*

$$c_1(\mathfrak{s})^2 + (n+1) \leq 4d(K(p), i) - 4d(U(p), i). \tag{9}$$

Furthermore, if X is sharp, then for every i there exists some extension $\mathfrak{s}$ that attains equality in (9)*.*

Proof. The fact that every spin^c structure on $K(p)$ extends across Z follows from the fact that $H_1(X)$ and $H_1(W)$ are torsion-free. Now fix some $i \in \mathrm{Spin^c}(K(p))$ and an extension $\mathfrak{s} \in \mathrm{Spin^c}(Z)$. From (7) we obtain

$$c_1(\mathfrak{s}|X)^2 + b_2(X) \leq 4d(K(p), i).$$

Observe that the maximum value of $c_1(\mathfrak{s}|W)^2 + 1$ depends only on the intersection pairing on W, which is independent of the knot K. Since $W_{-p}(U)$ is sharp, it follows that this value equals $4d(U(-p), i)$. Therefore,

$$c_1(\mathfrak{s}|W)^2 + 1 \leq -4d(U(p), i).$$

Summing these two inequalities results in (9).

We obtain equality in (9) under the assumption that X is sharp by taking an extension of i to some $\mathfrak{s}_X \in \mathrm{Spin^c}(X)$ that attains equality in (7) and gluing it to an extension $\mathfrak{s}_W \in \mathrm{Spin^c}(W)$ that attains the value $-4d(U(p), i)$. q.e.d.

Let K denote an L-space knot. We aim to use (9) to obtain information about the knot genus. Consider the Alexander polynomial of K,

$$\Delta_K(T) = \sum_{j=-g}^{g} a_j \cdot T^j, \quad g := \deg(\Delta_K),$$

and define the *torsion coefficient*

$$t_i(K) = \sum_{j \geq 1} j \cdot a_{|i|+j}.$$

Since K is an L-space knot, [**29**, Theorem 1.2] implies that the knot Floer homology group $\widehat{HFK}(K)$ is uniquely determined by the Alexander polynomial Δ_K. In particular, the maximum Alexander grading in which this group is supported is equal to the degree g of Δ_K. On the other hand, [**27**, Theorem 1.2] implies that this grading equals the knot genus. Furthermore, [**29**, Theorem 1.2] implies that the non-zero coefficients of the Alexander polynomial take values ± 1 and alternate in sign, beginning with $a_g = 1$. It follows that for all $i \geq 0$, the quantity

$$t_i(K) - t_{i+1}(K) = \sum_{j \geq 1} a_{i+j}$$

is always 0 or 1, so the values $t_i(K)$ form a sequence of monotonically decreasing, non-negative integers for $i \geq 0$. Therefore, we obtain

$$t_i(K) = 0 \text{ if and only if } |i| \geq g. \tag{10}$$

Owens-Strle state the following result explicitly; it slightly extends the case $q = 1$ of [**31**, Theorem 1.2] in that it applies when $p < 2g - 1$, so that it is not necessary to assume that $K(p)$ is an L-space (see also [**21**, Theorem 8.5] or the identical [**25**, Corollary 7.5]).

Theorem 2.4 (Theorem 6.1, [**24**])**.** *Let K denote an L-space knot and p a positive integer. Then the torsion coefficients and correction terms satisfy the relation*

$$-2t_i(K) = d(K(p), i) - d(U(p), i), \text{ for all } |i| \leq p/2. \tag{11}$$

In [**21**, p. 538], the version of Theorem 2.4 stated there is used in conjunction with (10) to enumerate the lens spaces obtained by surgery along a knot K with genus $g \leq 5$. By using this approach in tandem with Lemma 2.3, we obtain the estimates presented in Theorem 1.1. To that end, we focus our attention to the left side of (9). Donaldson's diagonalization theorem implies that $H_2(Z) \approx -\mathbb{Z}^{n+1}$, where $-\mathbb{Z}^{n+1}$ denotes the integer lattice equipped with *minus* the standard Euclidean inner product [**6**]. Choose an orthonormal basis $\{e_0, \ldots, e_n\}$ for $-\mathbb{Z}^{n+1}$: $\langle e_i, e_j \rangle = -\delta_{ij}$ for all i, j, where δ_{ij} denotes the Kronecker delta. The first Chern class map

