Examples of isometric immersions of $\mathbf{R}^2$ into $\mathbf{R}^4$ with vanishing normal curvature

Dedicated to the memory of Prof. Takashi OKAYASU

By Hiroshi MORI*)

Professor Emeritus, Joetsu University of Education

(Communicated by Kenji FUKAYA, M.J.A., Feb. 12, 2019)

Abstract: We construct a family of isometric immersions of $\mathbf{R}^2$ into $\mathbf{R}^4$ with vanishing normal curvature.

Key words: Isometric immersions; normal curvature.

1. Introduction and result. Hartman [2] showed that, for each pair of integers (n,p) with $1 \le p < n$, an isometric immersion f of $\mathbf{R}^n$ into $\mathbf{R}^{n+p}$ is reduced to an isometric immersion h of $\mathbf{R}^p$ into $\mathbf{R}^{2p}$, $f = B \circ (1 \times h) \circ A$, where A is an isometry of $\mathbf{R}^n$, B is an isometry of $\mathbf{R}^{n+p}$, and 1 is the identity mapping of $\mathbf{R}^{n-p}$. For $p = 1$, every h is completely charaterized by a real-valued function of a single variable (see Dajczer *et al.* [1]). For $p \ge 2$, the problem of describing all h remains elusive, even for $p = 2$.

Few isometric immersions of $\mathbf{R}^2$ into $\mathbf{R}^4$ are known. In this paper, we construct a family of new isometric immersions with vanishing normal curvature by getting solutions of a system of second order partial differential equations of hyperbolic type. The definition of the normal curvature R_n is given in [3], p. 526.

We are in the C^ω-category, unless otherwise is stated.

Proposition 1. *There exists a family of isometric immersions of $\mathbf{R}^2$ into $\mathbf{R}^4$ with vanishing normal curvature, each of which depends on four real parameters s, a, b, c and an analytic function w on $\mathbf{R}^2$.*

Corollary. *Except for one, every immersion f in the family is not a Riemannian product of two curves in $\mathbf{R}^4$ (see Remark 1 below). As $\mathbf{R}^4$-valued functions, every such f is an analytic function on $\mathbf{R}^2$ everywhere.*

2. Preliminaries. We recall basic results (see [3]). Let (x,y) be a standard coordinate system of $\mathbf{R}^2$ and D a domain in $\mathbf{R}^4$. For real-valued functions u_1 and u_2 defined on $\mathbf{R}^2 \times D$, let us consider a system of total differential equations for $\mathbf{R}^4$-valued functions f, e_1, e_2, e_3 and e_4.

$$
\begin{aligned}
(1)\qquad df &= (dx)e_1 + (dy)e_2,\\
de_1 &= d(\partial_x u_2)e_3 - d(\partial_x u_1)e_4,\\
de_2 &= d(\partial_y u_2)e_3 - d(\partial_y u_1)e_4,\\
de_3 &= -d(\partial_x u_2)e_1 - d(\partial_y u_2)e_2,\\
de_4 &= d(\partial_x u_1)e_1 + d(\partial_y u_1)e_2.
\end{aligned}
$$

The integrability condition of equations (1) is interpreted as a system of the following partial differential equations of hyperbolic type.

$$
\begin{aligned}
(2)\quad &(\partial_x^2 u_2 - \partial_y^2 u_2)(\partial_x\partial_y u_1) = (\partial_x^2 u_1 - \partial_y^2 u_1)(\partial_x\partial_y u_2),\\
&(\partial_x^2 u_1)(\partial_y^2 u_1) + (\partial_x^2 u_2)(\partial_y^2 u_2)\\
&\qquad = (\partial_x\partial_y u_1)^2 + (\partial_x\partial_y u_2)^2.
\end{aligned}
$$

Getting solutions u_1 and u_2 of (2) and applying Proposition 1.1 of [3], we shall prove our main results in the next section.

