NOTES ON THE DUALITY THEOREM OF NON-COMMUTATIVE TOPOLOGICAL GROUPS

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(Received June 30, 1963)

Duality theorems of topological groups are well known as one of the most beautiful theorems in the modern mathematics. As an application of these theorems, the author proved formerly the following fact: let G_1, G_2 be commutative locally compact topological groups with character groups χ_1, χ_2, F the whole set of homomorphic mappings of G_1 into G_2, and Φ the whole set of homomorphic mappings of χ_2 into χ_1, then F and Φ become topological groups and are isomorphic to each other.

Professor T.Tannaka suggested the author to prove the analogy of the above mentioned result for non-commutative topological groups. The purpose of this paper is to prove this analogy. The author wishes to express his hearty thanks to Professor T.Tannaka for his kind instructicns and advices.

1. In this paper we shall assume that a group G is a non-commutative topological group (with the Hausdorff topology), and a representation of a group G is a continuous bounded representation. In the beginning we shall refer to the duality theorem of non-commutative topological groups by T. Tannaka [3].

Let G be a compact group and G^* the whole set of representations of G. In G^* the following three operations are admitted:

(1) $D^{(1)} \times D^{(2)}$ (Kronecker composition)

(2) $\begin{pmatrix} D^{(1)} & \\ & D^{(2)} \end{pmatrix}$ (direct sum)

(3) $P^{-1}DP$ (similar representation)

(4) $\bar{D}$ (conjugate complex representation).

Now we understand by a representation of G^* a mapping

$$D \xrightarrow{A} A(D)$$

(D being an arbitrary element of G^*, $A(D)$ a non-singular matrix with same degree as D), with the following conditions:

(1) $A(D^{(1)} \times D^{(2)}) = A(D^{(1)}) \times A(D^{(2)})$

(2) $A\begin{pmatrix} D^{(1)} & \\ & D^{(2)} \end{pmatrix} = \begin{pmatrix} A(D^{(1)}) & \\ & A(D^{(2)}) \end{pmatrix}$

(3) $A(P^{-1}DP) = P^{-1}A(D)P$

(4) $A(\overline{D}) = \overline{A(D)}$

Let G^{**} be the whole set of representations of G^*. If we define the product of G^{**} by $AB(D) = A(D)B(D)$ where A and B belong to G^*, and introduce a neighbourhood-basis of an element A_0 of G^{**} by

$$U(A_0, D^{(1)}, \cdots, D^{(s)}; \varepsilon) = \{A; \|A(D^{(i)}) - A_0(D^{(i)})\| < \varepsilon, i = 1, \cdots, s\},$$

where $D^{(i)}$ are elements of G^*, and $\|C\|$ is the usual matrix norm, then G^{**} becomes a topological group. Let $D(a) = A_a(D)$ for an element a of G, then A_a belongs to G^{**} and the correspondence $a \to A_a$ is an isomorphic mapping of G onto G^{**}.

2. Definition. A mapping φ of G_1^* into G_2^*, where G_1, G_2 are compact groups, is called a homomorphic mapping if and only if the following conditions are satisfied:

(1) $\varphi(D^{(1)} \times D^{(2)}) = \varphi(D^{(1)}) \times \varphi(D^{(2)})$

(2) $\varphi\begin{pmatrix} D^{(1)} & \\ & D^{(2)} \end{pmatrix} = \begin{pmatrix} \varphi(D^{(1)}) & \\ & \varphi(D^{(2)}) \end{pmatrix}$

(3) $\varphi(P^{-1}DP) = P^{-1}\varphi(D)P$

(4) $\varphi(\overline{D}) = \overline{\varphi(D)}$

(5) $\varphi(D)$ has the same degree as D.

Let G_1, G_2 be compact groups and f be a homomorphic mapping of G_1 into G_2. If $D_2 f = D_1$ for an element D_2 of G_2^*, then it is clear that D_1 is an element of G_1^*, and we can verify that a mapping $\varphi: D_2 \to D_1 = D_2 f$ is a homomorphic mapping of G_2^* into G_1^*. Then this mapping φ is called the conjugate mapping of f.