$$c_1 : \mathrm{Spin^c}(Z) \to H^2(Z)$$

has image the set of *characteristic covectors* for the inner product space $H_2(Z)$. Identify $H_2(Z) \approx H^2(Z)$ by Poincaré duality; then Donaldson's diagonalization theorem implies that this set corresponds to

$$\mathrm{Char}(-\mathbb{Z}^{n+1}) = \left\{ \mathfrak{c} = \sum_{i=0}^{n} \mathfrak{c}_i e_i \mid \mathfrak{c}_i \text{ odd for all } i \right\}.$$

Write

$$\sigma = \sum_{i=0}^{n} \sigma_i e_i$$

for the image of the class $[\Sigma]$ under the inclusion $H_2(W) \hookrightarrow H_2(Z) \approx -\mathbb{Z}^{n+1}$.

With the preceding notation in place, the following lemma follows on combination of Lemma 2.3 with Theorem 2.4.

Lemma 2.5. *Let K denote an L-space knot, and suppose that $K(p)$ bounds a smooth, negative definite 4-manifold X with $H_1(X)$ torsion-free. Then*

$$\mathfrak{c}^2 + (n+1) \leq -8t_i(K) \tag{12}$$

for all $|i| \leq p/2$ and $\mathfrak{c} \in \mathrm{Char}(-\mathbb{Z}^{n+1})$ such that $\langle \mathfrak{c}, \sigma \rangle + p \equiv 2i \pmod{2p}$. Furthermore, if X is sharp, then for every $|i| \leq p/2$, there exists $\mathfrak{c}$ that attains equality in (12).

3. Genus bounds

In this section, we establish the bounds appearing in Theorem 1.1. Both bounds stem from the following result, whose proof and application are elementary. Recall that $|\sigma|_1$ denotes the L^1-norm $\sum_{i=0}^{n} |\sigma_i|$.

Proposition 3.1. *Under the hypotheses of Lemma* 2.5*,*

$$2g \leq p - |\sigma|_1, \tag{13}$$

with equality if X is sharp.

Proof. Select $\mathfrak{c} \in \{\pm 1\}^{n+1}$ and define i by the conditions that $\langle \mathfrak{c}, \sigma \rangle + p \equiv 2i \pmod{2p}$ and $|i| \leq p/2$. Lemma 2.5 implies that $t_i(K) = 0$, noting that the torsion coefficients are non-negative, and (10) implies in turn that $|i| \geq g$. Since $|\langle \mathfrak{c}, \sigma \rangle| \leq |\langle \sigma, \sigma \rangle| = p$, we have $\langle \mathfrak{c}, \sigma \rangle + p = 2i$ or $2i + 2p$ according to whether $i \geq 0$ or $i < 0$. If $i \geq 0$, then $\langle \mathfrak{c}, \sigma \rangle + p = 2i \geq 2g$, while if $i < 0$, then $\langle \mathfrak{c}, \sigma \rangle + p \geq p \geq 2|i| \geq 2g$. It follows that for all $\mathfrak{c} \in \{\pm 1\}^{n+1}$, we have

$$2g \leq \langle \mathfrak{c}, \sigma \rangle + p. \tag{14}$$

The minimum value of the right side of (14) over all $\mathfrak{c}$ is attained by the *sign vector* $s(\sigma)$ defined by

$$s(\sigma)_j := \begin{cases} +1, & \text{if } \sigma_j \geq 0; \\ -1, & \text{otherwise.} \end{cases}$$

For it, (14) produces the desired bound (13). The equality under the assumption that X is sharp follows immediately. q.e.d.

