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APPROXIMATE CONFIDENCE LIMITS FOR COMPLEX SYSTEMS WITH EXPONENTIAL COMPONENT LIVES

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The asymptotic distribution of the log-likelihood ratio is shown to provide a method of determining approximate confidence bounds for the reliability function of any coherent system when each component has an exponential life with unknown failure rate and component performance data are provided in the form: number of failures (minimum of one) and total operating time. Thus the method applies under all general types of censoring. This extends the results of the authors, *Ann. Math. Statist.* (1968), on confidence limits for coherent structures with binomial data on the component's reliability. Methods similar to those previously utilized are combined with some special properties of the exponential distribution to obtain the results.

0. Introduction. The problem of establishing confidence limits for the reliability of systems has now extended over a decade. The first results were those of Buehler [2] in 1957. The problem he considered was equivalent with finding exact confidence limits for two components in series with binomial data on each component. The construction of tables of exact bounds for up to three component series systems with binomial data of various sample sizes for the components was done by Lipow and Riley [7]. This work in two volumes was published by the Defense Documentation Center.

However, the bulk of the tables for even such small numbers of components made the use of simpler approximate confidence limits quite appealing. Madansky in [9] utilized the asymptotic distribution of the likelihood ratio and the usual practice of inverting a test to obtain a confidence bound, to yield approximate confidence bounds for series, parallel and series-parallel systems.

Lentner and Buehler [6] used the Lehmann–Scheffé theory of exponential families to find exact confidence limits for the specific case of components in series. Due to the difficulty in computing these limits, El Mawaziny and Buehler [4] give an approximation to this exact solution for the case where the sample sizes for all components are large and the failure law for each component is exponential.

Approximate confidence intervals for the reliability of any system (or structure) which can be represented by a monotone Boolean function of Bernoulli variates were obtained by Myhre and Saunders [11]. There the component failure data were the outcome of a number of Bernoulli trials for performance or nonperformance. That paper was an extension of the results of Madansky *loc. cit.*, for series systems and it depended upon the adequacy of the asymptotic distribution of the likelihood ratio. Here we will follow the same general lines of argument used in [11] to obtain approximate confidence intervals for system reliability from samples of component

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life lengths but the assumption of binomial data on performance of each component is replaced by the assumption of component life length being exponentially distributed.

This work also is closely related to the paper by Madansky and Olkin [10], but we concentrate here on the problems special to confidence intervals for reliability of systems, while they are concerned with other constraint parameters.

1. The general coherent system and the likelihood ratio. Let the number of components in a given system be m. The state of components, at any given time $t > 0$, is the random vector $\mathbf{Y}(t) = (Y_1(t), \cdots, Y_m(t))$ where $Y_j(t)$, a Bernoulli random variable, is the indicator of performance for the jth component at that time. It is assumed that the system has a unique representation as a nondecreasing Boolean function Φ, the functional value of which, $\Phi(\mathbf{Y}(t))$, is the indicator of the state of the system. We assume without loss of generality that each component of Φ is essential, [1], page 64.

The reliability of the jth component is $EY_j(t)$ and similarly the reliability of the system is $E\Phi(\mathbf{Y}(t))$ at any time $t > 0$.

If the life length of the jth component $X_j = \sup\{t > 0 \colon Y_j(t) = 1\}$ is exponentially distributed for $j = 1, \cdots, m$, then the density of X_j is

$$f_j(t) = \lambda_j \exp(-\lambda_j t) \qquad \text{for } t > 0. \tag{1.1}$$

Let $\lambda = (\lambda_1, \cdots, \lambda_m)$ be a point in the parameter space

$$\mathscr{H} = \{(\lambda_1, \cdots, \lambda_m) \colon 0 < \lambda_j < \infty, j = 1, \cdots, m\}.$$

Since we are primarily interested in the system reliability at a prescribed time (in certain cases called the mission length) we can without loss of generality take it to be unity. We now set $h(\lambda) = E\Phi[Y(1)]$. One sees that

$$h(\lambda) = \textstyle\sum_y \Phi(y) \prod_{j=1}^m \{y_j \exp(-\lambda_j) + (1-y_j)[1-\exp(-\lambda_j)]\} \tag{1.2}$$

where $y = (y_1, \cdots, y_m)$ is a vertex of the m-dimensional unit hypercube and the summation is over all such vertices.

Suppose that $n_j \geqq 1$ identical replications of the jth component are tested for $j = 1, \cdots, m$. Consequently, observations are made on independent random variables identically distributed as X_j, which by our convention are life lengths expressed as multiples (possibly less than one) of the given mission length.