3. Proof of the result.

3.1. Solutions of partial differential equations (2). Let a, b, c, $(c < b)$ be positive constants and s a constant. We define the real numbers α, β, e by

$$
\begin{aligned}
(3)\qquad \alpha &= \frac{1 - ab + (a+b)c}{2\sqrt{c^2+1}},\\
\beta &= \frac{-(1-ab)c + a + b}{2\sqrt{c^2+1}},\\
e &= \sqrt{c^2+1} - c.
\end{aligned}
$$

Let us define a function $w(x,y)$ on $\mathbf{R}^2$ by $F(x+ey)$,

---

2010 Mathematics Subject Classification. Primary 53C42.

*) Present address: 2-5-45 Kitashiro-cho, Joetsu, Niigata 943-0824, Japan.

doi: 10.3792/pjaa.95.21

where $F(t)$ is a real analytic function defined on whole $\mathbf{R}$ with $F(0) = F'(0) = F''(0) = 0$.

Lemma 1. *For each function $w(x,y)$ as above, the functions*

$$u_1 = s\{(b/2+\alpha)x^2 + xy + (-b/2+\alpha)y^2 + w(x,y)\},$$
$$u_2 = s\{(ab/2+\beta)x^2 + axy + (-ab/2+\beta)y^2 + aw(x,y)\} \tag{4}$$

are solutions of the equations (2).

Proof. Denote by h the partial derivative $\partial_x^2 w(x,y)$. Then, we have the identities $\partial_x\partial_y w(x,y) = eh$ and $\partial_y^2 w(x,y) = e^2 h$ from which the identities

$$(\partial_x^2 - \partial_y^2)u_2 = a(\partial_x^2 - \partial_y^2)u_1 = as\{2b + (1-e^2)h\},$$
$$\partial_x\partial_y u_2 = a(\partial_x\partial_y u_1) = as(1+eh)$$

follows, and hence u_1, u_2 are solutions of the first equation of (2).

Since the function h has no constant terms, the left-hand side of the second equation (2) is of the form.

$$s^2\{\lambda + \mu h + \nu h^2\}, \tag{5}$$

where

$$\lambda = 4(\alpha^2+\beta^2) - (a^2+1)b^2,$$
$$\mu = (a^2+1)b(e^2-1) + 2(\alpha + a\beta)(e^2+1),$$
$$\nu = (a^2+1)e^2.$$

The right-hand side of the second equation (2) is of the form.

$$(a^2+1)s^2(1 + 2eh + e^2h^2). \tag{6}$$

From (5) and (6) together with (3) it follows that u_1, u_2 are solutions of the second equation of (2). $\square$

3.2. Equivalence relation. We recall here classical isometric immersions of $\mathbf{R}^2$ into $\mathbf{R}^4$, and introduce an equivalence relation.

Example 1. The mapping α, $\alpha(x,y) = (c(x), y)$, is an isometric immersion of $\mathbf{R}^2$ into $\mathbf{R}^4$, where $c(x)$ is a curve in $\mathbf{R}^3 \cong \mathbf{R}^3 \times \{0\}$, x being the arc length parameter.

Example 2. The mapping β, $\beta(x,y) = (c_1(x), c_2(y))$ is an isometric immersion of $\mathbf{R}^2$ into $\mathbf{R}^4$, where, $c_1(x)$ (resp. $c_2(y)$) is a curve in $\mathbf{R}^2 \cong \mathbf{R}^2 \times \{(0,0)\}$ (resp. $\mathbf{R}^2 \cong \{(0,0)\} \times \mathbf{R}^2$), x and y being the arc length parameters. In Examples 1 and 2, we mean $\cong$ by a congruent under the action of $O(4)$ on $\mathbf{R}^4$.

An isometry ϕ of $\mathbf{R}^n$ onto itself is given by

$$\phi(x^1, \cdots, x^n) = (x^1, \cdots, x^n)\tau + (b^1, \cdots, b^n),$$

where b^i are constants, $\tau = (a^i_j)$ is in $O(n)$, the orthogonal group.

Denote by $\mathcal{F}$ the space of all isometric immersions of $\mathbf{R}^2$ into $\mathbf{R}^4$, and introduce a relation $\sim$ in $\mathcal{F}$. Two elements f and h of $\mathcal{F}$ are said to be *equivalent* if and only if $h \circ \phi = \psi \circ f$ with an isometry ϕ of $\mathbf{R}^2$ and an isometry ψ of $\mathbf{R}^4$. The relation $\sim$ is an equivalence relation.

