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Tangents of a continuum and differentiable maps

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0. Introduction

Let $f: M \to N$ be a C^r map between manifolds and A be a subset of M. Sard has shown that if the tangent map Tf vanishes on the tangent spaces over A then $f(A)$ is small (Theorem 1). Moreover one may well expect that:

If Tf vanishes at least on the tangents of A, f(A) is small.
In this note we remark that Glaeser's linearized paratingents immediately enable us an accurate description of the above intuitive statement as a corollary of Sard's theorem. As for the case when A is connected and f is sufficiently smooth, we obtain a necessary and sufficidnt condition for $f(A)$ to be one point (Theorem 2). Then lightness of a sufficiently smooth map can be expressed by a property of Tf (or first derivatives)*.

Now we prepare the notation. Through this note, by a manifold M we mean a finite dimensional separable Hausdorff C^q manifold ($1 \leqq q \leqq \omega$*) without boundary. The dimension m of M is indicated as M^m. A submanifold means always a regular submanifold. TM denotes the tangent bundle of M. π is its projection onto M and $T_xM =$

* Church pointed out the importance of assuming lightness in the study of the structure of differentiable open maps. For, non-light maps are often too complicated to admit some positive assertions. A few simple and natural sufficient conditions are known for light maps; see [4].
* ω means real analyticity.

$\pi^{-1}(x)$. When we mention of C^r map $f: M \to N$, M and N are assumed to be C^q manifolds with $1 \leqq r \leqq q$. f canonically induces the tangent map $Tf: TM \to TN$.

1. Paratingent spaces

First, let us review the notion of the (*linearized*) *paratingent* of order s* in a form adapted to a manifold. Let A be a subset of a C^q manifold M and x a point of M. Condiser the set $\{V_\lambda\}_{\lambda \in \Lambda}$ of all C^s submanifolds ($1 \leqq s \leqq q$) which include A in a neighbourhood of x (i.e. the germ of V_λ at x includes one of A). The map of inclusion $\iota_\lambda: V_\lambda \to M$ induces the tangent map $T_x\iota_\lambda: T_xV_\lambda \to T_xM$. We put

$$P^s_{A,x} = \bigcap_{\lambda \in \Lambda} T_x\iota_\lambda(T_xV_\lambda) \subset T_xM,$$

$$P^s_A = \bigcup_{x \in M} P^s_{A,x} \subset TM.$$

We call $P^s_{A,x}$ the partingent space of order s of A at x and its element a paratingent of order s. It is obvious that

$$P^s_{\bar{A}} = P^s_A,$$

$$P^1_A \subset P^2_A \subset \cdots \subset P^q_A,$$

$$P^s_A \supset P^s_B \quad \text{if } A \supset B.$$

Proposition 1. *If V is an element of minimal dimension among the C^s submanifolds including A in neighbourhoods of x then $T_x\iota(T_xV) = P^s_{A,x}$, where $\iota: V \to M$ denotes the canonical injection.*

Corollary 1. *P^s_A is a closed subset of TM.*

These proofs are standard and we omit them.

If we put

$$A_\nu = A^s_\nu = \{x \in M : \dim P^s_{A,x} = \nu\}, \qquad \tilde{A}_\nu = \tilde{A}^s_\nu = A^s_\nu \cap A,$$

it is also easy to see the following:

* The notion of linearized paratingent of order s is due to Glaeser [5]. His $P^s_{tgl}(x)$ denotes the set of directions while our $P^s_{A,x}$ expresses the set of vectors. Here we introduce only a part of his theory on linearized paratingents.

Corollary 2. $\bigcup_{\nu=j}^{i} A_\nu$ *is an open subset of* $\bigcup_{\nu=j}^{m} A_\nu$ *for* $-1 \leqq j \leqq i \leqq m$. $\bigcup_{\nu=-1}^{m} A_\nu$ *(resp.* $\bigcup_{\nu=0}^{m} A_\nu$, $\bigcup_{\nu=1}^{m} A_\nu$*) is* M *(resp.* $\bar{A}$, *the derived set of* A*).*

The following is foundamental for paratingents.

Proposition 2. *Let* $f: M^m \to N^n$ *be a* C^r*-map and* A *be a subset of* M. *Then* Tf *maps* P_A^s *into*P_B^s, *where* $B=f(A)$ *and* $1 \leqq s \leqq r$.