THEOREM. *Let G_1, G_2 be compact groups, F the whole set of homomorphic mappings of G_1 into G_2, and Φ the whole set of homomorphic mappings of G_2^* into G_1^*. If φ is the conjugate mapping of f, where f is an element of F, then a mapping $f \to \varphi$ is a one-to-one correspondence of F onto Φ.*

PROOF. If f_1, f_2 are elements of F and $f_1 \neq f_2$, then there is an element a of G_1 with $f_1(a) \neq f_2(a)$. There exists an element D_2 of G_2^* with $D_2\{f_1(a)\} \neq D_2\{f_2(a)\}$, that is, $\varphi_1(D_2) \neq \varphi_2(D_2)$. Therefore the mappings φ_1, φ_2 which are conjugate to f_1, f_2, are different from each other.

Let φ be any element of Φ, and A_1 an element of G_1^{**}. If we denote $A_1\varphi = A_2$, then A_2 clearly belongs to G_2^{**}. The mapping $f: A_1 \rightarrow A_2$ transforms G_1^{**} into G_2^{**}. It is evident that f is an algebraically homomorphic mapping. Let any neighbourhood of A_2 be

$$U(A_2\,; D_2^{(1)}, \dots, D_2^{(s)}\,; \varepsilon) = \{A_2'\,; \|A_2'(D_2^{(i)}) - A(D_2^{(i)})\| < \varepsilon,\ i = 1, \dots, s\}$$

where $D_2^{(i)}$ belong to G_2^*. If we denote $\varphi(D_2^{(i)}) = D_1^{(i)}$ and make a neighbourhood of A_1, $U(A_1\,; D_1^{(1)}, \dots, D_1^{(s)}\,; \varepsilon)$, then it is clear that f transforms $U(A_1\,; D_1^{(1)}, D_1^{(s)}\,; \varepsilon)$ into $U(A_2\,; D_2^{(1)}, \dots, D_2^{(s)}\,; \varepsilon)$. That is, f is a continuous mapping of G_1^{**} into G_2^{**}. Let f_i be an isomorphic mapping of G_i onto G_i^{**}, $i = 1, 2$, obtained by the duality theorem. If we set $f_1(a) = A_a$ for $a \in G$ and $f' = f_2^{-1} f f_1$, then f' is a homomorphic mapping G_1 into G_2 and it holds that

$$A_a\{\varphi(D_2)\} = \{f(A_a)\}(D_2) \qquad \text{for } D_2 \in G_2,$$

hence we have

$$\{\varphi(D_2)\}(a) = D_2\{f'(a)\}.$$

Therefore φ is the conjugate mapping of f', and thus we have established a one-to-one correspondence between F and Φ.

3. Let G_1, G_2 be compact groups, and F, Φ have the meaning as above. We shall introduce a topology in each of F and Φ as follows. Let K be a compact subset of G_1, U an open subset of G_2 and $W(K, U)$ the set of elements of F which maps K into U. If $\sum_1$ is the whole set of $W(K, U)$, then it is easily proved that F is a topological space with $\sum_1$ as an open sub-basis.

On the other hand, if D_1 is an element of G_1^*, then $\|D_1(x)\|$ attains its maximum value at some element x of G_1, because G_1 is a compact set. Therefore we write this maximum value as $\|D_1\|$. Let φ be an element of Φ. For any set of elements $D_2^{(i)}$ element $D_2^{(i)}$ of G_2^*, we put

$$U_\varphi = \{\varphi'\,; \|\varphi'(D_2^{(i)}) - \varphi(D_2^{(i)})\| < \varepsilon,\ i = 1, \dots, s\}$$

where ε is an arbitrary positive real number. If $\sum_\varphi$ is the whole set of U_φ, then Φ is a topological space with $\sum_\varphi$ as a neighbourhood-basis of φ.

THEOREM. *Let G_1 and G_2 be compact groups, F the whole set of homomorphic mappings of G_1 into G_2, and Φ the whole set of homomorphic mappings of G_2^* into G_1^*, then F and Φ are homeomorphic to each other.*

PROOF. Let f be an element of F. If φ is the conjugate mapping of f, then it is deduced from the above theorem that the correspondence of f to φ is a one-to-one mapping of F onto Φ. Let U_φ be any neighbourhood-base of φ with

$$U_\varphi = \{\varphi' ; \|\varphi'(D_2^{(i)}) - \varphi(D_2^{(i)})\| < \varepsilon,\ i = 1,\cdots,s\}.$$