If we let t_j denote the total observed test time for the jth component and $s_j (1 \leqq s_j \leqq n_j)$ denote the observed number of failures during the time t_j, then it is known that the log-likelihood, except for some constant not depending upon λ, is merely

$$L^*(\lambda) = \textstyle\sum_{j=1}^m [s_j \ln \lambda_j - \lambda_j t_j].$$

The s_j and t_j may both be random and censoring may also be random independent of the life distributions. This fact for exponential distributions was first given by

Herd [5] and Sampford [13]. Also in this connection, see the discussion by Cohen in [3]. The logarithm of the likelihood ratio, say $L(r)$, is given by

$$L(r) = \sup_{\{\lambda:\, h(\lambda)=r\}} L^*(\lambda) - \sup_{\lambda \in \mathscr{H}} L^*(\lambda).$$

We now follow the usual method of inverting a test, in this case the likelihood ratio test, in order to obtain a confidence interval.

Proceeding as we have done in [11] we utilize Wilks' theorem [14] on the asymptotic behavior of the logarithm of the likelihood ratio to obtain a confidence set of level γ for the system reliability at the mission length. This is $\{r\colon -2L(r) \leqq \chi_\gamma^2(1)\}$ where $\chi_\gamma^2(1)$ is the γth quantile of the Chi-square distribution with one degree of freedom.

Since the maximum likelihood estimate of λ_j is $\hat{\lambda}_j = (s_j/t_j)$ for $j = 1, \cdots, m$ we see

$$L^*(\hat{\lambda}) = \sup_{\lambda \in \mathscr{H}} L^*(\lambda) = \textstyle\sum_{j=1}^m [s_j \ln s_j - s_j(\ln t_j + 1)].$$

To maximize $L^*(\lambda)$ subject to the restriction $h(\lambda) = r$, we proceed as in [11] and use a Lagrange multiplier δ, take partial derivatives, and equate to zero. This yields the system of equations

$$\frac{s_j}{\lambda_j} - t_j = \delta \partial_j h(\lambda) \qquad (j = 1, \cdots, m) \tag{1.3}$$

where $\partial_j h$ is the partial derivative of h with respect to its jth argument. The existence of $\partial_j h$ follows from the definition of h. For given δ denote the vector solution of (1.3) by $\tilde{\lambda}(\delta)$, assuming presently that it exists and is unique within $\mathscr{H}$, which we shall later prove in Theorem 3. Note that $\tilde{\lambda}(0) = \hat{\lambda}$.

Since Lagrange multipliers are being used, the confidence set may conveniently be written in terms of the multiplier δ rather than in terms of the reliability r. If we define

$$\Lambda(\delta) = L^*[\tilde{\lambda}(\delta)] - L^*(\hat{\lambda})$$

for those values of δ for which $\tilde{\lambda}(\delta)$ exists, then we can obtain

THEOREM 1. *If $h\tilde{\lambda}(\cdot)$ is monotone decreasing across an interval $[\delta^-, \delta^+]$ where $\delta^- < 0 < \delta^+$ are two values of δ for which $\Lambda(\delta) = -\frac{1}{2}\chi_\gamma^2(1)$ then*

$$\{r\colon h\tilde{\lambda}(\delta^-) > r > h\tilde{\lambda}(\delta^+)\} = \{r\colon -2L(r) \leqq \chi_\gamma^2(1)\}. \tag{1.4}$$

The proof is analogous to the corresponding result of Theorem (1.7) given in [11] and so will not be given here.

To complete this argument we must show that there exists values of δ for which $\tilde{\lambda}(\delta)$ exists uniquely, and hopefully can be easily found, and also that there exists values of δ in an interval about zero for which $h\tilde{\lambda}(\cdot)$ is decreasing. We turn to these tasks now.

2. The contractive operator and the iterative procedure. The transformation A from $\mathscr{H}$ into $\mathscr{H}$ for fixed δ we define by setting its jth component for $j = 1, \cdots, m$

$$A_j(\lambda; \delta) = \frac{s_j}{t_j + \delta \partial_j h(\lambda)}.$$

This is suggested by solving (1.3) as if $\partial_j h$ were constant. It is clear that $A_j(\lambda;\delta)$ is continuous in δ, where defined. Since $\partial_j h(\lambda) < 0$, see equation (1.7) below, $A_j(\lambda;\cdot)$ is an increasing function of δ for $t_j > -\delta\,\partial_j h(\lambda)$. We will now show that A is a contractive map of the complete metric space $\mathscr{H}$ into itself and hence has a unique fixed point, $\tilde{\lambda}(\delta)$.