Definition. An element f of $\mathcal{F}$ is said to be a *Riemannian product* of two curves in $\mathbf{R}^4$ if f is equivalent to an isometric immersion α in Example 1, or to an isometric immersion β in Example 2.

Lemma 2. *Let $f(a,b,c,s,w(x,y))$ (denote by f^*) be an element of $\mathcal{F}$, which is constructed by functions u_1 and u_2 as in Lemma 1. If $s \neq 0$, then the immersion f^* is not a Riemannian product of two curves in $\mathbf{R}^4$.*

Proof. We prove that f^* is not related to one in Example 1. Similarly, f^* will not be related to an isometric immersion in Example 2.

By using (1), it can be easily shown that an isometric immersion f given in Proposition 2 is related to one in Example 1 if and only if for a constant $\theta, (-\pi < 2\theta \le \pi)$, the following equations hold identically.

$$(\sin^2\theta)\partial_x^2 u_i + (\cos^2\theta)\partial_y^2 u_i - (\sin 2\theta)\partial_x\partial_y u_i = 0, \quad (i = 1,2). \tag{7}$$

We now prove Lemma 2 by reduction to absurdity. Suppose that for a constant θ, all the equations (7) (with u_j in (3)) hold identically. The constant terms in (7) are of the form.

$$s\{(\sin^2\theta)(b/2+\alpha) - (\sin 2\theta) + (\cos^2\theta)(-b/2+\alpha)\} = 0,$$
$$s\{(\sin^2\theta)(ab/2+\beta) - a(\sin 2\theta) + (\cos^2\theta)(-ab/2+\beta)\} = 0. \tag{8}$$

Multiplying a on the first equation of (8) and subtracting the second one of (8), we have an equality $s(a\alpha - \beta) = 0$. The equality is inconsistent with the condition $s(b-c) \neq 0$ by virtue of (3). $\square$

Lemmas 1 and 2 imply that Proposition 1 is valid.

Remark 1. If $s = 0$ in Lemma 1, the functions u_1 and u_2 are identically zero, and hence f is a standard isometric imbedding of $\mathbf{R}^2$ into $\mathbf{R}^4$ with a standard basis $\{e_1, e_2, e_3, e_4\}$ along the isometric immersion f.

By using Lemma 3 below, $f(a, b, c, s, w(x, y))$ depends only on parameters a, b, c and s, and an analytic function $w(x, y)$.

3.3. Solutions of partial differential equations (1). Next lemma is given by Prof. N. Shimakura.

Lemma 3. *Let $\{e_1, e_2, e_3, e_4\}$ be functions of class C^2 of (x, y) defined in an open subset Ω in $\mathbf{R}^2$ with values in $\mathbf{R}^4$ which satisfy the equations*

$$(9) \quad \partial_x e_j = \sum_{k=1}^{4} s_{jk}(x, y) e_k, \quad \partial_y e_j = \sum_{k=1}^{4} t_{jk}(x, y) e_k \quad (j = 1, 2, 3, 4).$$

If s_{jk} and t_{jk} are real-analytic functions of (x, y) in Ω, then e_1, e_2, e_3, e_4 are real-analytic functions of (x, y) in Ω.

Proof. If ω is an open subset of Ω whose closure is a compact subset of Ω, there exist positive number μ_1 and ρ_1 independent of x, y such that

$$(10) \quad |\partial_x^p \partial_y^q s_{jk}| + |\partial_x^p \partial_y^q t_{jk}| \le \mu_1 \rho_1^{p+q} p! q! \quad (j, k = 1, 2, 3, 4)$$

for all integers $p \ge 0$ and $q \ge 0$ if $(x, y) \in \omega$. Let us show that there exist positive numbers μ_2 and ρ_2 independent of x, y such that

$$(11) \quad \|\partial_x^p \partial_y^q e_j\| \le \mu_2 \rho_2^{p+q} p! q! \quad (j = 1, 2, 3, 4)$$

for all integers $p \ge 0$ and $q \ge 0$ if $(x, y) \in \omega$.

