TRANSFORMATION GROUPS ON A K(π, 1), II

P. E. Conner

1. INTRODUCTION

This note is concerned with the study, by methods introduced in [3], of involutions on a finite-dimensional K(π, 1). For the sake of simplicity, we shall restrict our discussion to involutions, although our methods apply equally well to cyclic transformations of prime order.

Throughout this note, we denote by (T, X) an involution with at least one fixed point. The fixed point set of (T, X) is denoted by $F \subset X$. We assume that X is locally compact, connected, separable metric, and locally contractible. We denote by Y the universal covering space of X, and

$$\alpha: Y \to X$$

is the covering map. If $\pi = \pi_1(X)$ is the fundamental group of X, then (π, Y) is the action of π on Y defining a principal fibre structure in the covering space. The elements of π are denoted by the letter σ, with suitable subscripts when it is necessary to consider more than one element at a time.

An *involution with base point,* $(T, (X, x))$, is an involution for which a fixed point x is chosen as the reference point. An involution with base point induces an automorphism $T_*: \pi_1(X, x) \to \pi_1(X, x)$ of period 2. It was shown in [3] that an involution with a base point $(T, (X, x))$ admits a covering involution (t, Y) such that $\alpha(t(y)) = T(\alpha(y))$ and

$$t(\sigma(y)) = T_*(\sigma)(t(y)),$$

for any element $\sigma \in \pi$ and any point $y \in Y$.

2. PRELIMINARY FORMULAS

We let $(T, (X, x))$ be a fixed involution with base point, and we let (t, Y) be the covering involution. We define a subset $C \subset \pi = \pi_1(X, x)$ by

$$C = \{ \sigma | \sigma(y) = t(y) \text{ for some } y \in Y\}.$$

Furthermore, for $\sigma \in C$, let

$$F(\sigma) = \{ y | y \in Y, \sigma(y) = t(y)\}.$$

The importance of the sets $F(\sigma)$ lies in the fact that $\alpha(y) \in F$ if and only if $y \in F(\sigma)$ for some $\sigma \in C$.

(2.1) LEMMA. *If* $\sigma \in C$, *then* $T_*(\sigma) = \sigma^{-1}$, *and the map* $t_\sigma: Y \to Y$ *defined by* $y \to \sigma^{-1}(t(y))$ *is an involution.*

Received July 2, 1959.

Let z be a point in $F(\sigma)$. By definition, $t(z) = \sigma(z)$. This means that

$$z = t(\sigma(z)) = T_*(\sigma)(t(z)) = (T_*(\sigma)\sigma)(z).$$

Since π operates freely on Y, it follows that $T_*(\sigma) = \sigma^{-1}$.

To show that $t_\sigma(t_\sigma(y)) = y$ for $y \in Y$, we consider

$$\sigma^{-1}(t(\sigma^{-1}(t(y)))) = \sigma^{-1}(T_*(\sigma^{-1}))(t^2(y)) = \sigma^{-1}\sigma(y) = y.$$

It follows from the definition of $F(\sigma)$ that $F(\sigma)$ can be thought of as the fixed point set of the involution t_σ. In particular, if Y is contractible and finite-dimensional, then $F(\sigma)$ is acyclic mod 2, by Smith's theorem [6, p. 364].

(2.2) LEMMA. *Let $\sigma_1, \sigma_2 \in C$ and let $\sigma \in \pi$ be an element for which*

$$\sigma F(\sigma_1) \cap F(\sigma_2) \neq \emptyset;$$

then $T_*(\sigma) = \sigma_2\sigma\sigma_1^{-1}$ *and* $\sigma F(\sigma_1) = F(\sigma_2)$.

Suppose that $z \in F(\sigma_2)$ is a point for which $\sigma(z) \in F(\sigma_2)$. By definition, $\sigma_2(\sigma(z)) = t(\sigma(z))$. This implies that

$$\sigma(z) = t(\sigma_2(\sigma(z)) = T_*(\sigma_2)(t(\sigma(z)) = \sigma_2^{-1}(T_*(\sigma)(t(z))) = (\sigma_2^{-1}T_*(\sigma)\sigma_1)z.$$

Since π acts freely on Y, $\sigma = \sigma_2^{-1}T_*(\sigma)\sigma_1$ or $\sigma_2\sigma\sigma_1^{-1} = T_*(\sigma)$.

We next show that $F(\sigma_1) \subset F(\sigma_2)$. The relation

$$t(\sigma(y)) = T_*(\sigma)(t(y)) = \sigma_2\sigma\sigma_1^{-1}(\sigma_1(y)) = \sigma_2(\sigma(y))$$

implies that $\sigma(y) \in F(\sigma_2)$. It is now simple to show that $\sigma^{-1}F(\sigma_2) \subset F(\sigma_1)$, so that $\sigma F(\sigma_1) = F(\sigma_2)$.