Proof. We follow Glaeser's proof of the next corollary. Suppose that there exists a vector $e \in P_{A,x}^s$ such that $Tf(e) \notin P_B^s$. Put

$$\dim P_{A,x}^s = t, \quad \dim P_{B,y}^s = u, \quad y = f(x).$$

We can choose a local coordinate system $(x^1, x^2, \dots, x^t, x^{t+1}, \dots, x^m)$ at x of M as a C^s manifold such that $(\partial/\partial x^1)_x, (\partial/\partial x^2)_x, \dots, (\partial/\partial x^t)_x$ form a basis of $P_{A,x}^s$ and the C^s submanifold V defined by

$$x^{t+1} = x^{t+2} = \cdots = x^m = 0$$

includes A in a neighbourhood of x. Let $(y^1, y^2, \dots, y^u, y^{u+1}, \dots, y^n)$ and W be the correspondents of $(x^1, x^2, \dots, x^t, x^{t+1}, \dots, x^m)$ and V, for $\{N, B, y\}$ instead of $\{M, A, x\}$. We may assume that

$$e = (\partial/\partial x^1)_x, \qquad Tf(e) = (\partial/\partial y^{u+1})_y$$

without loss of generality. This means $(\partial/\partial x^1)_x(y^{u+1} \circ f) = 1$ and hence the equation $y^{u+1} \circ f = 0$ defines an $(t-1)$-deminsional C^s submanifold of V which includes A in a neighbourhood of x. Then $\dim P_{A,x}^s \leqq t-1$, a contradiction. Thus we have proved

$$Tf(P_{A,x}^s) \subset P_{B,y}^s.$$

Corollary [5, p. 55]. *Let* $f: M \to N$ *be a* C^r *map and let* A *be a subset of* M. *If* $f(A)$ *is one point then* $Tf(P_A^s) = 0$ *for* $1 \leqq s \leqq r$.

2. Hausdorff measure of $f(A)$

Since our manifold admits a Riemannian metric we can canonically define the (outer) t-dimensional *Hausdorff measure* μ_t on M for $t \geqq 0$*. A subset A of M is called *t-null*, *t-finite* or *t-sigmafinite* respectively when $\mu_t A=0$, when $\mu_t A<\infty$ or when A is a countable union of t-finite sets. t-nullity and t-sigmafiniteness are invariants of changes of the Riemannian metric and regular imbeddings of M into other manifolds. Sard has proved the following:

Theorem 1 [8, *Theorem* 2]. *Let* $f: M^m \to N$ *be a* C^r *map. If* $A \subset M$ *is t-sigmafinite* $(t>0)$ *and if* $Tf(\pi^{-1}A)=0$, *then* $f(A)$ *is* (t/r)*-null.*

Remark. If $\dim P^1_{A,x} \leqq t$ for any $x \in A$, especially if $m \leqq t$, then A is t-sigmafinite.

Putting $\tilde{P}^s_A = P^s_A \cap \pi^{-1}(A)$, we can sharpen the theorm as follows.

Corollary 1. *Let* $f: M \to N$ *be a* C^r *map and let* A *be a t-sigmafinite* $(t>0)$ *subset of* M. If $Tf(\tilde{P}^s_A)=0$, $f(A)$ *is* (t/u)*-null* ($u=\min(r, s)$).

Proof. Every $x \in A_v \cap A$ has an open neighbourhood B_x in A which is included in a v-dimensional C^s submanifold V_x, where A_v is the set defined in § 1. Let $\iota: V_x \to M$ be the canonical injection and $\pi': TV_x \to V_x$ be the canonical projection. If $y \in B_x \cap A_v$ then $T\iota(T_y V_x) = \tilde{P}^s_{A,y}$ by Proposition 1. Hence we have

$$T(f \circ \iota)(T_y V_x) = Tf\{T\iota(T_y V_x)\} = Tf(\tilde{P}^s_{A,y}) = 0$$

and

$$T(f \circ \iota)\{\pi'^{-1}(B_x \cap A_v)\} = 0.$$

Since $f \circ \iota$ is a C^u map on V_x, $f(B_x \cap A_v) = f \circ \iota(B_x \cap A_v)$ is (t/u)-null by the theorem. A is a countable union of sets of the form $B_x \cap A_v$. Then $f(A)$ is also (t/u)-null.

A continuous map between topological spaces is defined to be *light* when the inverse image of any one point does not contain a

* See [8] or [6, p. 102] for the definition.

continuum (a connected compact set containing at least two points). If the space of definition of the map is a separble metric and localy compact space, the map is light if and only if the inverse image of one point is either empty or 0-dimensional.