(11) is true for $p = q = 0$ with $\rho_2 = 1$ and a $\mu_2 > 0$. Given n, assume (11) for all p, q satisfying $p + q \le n$ with a $\mu_2 > 0$ and a ρ_2. The Leibniz formula

$$\partial_x^{p+1} \partial_y^q e_j = \sum_{k=1}^{4} \sum_{p'=0}^{p} \sum_{q'=0}^{q} \frac{p! q!}{p'!(p-p')! q'!(q-q')!} \times (\partial_x^{p'} \partial_y^{q'} s_{jk})(\partial_x^{p-p'} \partial_y^{q-q'} e_k)$$

and (10), (11) with $p + q \le n$ yield

$$\begin{aligned} \|\partial_x^{p+1} \partial_y^q e_j\| &\le \sum_{k=1}^{4} \sum_{p'=0}^{p} \sum_{q'=0}^{q} p! q! \mu_1 \rho_1^{p'+q'} \mu_2 \rho_2^{p-p'+q-q'} \\ &= 4 \mu_1 \mu_2 p! q! \sum_{p'=0}^{p} \sum_{q'=0}^{q} \rho_1^{p'+q'} \rho_2^{p-p'+q-q'}. \end{aligned}$$

If we choose a ρ_2 greater than ρ_1, the right-hand side is smaller than

$$\begin{aligned} &4 \mu_1 \mu_2 (p+1)! q! \rho_2^p \sum_{q'=0}^{q} \rho_1^{q'} \rho_2^{q-q'} \\ &< 4 \mu_1 \mu_2 (p+1)! q! \rho_2^{p+q+1} / (\rho_2 - \rho_1). \end{aligned}$$

If we choose again a ρ_2 greater than $\rho_1 + 4\mu_1$, we have

$$\|\partial_x^{p+1} \partial_y^q e_j\| < \mu_2 (p+1)! q! \rho_2^{p+q+1}.$$

We can prove for this choice of μ_2 and ρ_2 also that

$$\|\partial_x^p \partial_y^{q+1} e_j\| < \mu_2 p! (q+1)! \rho_2^{p+q+1}$$

starting from $\partial_y e_j = \sum_k t_{jk} e_k$. So, (11) is true for all p and q satisfying $p + q \le n + 1$, and hence for all $p \ge 0$ and $q \ge 0$ if $(x, y) \in \omega$. □

Proof of the corollary. Let $u_j, (j = 1, 2)$ be functions given in (3). By applying Lemma 3 in case where

$$\begin{array}{lll} s_{12} = 0, & t_{12} = 0, & s_{23} = t_{13} = \partial_x \partial_y u_2, \\ s_{13} = \partial_x^2 u_2, & t_{23} = \partial_y^2 u_2, & s_{24} = t_{14} = -\partial_x \partial_y u_1, \\ s_{14} = -\partial_x^2 u_1, & t_{24} = -\partial_y^2 u_1, & s_{34} = t_{34} = 0, \\ s_{jk} = -s_{kj}, & t_{jk} = -t_{kj}, & (j, k = 1, 2, 3, 4), \end{array}$$

the solutions e_1, e_2, e_3, e_4 of (1) are real-analytic functions on $\mathbf{R}^2$, so is the solution f of (1). □

Acknowledgements. The present author would like to thank to Prof. Norio Shimakura for his constant encouragement and for his useful comments. He would like to thank to the referee for his useful comments.

References

[1] M. Dajczer, M. Antonucci, G. Oliveira, P. Lima-Filho, *et al.*, *Submanifolds and isometric immersions*, Mathematics Lecture Series, 13, Publish or Perish, Inc., Houston, TX, 1990.

[2] P. Hartman, On the isometric immersions in Euclidean space of manifolds with nonnegative sectional curvatures. II, Trans. Amer. Math. Soc. **147** (1970), 529–540.

[3] H. Mori and N. Shimakura, Isometric immersions of Euclidean plane into Euclidean 4-space with vanishing normal curvature, Tohoku Math. J. (2) **61** (2009), no. 4, 523–550.