As a particular case, we note that the relation $F(\sigma_1) \cap F(\sigma_2) \neq \emptyset$ implies that $F(\sigma_1) = F(\sigma_2)$, since we may apply (2.2) for $\sigma = e$, the identity element of π. If $\sigma \in C$, let

$$H(\sigma) = \{\sigma_1 \mid \sigma_1 \in \pi,\ \sigma_1 F(\sigma) = F(\sigma)\}.$$

Of course, $H(\sigma)$ is a subgroup of π. We may use (2.2) to characterize $H(\sigma)$ as follows:

$$H(\sigma) = \{\sigma_1 \mid \sigma_1 \in \pi,\ T_*(\sigma_1) = \sigma\sigma_1\sigma^{-1}\}.$$

Let us note in particular that if π is abelian, then $H(\sigma)$ is independent of $\sigma \in C$ and it consists of elements σ_1 for which $T_*(\sigma_1) = \sigma_1$. For an abelian group, too, C is a subgroup.

We define a set $L \subset Y$ by $\alpha^{-1}(F) = L$. If $y \in L$, then there is a unique element $\sigma \in C$ for which $y \in F(\sigma)$. Note that L is invariant under the action of π on Y.

(2.3) LEMMA. *The set* $F(\sigma) \subset L$ *is both open and closed in* L.

Suppose $y \in F(\sigma)$. Let V_y be an open neighborhood in Y for which

$$\sigma_1 V_y \cap \sigma_2 V_y \neq \emptyset$$

implies that $\sigma_1 = \sigma_2$. Let $U_y = V_y \cap L$; then $t(V_y) \cap \sigma(U_y)$ contains $t(y) = \sigma(y)$, so that $t(V_y) \cap (U_y)$ is an open neighborhood of $\sigma(y)$ (in L). By continuity, there is an open neighborhood N_y in L such that $N_y \subset U_y \subset V_y$, and $t(N_y) \subset \sigma(U_y)$. If $z \in N_y$, let z be in $F(\sigma_1)$; then $t(z) = \sigma_1(z)$, and $\sigma_1(z) \in t(N_y) \subset \sigma(U_y)$, so that $\sigma_1(V_y) \cap \sigma(V_y) \neq \emptyset$ and $\sigma = \sigma_1$. This implies that $N_y \subset F(\sigma)$ and that $F(\sigma)$ is open in L. The set $F(\sigma)$ is closed in Y, since it is the fixed point set of the involution t_σ.

(2.4) LEMMA. *Let* $F_c \subset X$ *be a component of the fixed point set* F. *Let* $y \in F(\sigma)$ *be a point such that* $\alpha(y) \in F_c$; *then* $F_c \subset \alpha(F(\sigma)) \subset F$.

Under our assumptions on X, it can be shown that F_c is arc-wise connected [5]. Let x be any point in F_c, and let $f(t)$ denote a path in F_c for which $f(0) = \alpha(y)$ and $f(1) = x$. By the covering homotopy theorem, there is some path $h(t)$ in Y which covers $f(t)$ and for which $h(0) = y$. Since $h(t)$ covers a path in F_c, it follows that $h(t)$ belongs to L. We have just seen, however, that $F(\sigma)$ is both open and closed in L, so that the path $h(t)$ must be in $F(\sigma)$. In particular, $h(1) \in F(\sigma)$ and $\alpha(h(1)) = x$.

An equivalence relation may be introduced in C. If $\sigma_1, \sigma_2 \in C$, then $\sigma_1 \sim \sigma_2$ if and only if there is an element $\sigma \in \pi$ such that $T_*(\sigma)\sigma_1 = \sigma_2\sigma$. Let $\hat{C}$ be the set of equivalence classes in C under this equivalence relation.

(2.5) LEMMA. *Two elements* $\sigma_1, \sigma_2 \in C$ *are equivalent if and only if there is an element* $\sigma \in \pi$ *such that* $\sigma F(\sigma_1) = F(\sigma_2)$.

This follows from (2.2) and the definition of equivalence. We shall take up the application of these lemmas to involutions with base point on finite-dimensional $K(\pi, 1)$-spaces. The $K(\pi, 1)$-space is particularly suitable for applying the foregoing remarks, because the universal covering space of a $K(\pi, 1)$ is contractible. We can apply Smith's results to the involutions t_σ on the covering space, to list their basic properties.