The bound (4) in Theorem 1.1 follows at once from Proposition 3.1 and the trivial inequality $p = |\langle \sigma, \sigma \rangle| \leq |\sigma|_1^2$. Now suppose that X is sharp. Then $2g = p - |\sigma|_1$ by Proposition 3.1, and its proof extends to show that for all $p - |\sigma|_1 \leq 2i \leq p$, there exists $\mathfrak{c} \in \{\pm 1\}^{n+1}$ with $\langle \mathfrak{c}, \sigma \rangle + p = 2i$. Replacing any such $\mathfrak{c}$ by its negative, we obtain this fact for all $p - |\sigma|_1 \leq 2i \leq p + |\sigma|_1$. In other words, for all $-|\sigma|_1 \leq j \leq |\sigma|_1$ with $j \equiv p \equiv |\sigma|_1 \pmod 2$, there exists $\mathfrak{c} \in \{\pm 1\}^{n+1}$ for which $\langle \mathfrak{c}, \sigma \rangle = j$. By a change of basis of $-\mathbb{Z}^{n+1}$, we may assume that the vector σ has the property that

$$0 \leq \sigma_0 \leq \cdots \leq \sigma_n.$$

Write a vector $\mathfrak{c} \in \{\pm 1\}^{n+1}$ in the form $(-1, \ldots, -1) + 2\chi$, where $\chi \in \{0,1\}^{n+1}$. Then we obtain that for every $0 \leq k \leq |\sigma|_1$, there exists $\chi \in \{0,1\}^{n+1}$ for which $|\langle \chi, \sigma \rangle| = -\langle \chi, \sigma \rangle = k$. In other words, for every such k, there exists a subset $S \subset \{0, \ldots, n\}$ for which $\sum_{i \in S} \sigma_i = k$.

Lemma 3.2. *Consider a sequence of integers $0 \leq \sigma_0 \leq \cdots \leq \sigma_n$. For every value $0 \leq k \leq \sigma_1 + \cdots + \sigma_n$, there exists a subset $S \subset \{0, \ldots, n\}$ such that $\sum_{i \in S} \sigma_i = k$ if and only if*

$$\sigma_i \leq \sigma_0 + \cdots + \sigma_{i-1} + 1 \text{ for all } 0 \leq i \leq n. \tag{15}$$

If we imagine the σ_i as values of coins, then Lemma 3.2 provides a necessary and sufficient condition under which one can make exact change from the coins in any amount up to their total value. We call such a vector $\sigma = (\sigma_0, \ldots, \sigma_n)$ a *changemaker* (cf. [**15**]); the concept was apparently first introduced under the term *complete sequence* in [**4**, **18**]. Before proceeding to the proof of Lemma 3.2, we emphasize what we have just established.

Theorem 3.3. *Let $K \subset S^3$ denote an L-space knot and suppose that $K(p)$ bounds a sharp 4-manifold X. Then $H_2(X) \oplus H_2(W)$ embeds as a full-rank sublattice of $-\mathbb{Z}^{n+1}$, where $n = b_2(X)$ and the generator of $H_2(W)$ maps to a changemaker σ with $\langle \sigma, \sigma \rangle = -p$.*

Proof of Lemma 3.2. ($\Longrightarrow$) We establish the contrapositive statement. If the inequality (15) failed for some i, then let k denote the value $\sigma_0 + \cdots + \sigma_{i-1} + 1$. For any $S \subset \{1, \ldots, n\}$, either $j < i$ for all $j \in S$, in which case $\sum_{j \in S} \sigma_j < k$, or there exists some $j \in S$ with $j \geq i$, in which case $\sum_{j \in S} \sigma_j \geq \sigma_i > k$. Therefore, there does not exist a subset S such that $\sum_{j \in S} \sigma_j = k$.

($\Longleftarrow$) We proceed by induction on n. The statement is obvious when $n = 0$. For the induction step, select any value $1 \leq k \leq \sigma_1 + \cdots + \sigma_n$ and pick the largest value j for which $k \geq \sigma_0 + \cdots + \sigma_{j-1} + 1$. By (15), $k - \sigma_j \geq 0$, and by the choice of j, we have $k - \sigma_j \leq \sigma_0 + \cdots + \sigma_{j-1}$. By induction on n, there exists $S' \subset \{1, \ldots, j-1\}$ (possibly the empty

