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A COUNTEREXAMPLE TO A LEMMA OF L. D. BROWN

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In his renowned paper on sufficient statistics, L. D. Brown uses the following lemma (see page 1464, Lemma 5.1) for the proof of his main theorems (Theorem 2.1, Theorem 8.1 and Theorem 8.1′):

Let (a, b) be a finite interval in $\mathbb{R}$ and $\Phi:(a, b) \times (a, b) \to \mathbb{R}$ a continuous function. Assume that there exists a measurable set $A \subset (a, b)$ such that, for all $y \in [c, d]$ with $a < c < d < b$ and all measurable sets B, $\mu(B) = 0$ implies $\mu\{x \in A : \Phi(x, y) \in B\} = 0$ (where μ denotes the Lebesgue measure).

Then for any fixed measurable set B, the function $y \to \mu\{x \in A : \Phi(x, y) \in B\}$ is continuous on $[c, d]$.

The proof of this lemma given by Brown contains an error on page 1465, lines 10 and 11. It is the purpose of this note to show by a counterexample that the lemma itself is wrong. This does not imply, of course, that Brown's Theorems 2.1, 8.1 and 8.1′ are necessarily wrong. The author suspects however, that additional assumptions on the sufficient statistic will be needed for a correct proof.[1]

1. To prepare the definition of an appropriate function Φ we shall introduce the following sequence of sets:

$$A_n = \bigcup_{k=1}^{2^n} (k2^{-n} - 2^{-2n}, k2^{-n}), \qquad n \in \mathbb{N}.$$

The following relations will be needed subsequently:

$$\text{(1)} \qquad \lim_{n\to\infty} 2^n \mu([0, r) \cap A_n) = r \qquad \text{for every} \quad r \in [0, 1].$$

This follows immediately from the inequalities

$$r - 1/2^n \leqq 2^n \mu([0, r) \cap A_n) \leqq r \qquad \text{for} \quad r \in [0, 1], n \in \mathbb{N}.$$

$$\text{(2)} \qquad \mu(\textstyle\bigcup_{n=1}^{\infty} A_n) \leqq \frac{3}{4}.$$

We have $\mu(\bigcup_{n=1}^{\infty} A_n) = \frac{1}{2} + \frac{1}{2}\mu(\bigcup_{n=2}^{\infty} A_n) \leqq \frac{1}{2} + \frac{1}{2}\sum_{n=2}^{\infty} \mu(A_n) = \frac{3}{4}$.

2. Let $\mathscr{B}_0$ denote the Borel field of $[0, 1]$ and let $P_n | \mathscr{B}_0$ be the measure having μ-density $2^n 1_{A_n}$. The function $r \to P_n[0, r)$ is continuous and we have by (1): $\lim_{n\to\infty} P_n[0, r) = r$ for all $r \in [0, 1]$.

3. Let $(s_n)_{n \in \mathbb{N}} \downarrow \frac{1}{2}$ be a decreasing sequence with $s_1 = 1$. For $x \in [0, 1]$ and $y \in [s_{n+1}, s_n]$ let

$$G(x, y) = \tfrac{1}{2}x + \tfrac{1}{2}\frac{y - s_{n+1}}{s_n - s_{n+1}} P_n[0, x) + \tfrac{1}{2}\frac{s_n - y}{s_n - s_{n+1}} P_{n+1}[0, x).$$

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[1] In the meantime, Mr. Brown submitted a paper "Sufficient statistics in the case of independent random variables" for publication.

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This defines a continuous function $G:[0,1]\times(\frac{1}{2},1]\to[0,1]$. Furthermore, $(x_k)_{k\in\mathbb{N}}\to x_0$ and $(y_k)_{k\in\mathbb{N}}\to\frac{1}{2}$ implies $(G(x_k,y_k))_{k\in\mathbb{N}}\to x_0$. This can be seen as follows: For every $k\in\mathbb{N}$, let n_k be such that $s_{n_k+1}<y_k\leqq s_{n_k}$. As $(P_{n_k}[0,x))_{k\in\mathbb{N}}\to x$ for every $x\in[0,1]$, we have

$$\limsup_{k\in\mathbb{N}} P_{n_k}[0,x_k)\leqq \limsup_{k\in\mathbb{N}} P_{n_k}[0,x)\leqq x \quad \text{for every} \quad x>x_0$$

and therefore $\limsup_{k\in\mathbb{N}} P_{n_k}[0,x_k)\leqq x_0$. Similarly, $x_0\leqq \liminf_{k\in\mathbb{N}} P_n[0,x_k)$ so that $\lim_{k\in\mathbb{N}} P_{n_k}[0,\ x_k)=x_0$. Hence $(\alpha_k\ P_{n_k}[0,\ x_k)+(1-\alpha_k)P_{n_k+1}[0,\ x_k))_{k\in\mathbb{N}}\to x_0$ for any sequence $\alpha_k\in[0,1]$, $k\in\mathbb{N}$. Applied for $\alpha_k=(y_k-s_{n_k+1})/(s_{n_k}-s_{n_k+1})$, this yields $(G(x_k,y_k))_{k\in\mathbb{N}}\to x_0$. Hence we may extend G to a continuous function on $[0,1]\times[\frac{1}{2},1]$ by defining $G(x,\frac{1}{2})=x$, $x\in[0,1]$.