3. APPLICATIONS TO A FINITE-DIMENSIONAL $K(\pi, 1)$

(3.1) THEOREM. *If* $(T, (K(\pi, 1), x))$ *is an involution on a finite-dimensional* $K(\pi, 1)$*-space, then* $C \subset \pi$ *consists of all elements which satisfy* $T_*(\sigma) = \sigma^{-1}$, *and the components of the fixed point set* F *are in one-to-one correspondence with* $\hat{C}$.

Since $\pi_i(K(\pi, 1), x) = 0$ for $i > 1$, it follows that the universal covering space Y of $K(\pi, 1)$ is contractible. If $\sigma \in \pi$ is an element for which $T_*(\sigma) = \sigma^{-1}$, then $t_\sigma(y) = \sigma^{-1}(t(y))$ is an involution. Since Y is acyclic and finite-dimensional, it follows that t_σ has a nonvoid fixed point set which is acyclic mod 2 [6, p. 363]. In particular, $\sigma \in C$, and $F(\sigma)$ is connected. From (2.4) it follows that $\alpha(F(\sigma)) \subset K(\pi, 1)$ is a component of the fixed point set of $(T, K(\pi, 1))$. Furthermore, if $\alpha(F(\sigma_1)) = \alpha(F(\sigma_2))$, then for some $\sigma \in \pi$, $\sigma F(\sigma_1) = F(\sigma_2)$, and $\sigma_1 \sim \sigma_2$ by (2.5). Therefore, the components of F are in one-to-one correspondence with $\hat{C}$.

For the next result, we shall make use of the cohomology of a discrete group. We refer to [1, Exposé 1-13] for this concept.

(3.2) THEOREM. *If* $(T, (K(\pi, 1), x))$ *is an involution with a base point, then for every component* F_c *of the fixed point set there is an element* $\sigma \in C$ *such that*

$$H^i(F_c; Z_2) \simeq H^i(H(\sigma); Z_2).$$

We are to understand that the Alexander-Wallace-Spanier cohomology groups of F_c are isomorphic to the cohomology groups of the subgroup $H(\sigma) \subset \pi$. We choose an element $\sigma \in C$ such that $\alpha(F(\sigma)) = F_c$. The subgroup $H(\sigma)$ consists of those elements $\sigma_1 \in \pi$ for which $\sigma_1 F(\sigma) = F(\sigma)$. Since (π, Y) is a proper transformation group, the induced transformation group $(H(\sigma), F(\sigma))$ is also proper, and the quotient space $F(\sigma)/H(\sigma)$ is homeomorphic to F_c. We may regard $(H(\sigma), F(\sigma))$ as a covering of F_c. As we mentioned earlier, it can be shown that $F(\sigma)$ is locally connected, arc-wise connected and acyclic mod 2. As pointed out by Cartan [1, pp. 10-11], there is a spectral sequence $\{E_r^{s,t}\}$, with $E_2^{s,t} \simeq H^s(H(\sigma); H^t(F(\sigma); Z_2))$, whose E_∞-term is associated with $H^*(F_c; Z_2)$. Since $F(\sigma)$ is acyclic mod 2, however,

$$H^i(H(\sigma); Z_2) \simeq H^i(F_c; Z_2).$$

For involutions with base point on a finite-dimensional $K(\pi, 1)$, the cohomology mod 2 of the components of the fixed point set is entirely determined by the automorphism T_*. In a way, P. A. Smith's theorem about the fixed point set of an involution on a contractible space is one case of (3.2). It seems plausible that both results are special cases of a more extensive relation between the homotopy groups of a space and the involutions on that space. In this direction, we have continued to study the influence of higher homotopy groups on the fixed point set of an involution [2].

4. EXAMPLES

In this section we shall give some specific examples to illustrate our results.

(4.1) THEOREM. *Let* $(T, (K(\pi, 1), x))$ *be an involution with base point on a finite-dimensional* $K(\pi, 1)$ *for which* π *is a free abelian group of rank* n. *There is an integer* $0 \leq r \leq n$ *such that each component of the fixed point set has the cohomology ring* mod 2 *of an* r*-dimensional torus. If one fixed point is isolated, there are exactly* 2^n *components in* F.

We note that $K(\pi, 1)$ has the homotopy type of the n-torus. The subgroup $H(\sigma) \subset \pi$ is independent of $\sigma \in C$, and it consists of those elements $\sigma_1 \in \pi$ for which $T_*(\sigma_1) = \sigma_1$. Of course, $H(\sigma)$ is a free abelian group of rank $0 \leq r \leq n$, so that F_c has the mod 2 cohomology of an r-torus, in view of (3.2). In this case, C is a subgroup of π. The components of the fixed point set are in one-to-one correspondence with the quotient of C by the elements of the form $\sigma - T_*(\sigma)$, where $\sigma \in \pi$.