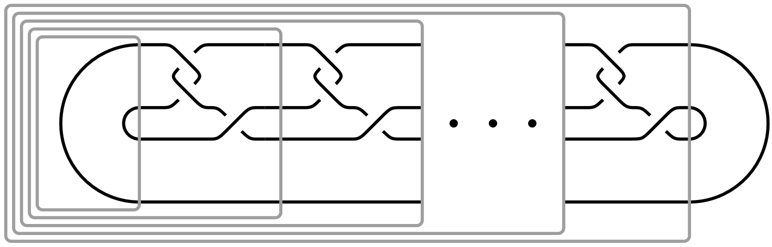

Figure 1. The knot κ_n.

set) for which $\sum_{i\in S'} \sigma_i = k - \sigma_j$; now $S = S' \cup \{j\}$ provides the desired subset with $\sum_{i\in S} \sigma_i = k$. q.e.d.

Returning to the case at hand, we appeal to Lemma 3.2 and invoke the inequality (15) for each $0 \le i \le n$ to obtain the estimate

$$\begin{aligned}(\sigma_0 + \cdots + \sigma_n + 1)^2 &= 1 + \sum_{i=0}^{n} \sigma_i^2 + 2\sigma_i(\sigma_0 + \cdots + \sigma_{i-1} + 1)\\ &\ge 1 + \sum_{i=0}^{n} 3\sigma_i^2 = 3p + 1.\end{aligned}$$

It follows that $|\sigma|_1 \ge \sqrt{3p+1} - 1$, and on combination with the equality in (13) we obtain the desired bound (5).

4. Iterated cables

In this section, we prove the final assertion of Theorem 1.1. Let $p_0 = 0$, and for $n \ge 0$, inductively define $a_{n+1} = 2p_n + 1$ and $p_n = 2a_n - 1$. Let $K_n = C(a_n, 2) \circ \cdots \circ C(a_1, 2) \circ U$.

Proposition 4.1. *For all $n \ge 1$, $K_n(p_n)$ is an L-space that bounds a sharp 4-manifold, and the bound in Equation* (5) *is attained for the pair (K_n, p_n).*

Proof. Let κ_n denote the alternating knot depicted in Figure 1. It contains n copies of the tangle $\mathcal{T}$ displayed in Figure 2. According to Proposition 2.2, $\Sigma(\kappa_n)$ is an L-space for all $n \ge 1$, and there exists a sharp 4-manifold X_n with $\partial X_n = \Sigma(\kappa_n)$. The space X_n admits a Kirby calculus description by attaching 2-handles along a framed link $\mathbb{L}_n \subset S^3 = \partial D^4$. Here $\mathbb{L}_n$ denotes a linear chain of $n-1$ unknots, with each component framed by -5 and oriented clockwise, and with each consecutive pair in the chain linked twice positively. This Kirby description derives from the one described on [**28**, p. 719] after canceling each 1-handle with an appropriate (-1)-framed 2-handle.

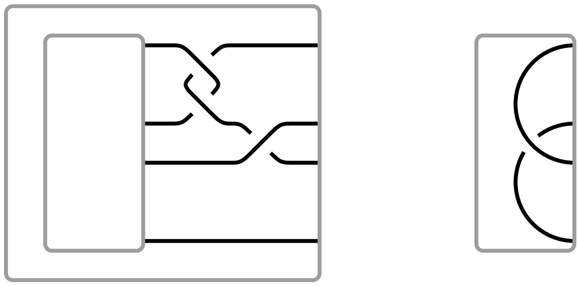

Figure 2. The pair of tangles $\mathcal{T}$ and $\mathcal{T}'$.

Let $\mathcal{T}''$ denote the inner-most tangle in the picture for κ_n and $\mathcal{T}_n$ the complementary tangle. Filling $\mathcal{T}$ with the rational tangle $\mathcal{T}'$ in Figure 2 yields a tangle isotopic as a marked tangle to $\mathcal{T}'$ itself. Thus, filling $\mathcal{T}_n$ with the rational tangle $\mathcal{T}''$ produces κ_n, while filling it with $\mathcal{T}'$ produces the unknot. It follows that $\Sigma(\mathcal{T}_n)$ is the complement of *some* knot $K_n \subset S^3$ for which p_n-surgery produces $\Sigma(\kappa_n)$ for *some* $p_n \in \mathbb{Q}$.

