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EXACT DISTRIBUTION OF THE PRODUCT OF INDEPENDENT GENERALIZED GAMMA VARIABLES WITH THE SAME SHAPE PARAMETER

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1. Introduction. Let X be a random variable whose frequency function is

$$f(x;\, a,\, d,\, p) = p(\Gamma(d/p)a^d)^{-1}x^{d-1}e^{-(x/a)^p}, \qquad x > 0;\ a,\, d,\, p > 0. \tag{1.1}$$

Form (1.1) is Stacy's [5] generalization of the gamma distribution. The familiar gamma, chi, chi-squared, exponential and Weibull variates are special cases, as are certain functions of normal variates. Form (1.1) is also a function introduced by L. Amoroso [1] in analyzing the distribution of income. Stacy [5] has studied some of the elementary properties of (1.1). Parr and Webster [3] have obtained expressions for the maximum likelihood estimators of the parameters of (1.1) and for their asymptotic variances and covariances. Malik [2] gives the exact distribution of the quotient of independent generalized gamma variables. In this note the exact distribution of the product of two independent generalized gamma variables with the same shape parameter is given.

2. Distribution of the product. Let X and Y be independently distributed with respective frequency functions $f(x;\, a_1,\, d_1,\, p)$ and $g(y;\, a_2,\, d_2,\, p)$. Let $U = XY$ where X and Y are defined as above.

The distribution of U is given by

$$f(u) = \int_0^\infty f(y)g(u/y)\,dy/y$$

$$= \int_0^\infty py^{d_1-1}e^{-(y/a_1)^p}[\Gamma(d_1/p)a_1^{d_1}]^{-1} \cdot p(u/y)^{d_2-1}[\Gamma(d_2/p)a_2^{d_2}]^{-1}e^{-(u/a_2y)^p}\,dy/y \tag{2.1}$$

$$= p^2u^{d_2-1}[\Gamma(d_1/p)\Gamma(d_2/p)a_1^{d_1}a_2^{d_2}]^{-1}\int_0^\infty \exp\left[-(y/a_1)^p - (u/a_2y)^p\right] \cdot y^{-(d_2-d_1+1)}\,dy. \tag{2.2}$$

If we make the transformation $(y/a_1)^p = t$, (2.2) can be written as

$$f(u) = pu^{d_2-1}[\Gamma(d_1/p)\Gamma(d_2/p)(a_1a_2)^{d_2}]^{-1} \cdot \int_0^\infty \exp\left[-t - (u^p/a_1^pa_2^pt)\right](t^{d_2/p-d_1/p+1})^{-1}\,dt. \tag{2.3}$$

It is well known in the theory of Bessel functions that [6]

$$\int_0^\infty e^{-t-z^2/4t}t^{-(v+1)}\,dt = 2(z/2)^{-v}K_v(z) \tag{2.4}$$

where $K_v(z)$ is the modified Bessel function of the second kind of order v. Thus, we have the exact distribution of the product $U = XY$.

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$$(2.5)\quad f(u) = 2pu^{d_2-1}[\Gamma(d_1/p)\Gamma(d_2/p)(a_1a_2)^{d_2}]^{-1} \cdot [(u/a_1a_2)^{p/2}]^{-(d_2/p-d_1/p)}K_{d_2/p-d_1/p}[2(u/a_1a_2)^{p/2}].$$

The distribution function of U is given by

$$(2.6)\quad \int_0^u f(t)\,dt = 2p[\Gamma(d_1/p)\Gamma(d_2/p)(a_1a_2)^{d_2}]^{-1} \cdot \int_0^u t^{d_2-1}[(t/a_1a_2)^{p/2}]^{-(d_2/p-d_1/p)}K_{d_2/p-d_1/p}[2(t/a_1a_2)^{p/2}]\,dt.$$

If we make the transformation $z = 2(t/a_1a_2)^{p/2}$, (2.6) can be written as

$$(2.7)\quad \int_0^u f(t)\,dt = 2^{2-d_1/p-d_2/p}[\Gamma(d_1/p)\Gamma(d_2/p)]^{-1} \cdot \int_0^{2(u/a_1a_2)^{p/2}} z^{d_1/p+d_2/p-1}K_{d_2/p-d_1/p}(z)\,dz.$$

It is well known in the theory of Bessel functions that [6], p. 80, for some integer $n \geqq 0$

$$(2.8)\qquad K_{n+\frac{1}{2}}(z) = (\pi/2z)^{\frac{1}{2}}e^{-z}\textstyle\sum_{r=0}^{n}(n+r)\,!\,[r\,!\,(n-r)\,!\,(2z)^r]^{-1}.$$

For the case $d_2/p - d_1/p = n + \frac{1}{2}$, we have

$$\begin{aligned} &\int_0^u f(t)\,dt \\ (2.9)\qquad &= 2^{2-d_1/p-d_2/p}[\Gamma(d_1/p)\Gamma(d_2/p)]^{-1}(\pi/2)^{\frac{1}{2}} \\ &\quad\cdot \textstyle\sum_{r=0}^{n}(n+r)\,!\,[r\,!\,(n-r)\,!\,2^r]^{-1}\Gamma(d_1/p + d_2/p - r - \frac{1}{2}) \\ &\quad\cdot I[2(u/a_1a_2)^{p/2}, d_1/p + d_2/p - r - \tfrac{3}{2}] \end{aligned}$$

where $I[2(u/a_1a_2)^{p/2}, d_1/p + d_2/p - r - \frac{3}{2}]$ is the incomplete gamma function [4].

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