4. As $x\to G(x,y)$ is increasing for all $y\in[\frac{1}{2},1]$, there exists a uniquely determined function $F:[0,1]\times[\frac{1}{2},1]\to[0,1]$ such that $G(F(x,y),y)=x$ for all $x\in[0,1]$, $y\in[\frac{1}{2},1]$.

Let $((x_k,y_k))_{k\in\mathbb{N}}\to(x_0,y_0)$. Given an arbitrary infinite subset $\mathbb{N}_0\subset\mathbb{N}$ we choose $\mathbb{N}_1\subset\mathbb{N}_0$ such that $(F(x_k,y_k))_{k\in\mathbb{N}_1}$ converges. Let r_1 denote the limit of this sequence. As G is continuous, $(G(F(x_k,y_k),y_k))_{k\in\mathbb{N}}$ converges to $G(r_1,y_0)$. As $G(F(x_k,y_k),y_k))=x_k$, we have $G(r_1,y_0)=x_0$ and therefore $r_1=F(x_0,y_0)$. This implies that F is continuous in (x_0,y_0). Furthermore $F(x,\frac{1}{2})=x$ for all $x\in[0,1]$.

5. For each $y\in(\frac{1}{2},1]$, let $Q_y\,|\,\mathscr{B}_0$ be the measure defined by

$$Q_y(B)=\tfrac{1}{2}\mu(B)+\tfrac{1}{2}\frac{y-s_{n+1}}{s_n-s_{n+1}}P_n(B)+\tfrac{1}{2}\frac{s_n-y}{s_n-s_{n+1}}P_{n+1}(B),\ B\in\mathscr{B}_0,$$

where n is chosen such that $y\in[s_{n+1},s_n]$.

We shall show that

$$\mu\{r\in(0,1):F(r,y)\in B\}=Q_y(B) \qquad \text{for all} \quad B\in\mathscr{B}_0,\ y\in(\tfrac{1}{2},1].$$

For each $x\in[0,1]$, $y\in(\frac{1}{2},1]$, we have

$$\begin{aligned}\mu\{r\in(0,1):F(r,y)\in[0,x)\}&=\mu\{r\in(0,1):F(r,y)<x\}\\&=\mu\{r\in(0,1):r<G(x,y)\}\\&=G(x,y)=Q_y[0,x).\end{aligned}$$

As the measures $B\to\mu\{r\in(0,1):F(r,y)\in B\}$ and $B\to Q_y(B)$ coincide for all $B=[0,r)$, $r\in[0,1]$, and trivially for $B=[0,1]$, they coincide for all $B\in\mathscr{B}_0$.

6. Now we define $\Phi:(0,1)\times(0,1)\to(0,1)$ by

$$\begin{aligned}\Phi(x,y)&=x & x\in(0,1),\ y\in(0,\tfrac{1}{2})\\&=F(x,y) & x\in(0,1),\ y\in[\tfrac{1}{2},1).\end{aligned}$$

As F is continuous and as $F(x,\frac{1}{2})=x$ for all $x\in(0,1)$, Φ is continuous.

Furthermore, $\mu(B) = 0$ implies $\mu\{x \in (0, 1) : \Phi(x, y) \in B\} = 0$. For $y \in (0, \frac{1}{2}]$ this is trivial. For $y \in (\frac{1}{2}, 1)$ this follows immediately from the fact that $P_n \ll \mu$ and therefore $Q_y \ll \mu$.

Now we shall show that $y \to \mu\{x \in (0, 1) : \Phi(x, y) \in \bigcup_{\nu=1}^{\infty} A_\nu\}$ is discontinuous at $y = \frac{1}{2}$.

For $y \in (0, \frac{1}{2}]$ we have

$$\mu\{x \in (0, 1) : \Phi(x, y) \in \textstyle\bigcup_{\nu=1}^{\infty} A_\nu\} = \mu(\bigcup_{\nu=1}^{\infty} A_\nu) \leqq \frac{3}{4}.$$

For $y \in (\frac{1}{2}, 1)$ we have

$$\mu\{x \in (0, 1) : \Phi(x, y) \in \textstyle\bigcup_{\nu=1}^{\infty} A_\nu\} = Q_y(\bigcup_{\nu=1}^{\infty} A_\nu) = \frac{1}{2} + \frac{1}{2}\mu(\bigcup_{\nu=1}^{\infty} A_\nu),$$

as $P_n(\bigcup_{\nu=1}^{\infty} A_\nu) = 1$ for all $n \in \mathbb{N}$.

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REFERENCE

[1] Brown, L. D. (1964). Sufficient statistics in the case of independent random variables. *Ann. Math. Statist.* **35** 1456–1474.