For a given structure Φ define the *criticality* of the jth component in the structure by

$$c_j = \textstyle\sum_{y_i,\, i\neq j} \Phi(y\,|\,j:1) - \Phi(y\,|\,j:0)$$

where $(y\,|\,j:x) = (y_1, \cdots, y_{j-1}, x, y_{j+1}, \cdots, y_m)$ for $x = 0, 1$.

We now have

THEOREM 2. *For each Φ (or h) and all δ such that*

$$\min(t_1, \cdots, t_m) > \delta > 0 \quad \text{and} \quad \sum_{j=1}^{m} \frac{s_j c_j}{(t_j-\delta)^2} < \frac{1}{\delta} \tag{2.1}$$

$$\delta < 0 \quad \textit{and} \quad \sum_{j=1}^{m} \frac{s_j c_j}{t_j^2} < \frac{1}{|\delta|}, \tag{2.2}$$

the transformation $A(\cdot\,;\delta)$ is a contractive map of the complete metric space $(\mathscr{H}, d)$ into itself where $d(\lambda,\mu) = 1/m\sum_{j=1}^{m} |\lambda_j - \mu_j|$ for $\lambda, \mu \in \mathscr{H}$.

PROOF. It must be shown that there exists $p \in [0, 1)$ such that $d[A(\lambda), A(\mu)] \leqq pd(\lambda, \mu)$ for all $\lambda, \mu \in \mathscr{H}$. We shall omit showing the dependence of both A and p upon δ in order to simplify the notation, thus

$$A_j(\lambda) - A_j(\mu) = \frac{s_j \delta[\partial_j h(\mu) - \partial_j h(\lambda)]}{[t_j + \delta\partial_j h(\lambda)][t_j + \delta\partial_j h(\mu)]}.$$

Following in turn the steps of the corresponding Theorem (2.4) proved in [11], which carry over directly to this case we obtain the inequalities

$$|\partial_j h(\mu) - \partial_j h(\lambda)| \leqq mc_j d(\lambda,\mu)$$

and

$$d(A(\lambda), A(\mu)) \leqq \sum_{j=1}^{m} \frac{s_j c_j \delta d(\lambda,\mu)}{[t_j + \delta\partial_j h(\lambda)][t_j + \delta\partial_j h(\mu)]}.$$

Now from known properties of coherent systems, see [1], each component of which is essential it follows that

$$\partial_j h(\lambda) = e^{-\lambda_j}[h(\lambda\,|\,j:0) - h(\lambda\,|\,j:1)] < 0. \tag{2.3}$$

It also follows from (2.3) that $\partial_j h(\lambda) \geqq -e^{-\lambda_j} \geqq -1$. Thus it is sufficient to require that (2.1) hold for $\delta > 0$ and (2.2) hold for $\delta < 0$. $\square$

For any $\lambda^0 \in \mathscr{H}$ we define the sequence $\lambda^n(\delta) = A(\lambda^{n-1}(\delta); \delta)$ $n = 1, 2, \cdots$ where $\lambda^0(\delta) = \lambda^0$.

THEOREM 3. *For every δ in the neighborhood of zero defined by the inequalities* (2.1) *and* (2.2), *a unique solution to the system of equations* (1.3) *exists, call it* $\tilde{\lambda}(\delta)$. *It can be found for any initial point* $\lambda^0 \in \mathscr{H}$ *as* $\lim_{n\to\infty} \lambda^n(\delta) = \tilde{\lambda}(\delta)$.

PROOF. That $A(\cdot\,;\delta)$ for δ within the prescribed neighborhood of zero has a unique fixed point

$$\tilde{\lambda}(\delta) = \lim \lambda^n(\delta) = A(\tilde{\lambda}(\delta):\delta)$$

follows from the known behavior of contractive maps, e.g. see [8], page 27. □

Thus we have established the first claim which was made, namely that $\tilde{\lambda}(\delta)$ exists uniquely. The proof of Theorem 1 was based on the fact that the existence of $\tilde{\lambda}_j(\delta)$ implies the existence of $\tilde{\lambda}_j'(\delta)$. We now prove a stronger result which will be used later.

THEOREM 4. *The function* $\tilde{\lambda}(\cdot)$ *is a continuously differentiable function within the neighborhood of zero prescribed by* (2.1) *and* (2.2).