Corollary 2. *Let* $f: M \to N$ *be a* C^r *map and let* A *be a connected subset of* M *satisfying* $Tf(\tilde{P}_A^s)=0$. *If* A *is u-sigmafinite* ($u=\min(r, s)>0$), *then* $f(\bar{A})$ *is one point and* f *is not light.*

Proof. By Corollary 1, $f(A)$ is 1-null and hence 0-dimensional [6, p. 104]. In other words $f(A)$ is totally disconnected. Since $f(A)$ is connected, it is one point. Then $f(\bar{A})$ is also one point. A locally compact connected set, if not a single point, contains a continuum [7, p. 83], hence f is not light.

3. Crushing conditions

Theorem 2. *Let* $f: M^m \to N$ *be a* C^r *map with* $r \geqq m$ *and let* A *be a connected subset of* M. *Then the following conditions are mutually equivalent.*

(i) $f(A)$ (*or* $f(\bar{A})$) *is one point.*

(ii) $Tf(P_A^r)=0$.

(iii) $Tf(\tilde{P}_A^r)=0$.

(iv) $Tf(P_A^{m-1})=0$.

(v) $Tf(\tilde{P}_A^{m-1})=0$.

Remark. If $r=m-1$, (ii) does not mean (i) by the example of Whitney [10]. (see also [5]).

Proof. Here we treat the case $m \geqq 2$. The case $m=1$ is easily justified by the definition of P_A^0 in §4. By Corollary of Proposition 2, (ii) follows from (i). It is obvious that (ii) means (iii), (iv) and each of them means (v). Therefore we have only to prove (i) assuming (v). If $x \in \bigcup_{\nu=0}^{m-1} \tilde{A}_\nu^{m-1}$ (see § 1) then x has an $(m-1)$-sigmafinite neighbourhood B_x in A. Then $f(B_x)$ is 1-null by Corollary 1 of Theorem 1. If $x \in \tilde{A}_m^{m-1}$, $P_{A,x}^{m-1}=T_xM$. Hence $Tf\{\pi^{-1}(\tilde{A}_m^{m-1})\}=0$ and $f(\tilde{A}_m^{m-1})$ is 1-null by

Theorem 1. Thus $f(\bar{A})=f(\bigcup_{\nu=0}^{m}\tilde{A}_\nu^{m-1})$ is 1-null and one point as in the proof of Corollary 2 of Theorem 1.

Now we have obtained the following.

Let $f\colon M^m\to N$ *be a* C^r*-map with* $r\geqq m$. *Then* f *is not light if and only if there exists a connected set* (*or continuum*) $A\subset M$ *satisfying any one* (*or each*) *of* (i)~(v) *in Theorem* 2.

We remark that the condition of open light maps can be also expressed by properties of the first derivatives. Because of the above result we have only to express the condition of openess for light maps. This is just done by the following theorem due to Titus-Young and Church*

Let $f\colon M^m\to N^m$ *be a light* C^r *map* ($r\geqq m$) *between m-dimensional manifolds. Then* f *is open if and only if there are covering charts* $\{(U_i, \varphi_i)\}$ *of* M *and* $\{(V_j\ \psi_j)\}$ *of* N *such that the jacobian of* $\psi_j\circ f\circ\varphi_i^{-1}$ *does not change sign on* $\varphi_i(U_i\cap f^{-1}(V_j))$.

It is also known that if f is a light open C^r map on M^m ($r\geqq m$) $f^{-1}(y)$ is discrete for any $y\in N^n$, see [3, (1.9)].

4. Classical paratigents

Let A be a closed subset of M. The *paratingents* of A was defined by Bouligand [1]. We write the set of Bouligand's paratigents $P_A^0=\bigcup_{x\in A}P_{A,x}^0$. Glaeser characterized P_A^1 as a minimal subset $L=\bigcup_{x\in A}L_x$ of TM satisfying the following:

(i) $L\supset P_A^0$.

(ii) L_x is a linear subspace of T_xM.

(iii) L has upper semi-continiuty of inclusion on A.

On the condition (ii), (iii) is equivalent to:

(iii)′ L is a closed subset of TM

[2, p. 67]. Let f be a C^r map ($t\geqq 1$) defined on M. Since the kernel of Tf satisfies (ii) and (iii)′, the condition $Tf(P_A^1)=0$ is equivalent to $Tf(P_A^0)=0$.

* "if" part has been proved by Titus-Young [9, Theorem 2] for C^1-maps. "only if" part has been established by Church [3, (1.7)].

If A is not closed we define $P_A^0 = P_{\bar{A}}^0$. (This is equivalent to Bouligand's definition.) Then it is trivial that Theorem 2 is justified also in the case $m=1$.

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