If a is any element of G_1 and an element b' of G_2 corresponds to an element A'_b of G_2^{**} in the isomorphic mapping G_2 onto G_2^{**}, then we can take a neighbourhood of $f(a)$ of the form

$$U_{f(a)} = \{b' ; \|A_{b'}(D_2^{(i)}) - A_{f(a)}(D_2^{(i)})\| < \frac{1}{2}\varepsilon,\ i = 1,\cdots,s\}.$$

There is a neighbourhood V_a of a with $f(\overline{V}_a) \subset U_{f(a)}$ where $\overline{V}_a$ is the closure of V_a. Then G is covered by finite open sets $V_{a_1},\cdots,V_{a_n}$, as $\{V_a | a \in G_1\}$ is an open-covering of G_1, and then $W_f = \bigcap_{j=1}^{n} W(\overline{V}_{a_j}, U_{f(a_j)})$ is a neighbourhood of f, as $f(\overline{V}_{a_j})$ is contained in $U_{f(a_j)}$ $j = 1,\cdots,n$. If f' is any element of W_f, and x is any element of G_1, then there exists $\overline{V}_{a_j}$ which contains x. Then $f(x)$ and $f'(x)$ belong to $U_{f(a_j)}$. Therefore

$$\|A_{f'(x)}(D_2^{(i)}) - A_{f(a_j)}(D_2^{(i)})\| < \frac{1}{2}\varepsilon$$

and

$$\|A_{f(x)}(D_2^{(i)}) - A_{f(a)}(D_2^{(i)})\| < \frac{1}{2}\varepsilon \quad \text{for } i = 1,\cdots,s,$$

then

$$\|A_{f'(x)}(D_2^{(i)}) - A_{f(x)}(D_2^{(i)})\| < \varepsilon \quad \text{for any } x \text{ of } G_1$$

whence

$$\|D_2^{(i)}\{f'(x)\} - D_2^{(i)}\{f(x)\}\| < \varepsilon$$

and therefore

$$\|\varphi'(D_2^{(i)}) - \varphi(D_2^{(i)})\| < \varepsilon.$$

Thus, φ' belongs to the neighbourhood U_φ of φ, accordingly the correspondence $f \to \varphi$ is continuous.

Let $W(K, U_{f(a)})$ be any neighbourhood sub-base of f, where

$$U_{f(a)} = \{b' ; \|A_{b'}(D_2^{(i)}) - A_{f(a)}(D_2^{(i)})\| < \varepsilon,\ i = 1,\cdots,s\}$$

for an element a of G_1. As $f(K)$ is compact, there is the maximum value of

$$\|A_{b'}(D_2^{(i)}) - A_{f(a)}(D_2^{(i)})\|$$

for b' in $f(K)$. Then there exists a neighbourhood $U'_{f(a)}$ of $f(a)$ of the form

$$U'_{f(a)} = \{b' ; \|A_{b'}(D_2^{(i)}) - A_{f(a)}(D_2^{(i)})\| < \varepsilon\lambda,\ 0 < \lambda < 1,\ i = 1,\cdots,s\}$$

and $f(K) \subset U'_{f(a)} \subset U_{f(a)}$. We now define a neighbourhood of φ by

$$U_\varphi = \{\varphi' ; \|\varphi'(D_2^{(i)}) - \varphi(D_2^{(i)})\| < \varepsilon(1-\lambda), \quad i = 1,\cdots,s\}.$$

If φ' is an element of U_φ and f' is such a mapping as $D_2 f' = \varphi'(D_2)$, then

$$\|D_2^{(i)}\{f'(x)\} - D_2^{(i)}\{f(x)\}\| < \varepsilon(1-\lambda)$$

for all element x of G_1, and thus

$$\|A_{f'(x)}(D_2^{(i)}) - A_{f(x)}(D_2^{(i)})\| < \varepsilon(1-\lambda).$$

On the other hand, if x is an element of K, we have

$$\|A_{f'(x)}(D_2^{(i)}) - A_{f(a)}(D_2^{(i)})\| < \varepsilon\lambda,$$

as $f(K)$ is contained in $U'_{f(a)}$. Accordingly

$$\|A_{f'(x)}(D_2^{(i)}) - A_{f(a)}(D_2^{(i)})\| < \varepsilon.$$

Hence f' belongs to W_f, and thus the correspondence $f \rightarrow \varphi$ is bi-continuous.

From the above mentioned proof, we conclude that F and Φ is homeomorphic to each other.

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