We claim that the pair (K_n, p_n) agrees with the pair stated in the proposition. We establish this fact by induction on n. When $n = 0$, κ_0 is a two-component unlink, $\Sigma(\kappa_0) \approx S^1 \times S^2$, and the claim follows easily by direct inspection. We proceed to the induction step.

The space $\Sigma(\mathcal{T})$ is the (unique) Seifert-fibered space over the annulus with a single exceptional fiber of multiplicity 2; equivalently, it is homeomorphic to a $C(p, 2)$ cable space (here p can denote any odd number) (cf. [**11**, Section 4.2]). On the left of Figure 3, we redraw $\mathcal{T}$ with emphasis on a collection of arcs on its boundary. Identify the picture of $\mathcal{T}$ in Figure 3 with the inner-most copy appearing in the diagram for κ_n. Let μ_n denote a meridian and λ_n a Seifert-framed longitude for K_n, oriented so that $\langle \mu_n, \lambda_n \rangle = +1$. Since filling $\Sigma(\mathcal{T}_n)$ with $\Sigma(\mathcal{T}')$ and $\Sigma(\mathcal{T}'')$ yields S^3 and $\Sigma(\kappa_n)$, respectively, it follows that $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ represent the meridian μ_n and surgery slope p_n, respectively.

The space $\Sigma(\mathcal{T}_n)$ consists of filling $\Sigma(\mathcal{T}_{n-1})$ with the cable space $\Sigma(\mathcal{T})$, where $\widetilde{\delta}_1$ gets identified with the meridian μ_{n-1} of K_{n-1}. It follows at once that $\Sigma(\mathcal{T}_n)$ is the complement of some 2-cable of K_{n-1}; it stands to determine which. Orienting $\widetilde{\delta}_3$ appropriately, we have

$$\langle \mu_{n-1}, \widetilde{\delta}_3 \rangle = \langle \widetilde{\delta}_1, \widetilde{\delta}_3 \rangle = 2$$

and

$$\langle \widetilde{\delta}_3, \lambda_{n-1} \rangle = \langle \widetilde{\delta}_3, \widetilde{\delta}_2 - p_{n-1} \cdot \mu_{n-1} \rangle = 1 + 2p_{n-1} = a_n,$$

applying the induction hypothesis to identify the surgery slope $\widetilde{\delta}_2$ with $p_{n-1} \cdot \mu_{n-1} + \lambda_{n-1}$. Thus, $\widetilde{\delta}_3$ represents the class $a_n \cdot \mu_{n-1} + 2\lambda_{n-1}$. On the other hand, $\widetilde{\delta}_3$ cobounds the annulus $\widetilde{D} \subset \Sigma(\mathcal{T})$ with $\widetilde{\gamma}_3$, and

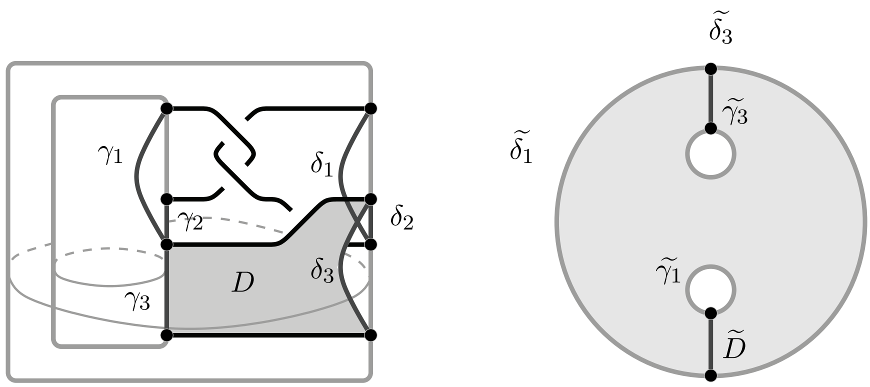

Figure 3. Arcs on the boundary of $\mathcal{T}$ and their lifts to curves on the boundary of $\Sigma(\mathcal{T}) \approx C(p, 2)$, a cross-section of which is shown.