If one fixed point is isolated, then $H(\sigma)$ consists only of the identity. Since $\sigma_1 + T_*(\sigma_1) \in H(\sigma)$, it follows that $T_*(\sigma_1) = -\sigma_1$ for all elements in π. Accordingly, $C = \pi$, and the components of F are in one-to-one correspondence with the quotient group $\pi/2\pi$, which has order 2^n.

Our results also apply to involutions on an aspherical 3-manifold. We denote by B^3 a connected, locally Euclidean 3-manifold (possibly open) for which

$$\pi_i(B^3) = 0 \quad (i > 1).$$

Such a manifold is obtained, for example, by removing from the 3-sphere a tamely imbedded simple closed curve. Involutions on such a manifold have special properties. Using only the fact that B is a locally Euclidean 3-space, Smith has observed that every component of the fixed point set is a 0-, 1-, or 2-dimensional manifold [6, p. 372]. If B^3 is orientable, and if T preserves the orientation, then each component of the fixed point set is either a simple closed curve or an open arc.

(4.2) THEOREM. *Let* $(T, (B^3, x))$ *be an orientation-preserving involution with base point on an orientable, aspherical* 3-*manifold. The subgroup of elements of* $\pi_1(B^3, x)$ *which satisfy* $T_*(\sigma) = \sigma$ *is either trivial or free cyclic.*

Let $F(e) \subset Y$ be the set of points fixed under the involution (t, Y). The subgroup $H(e)$ consists of exactly those elements $\sigma \in \pi_1(B^3, x)$ for which $T_*(\sigma) = \sigma$. The fixed point set $F(e)$ is an open arc, and therefore $(H(e), F(e))$ represents the universal covering space of $F_c = \alpha(F(e))$. Now F_c is an open arc or a simple closed curve, and accordingly $H(e)$ is trivial or free cyclic since it is isomorphic to the fundamental group of F_c. We point out that in this case F_c is that component of the fixed point set which happens to contain the base point x.

R. H. Fox raised a question about the type of knot which can be imbedded in the 3-sphere in such a way that it is invariant under some involution of the sphere. H. F. Trotter pointed out an application of (4.2) to this type of question.

(4.3) THEOREM. *Let* (T, S^3) *be an orientation-preserving involution on the* 3-*sphere with a simple closed curve of fixed points. If* $K \subset S^3$ *is a torus knot of type* (m, n) *in the complement of the fixed point set of* (T, S^3) *which is carried onto itself by* T, *then* mn *is even.*

We take B^3 to be the complement $S^3 \setminus K$ of the torus knot. We select any fixed point, and we apply (4.2) to the involution $(T, (B^3, x))$ to conclude that the automorphism $T_*: \pi_1(B^3, x)$ is nontrivial. The involution $(T, (B^3, x))$ is orientation-preserving; therefore, by Schreier's structure theorems for a torus knot group, T_* is an inner automorphism [4]. The group of all the inner automorphisms of a torus knot group is isomorphic to the free product of the cyclic groups Z_n and Z_m. This free product can contain a nontrivial element of order 2 only if m or n is even.

This remark does not require that the involution be linear, or even that the fixed point set be tamely imbedded. Trotter also pointed out that in (4.3) an involution may be replaced by any cyclic transformation group of order k. If K is an invariant torus knot of type (m, n) in the complement of the fixed point set of the cyclic transformation, then m or n is divisible by k.

REFERENCES

1. Séminaire H. Cartan de l'École Normale Supérieure, 1950/1951, *Cohomologie des groupes, suite spectrale, faisceaux,* Second edition, mimeographed. Secrétariat mathématique, 11 rue Pierre Curie, Paris (1955).

2. P. E. Conner, *A special homotopy theory for involutions* (to appear).

3. P. E. Conner and D. Montgomery, *Transformation groups on a* $K(\pi, 1)$, *I.* Michigan Math. J. 7 (1960), 405-412.

4. O. Schreier, *Über die Gruppen* $A^a B^b = 1$, Abh. Math. Sem. Univ. Hamburg 3 (1924), 167-169.

5. P. A. Smith, *Transformations of finite period, II,* Ann. of Math. (2), 40 (1939), 690-711.

6. ———, *Fixed points of periodic transformations,* Appendix B, pp. 350-373, of *Algebraic topology,* by S. Lefschetz, Amer. Math. Soc. Colloqiuum Publication 27 (1942).

University of Virginia