PROOF. Fix δ within the prescribed neighborhood and let B be the vector valued function with its jth coordinate defined by

$$B_j(\lambda:\delta) = \delta \partial_j h(\lambda) + t_j - \frac{s_j}{\lambda_j}.$$

By Theorem 3 the equation $B(\lambda:\delta) = 0$ has a unique solution, call it $\tilde{\lambda}$. Thus the Jacobian is not zero, i.e. $\det \partial_i B_j(\tilde{\lambda};\delta) \neq 0$. From the implicit function theorem the continuous differentiability of the function B is inherited by $\tilde{\lambda}$. Thus if $\tilde{\lambda}$ exists it is continuously differentiable. □

In view of Theorem 3 and Theorem 4 it remains only to show that $h\tilde{\lambda}(\delta)$ is a decreasing function of δ. We then have the result that for any coherent structure Φ an approximate confidence interval for the reliability function h is given by (1.4) in Theorem 1.

THEOREM 5. *For any coherent structure* Φ *with reliability function* h *and any failure data such that* $s_j \geqq 1$, *there exists a neighborhood of zero in* δ *across which* $h\tilde{\lambda}(\delta)$ *is decreasing in* δ.

PROOF. Take the derivative of both sides of (1.3) with respect to δ, primes denoting such differentiation. We obtain for $j = 1, \cdots, m$

$$\frac{-s_j}{(\tilde{\lambda}_j)^2} \cdot \tilde{\lambda}_j' = \partial_j h\tilde{\lambda} + \delta \sum_{i=1}^{m} \partial_{ij} h(\tilde{\lambda}) \cdot \tilde{\lambda}_i' \tag{2.4}$$

where we have omitted the argument δ. Multiply by $\tilde{\lambda}_j'$ on both sides of (2.4) and sum to obtain

$$-\frac{d}{d\delta} h[\tilde{\lambda}(\delta)] = \sum_{j=1}^{m} \frac{s_j}{(\tilde{\lambda}_j)^2} (\tilde{\lambda}_j')^2 + \delta \sum_{i=1}^{m} \sum_{j=1}^{m} \partial_{ij} h(\tilde{\lambda}) \tilde{\lambda}_i' \tilde{\lambda}_j'. \tag{2.5}$$

For $\delta = 0$, the left side of (2.5) is positive and from its continuity it follows that there exists a neighborhood of δ about zero such that $h\tilde{\lambda}(\delta)$ is a decreasing function of δ. $\square$

3. Concluding remarks. An alternate method that may suggest itself for construction of confidence limits uses the theory of asymptotic normality of maximum likelihood estimates. Such intervals would be of the form $h(\hat{\lambda}) \pm c\hat{\sigma}$ where $\hat{\sigma}^2$ is the maximum likelihood estimate of the variance of $h(\hat{\lambda})$. This method seems deceptively simple and direct. We suggest the likelihood ratio method, however, for two reasons.

Firstly, in many of the current applications, e.g., space launch vehicles, the reliability is calculated at such short times relative to the test data that the maximum likelihood upper bounds, because of their symmetry, may exceed unity. Such behavior has been observed in [12]. This tends to reduce our assurance in its accuracy. The likelihood ratio bounds are not symmetric when the reliability is close to unity and do not suffer from this defect.

Secondly, we feel that the computation necessary to use the maximum likelihood method for general systems of large order, as defined in (1.2), presents greater computational difficulty than does the likelihood ratio method in many instances. For example, if one can use the fact that the sample size for each component is large and equal, say to n, then the assumption $h(\hat{\lambda})$ is normal with mean $h(\lambda)$ and variance $1/n \sum_{i=1}^{m} \sigma_i^2 [\partial_i h(\lambda)]^2$, where σ_i^2/n is the asymptotic variance of $\hat{\lambda}_i$, might well be preferable.

However, the usual case is that the sample sizes for each component are unequal and small, while the order of the system is large. In this case $Eh(\hat{\lambda})$ may not be close to $h(\lambda)$, the asymptotic variance is not applicable, and computing the exact variance involving $Eh^2(\hat{\lambda})$ would require machine computation in the form of Bessel functions and would still contain the unknown λ_i and other parameters depending upon random censoring.

Although the likelihood ratio procedure does require some analysis, it seems to result in a method which is practical, even for large complex systems under general censoring conditions, since iteration by contractive operators converges so rapidly.

Lastly, we have continually used the phrase, "number of failures (minimum of one)," because contrary to the supposition that the asymptotic results require the number of failures for each component to be large, we conjecture that under certain conditions it is sufficient to have only the order of the system large for the asymptotic theory to apply. Although the investigation of the conditions necessary for asymptotic convergence is not complete in either case, some results of this nature are known at present and will be reported subsequently.

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