$\widetilde{\gamma}_3$ is a longitude for K_n, since it meets μ_n in a single point. It follows that $K_n \simeq C(a_n, 2) \circ K_{n-1}$. To complete the induction step, we must identify the slope p_n with the value stated in the proposition. The curve $\widetilde{\gamma}_3$ represents the class $2a_n \cdot \mu_n + \lambda_n$ (cf. [**11**, p. 32]). Orienting $\widetilde{\gamma}_2$ appropriately, we have

$$\langle \mu_n, \widetilde{\gamma}_2 \rangle = \langle \widetilde{\gamma}_1, \widetilde{\gamma}_2 \rangle = 1$$

and

$$\langle \widetilde{\gamma}_2, \lambda_n \rangle = \langle \widetilde{\gamma}_2, \widetilde{\gamma}_3 - 2a_n \cdot \mu_n \rangle = -1 + 2a_n = p_n,$$

so $\widetilde{\gamma}_2 = p_n \mu_n + \lambda_n$, as desired.

Lastly, it stands to verify that $2g(K_n) - 1 = p_n - \sqrt{3p_n + 1}$. This follows easily from the behavior of the knot genus under cabling. An alternative argument runs as follows. Since $b_2(X_n) = n - 1$, the vector σ belongs to $-\mathbb{Z}^n$. Furthermore, $|\langle \sigma, \sigma \rangle| = p_n$. In light of (15), it follows that $\sigma = \sum_{i=1}^n 2^{i-1} e_i$. The formula for $g(K_n)$ now follows on application of (13). q.e.d.

Fintushel-Stern gave a construction for a Kirby diagram of an iterated cable [**7**]. It would be illuminating to identify the spaces $\Sigma(\kappa_n)$ and $K_n(p_n)$ using their technique.

5. Concluding remarks

5.1. Iterated cables. We discovered the construction in Proposition 4.1 in the following indirect way. Suppose that (K, p) attains equality in (5), where $K(p)$ bounds a sharp 4-manifold X. It follows that the vector σ representing the class $[\Sigma]$ must attain equality in (15) for all i. Thus, σ takes the form $\sum_{i=1}^n 2^{i-1} e_i$ for some $n \geq 1$, and $p = |\langle \sigma, \sigma \rangle| = p_n$.

Now, $H_2(X)$ embeds in $-\mathbb{Z}^n$ as the orthogonal complement $(\sigma)^\perp$. This subspace is spanned by the vectors $2e_i - e_{i+1}$, for $i = 1, \ldots, n-1$. With respect to this basis, the intersection pairing on X equals the linking matrix for $\mathbb{L}_n$. Thus, the simplest choice for X is the result of attaching 2-handles to D^4 along the framed link $\mathbb{L}_n$. The knot κ_n results from reverse-engineering the process for producing a sharp 4-manifold from the branched double-cover of a non-split alternating link [**28**, p. 719]. The family of knots K_n follows in turn.

It appears difficult to address whether the family of knots K_n attaining equality in (5) is unique. Any other candidate knot must have the same torsion coefficients, and hence knot Floer homology groups, as some K_n. Examples of distinct L-space knots with identical knot Floer homology groups do exist, but not in great abundance (cf. [**16**, §1.1.3]).

5.2. The Goda-Teragaito conjecture. Inequality (3) implies the second bound in (1) for all $p \geq 20$. Furthermore, a quick analysis of changemakers of norm 18, coupled with an application of Proposition 3.1, settles (1) for this value of p as well. The values $p \leq 17$, with the exception of $p = 14$, fall to a theorem of Baker [**1**, Theorem 1.6]. Combining these results, the second bound in (1) follows for all except the two values $p \in \{14, 19\}$. Part of the difficulty in handling these remaining cases owes to the fact that 14-surgery along the $(3, 5)$-torus knot and 19-surgery along the $(4, 5)$-torus knot both produce lens spaces, while neither of these knots is hyperbolic. The best that our methods establish is that any putative counterexample to (1) must have the same knot Floer homology groups as one of these two knots.

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