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ABSOLUTE CONTINUITY AND RADON-NIKODYM DERIVATIVES FOR CERTAIN MEASURES RELATIVE TO WIENER MEASURE[1]

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We give sufficient conditions for the absolute continuity relative to Wiener measure, P_w, of a measure, P_y, induced by the sum, $y(t)$, of a Wiener process and a non-anticipating and differentiable "signal" process. When the signal process is a measurable function of y, we also give expressions for dP_y/dP_w and dP_w/dP_y.

1. Introduction. In this note we shall present some results on the absolute continuity with respect to Wiener measure of certain measures induced by the sum of a "signal" process and a Wiener process. Our interest in these measures arises from certain signal detection problems [7], [8].

To describe the measures involved it will be convenient to begin with a probability space $(\Omega, \mathscr{B}, \mathscr{P})$ with three random functions on it related by the equation

$$y(t,\omega) = \int_0^t z(s,\omega)\,ds + w(t,\omega), \qquad 0 \leqq t \leqq T. \tag{1}$$

The functions $y(\cdot,\cdot)$, $z(\cdot,\cdot)$ and $w(\cdot,\cdot)$ will be assumed to be each jointly measurable processes. Furthermore, we shall denote by $\mathscr{B}_t$ a monotone increasing family of subsigma-fields of $\mathscr{B}$ such that for all $t \in [0, T]$, the paths $z_0{}^t = \{z(s,\omega), 0 \leqq s \leqq t\}$ are measurable with respect to $\mathscr{B}_t$. We further assume that

(i) $w(t,\omega)$ is a Wiener process with

$$E[w(t,\omega)] = 0, \qquad E[w^2(t,\omega)] = t. \tag{2}$$

(ii) $z(t,\omega)$ is a not necessarily Gaussian process such that

$$w(t,\omega) - w(s,\omega) \perp\!\!\!\perp \mathscr{B}_u, \qquad u \leqq s \leqq t \leqq T. \tag{3}$$

where, following K. Itô, the symbol $\perp\!\!\!\perp$ is used to denote independence. All sigma-fields will be assumed to be complete relative to $\mathscr{P}$. An important family of sigma-fields is the one generated by the random variables $\{y(t,\omega)\}$,

$$\mathscr{F}_t \equiv \sigma\{y(s,\omega),\ 0 \leqq s \leqq t\}, \qquad 0 \leqq t \leqq T.$$

Let $\mathscr{P}_y$ denote the restriction of the measure $\mathscr{P}$ to the field $\mathscr{F}_T$. The measure induced by $\mathscr{P}_y$ on the space of continuous functions (in the $[0, T]$ interval) will be denoted by P_y. P_w will denote the Wiener measure on the space of continuous functions.

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Our proofs in this paper will rely heavily on a fundamental theorem of Girsanov [3, Theorem 1]. For convenience, this theorem is restated (and somewhat amplified) in the Appendix. Another result that we have often used below is that a stochastic integral can be transformed to a Wiener process by means of a random time substitution (cf. McKean [4, section 2.5]). With these preliminary remarks, we can begin to state our results.

THEOREM 1. *If*

$$\int_0^T z^2(t,\omega)\,dt < \infty \quad \text{a.s.} \quad \mathscr{P}, \tag{4}$$

then $P_y \ll P_w$, i.e. P_y is absolutely continuous with respect to P_w.

Our proof of this result is based on the theorem of Girsanov [restated in Appendix I] and on two simple lemmas (Lemma 1 and Lemma 2), which may have some independent interest. From these lemmas, it in fact follows easily (Proposition 1 and Proposition 2) that the measures P_y and P_w are mutually absolutely continuous ($P_y \sim P_w$) if (a) $z(\cdot,\cdot)$ is completely independent of $w(\cdot,\cdot)$ or (b) $z(\cdot,\cdot)$ is uniformly bounded in amplitude,[4] i.e.

$$\sup_{0\leqq t\leqq T} |z(t,\omega)| \leqq k < \infty \quad \text{a.s.} \quad \mathscr{P} \tag{5}$$

or, more generally, (c) $z(\cdot,\cdot)$ is uniformly bounded in energy, i.e.

$$\int_0^T |z(t,\omega)|^2\,dt \leqq K < \infty \quad \text{a.s.} \quad \mathscr{P}. \tag{6}$$

THEOREM 2. *If, in addition to the assumptions of Theorem* 1, *$z(t,\omega)$ is measurable with respect to $\mathscr{F}_t$, say*[5]

$$z(t,\omega) = \phi(t,\omega) \tag{7}$$

then we can write

$$\frac{dP_y}{dP_w} = \exp\left\{\int_0^T \phi(t,\omega)\,dy(t,\omega) - \tfrac{1}{2}\int_0^T \phi^2(t,\omega)\,dt\right\}, \qquad on \quad A \tag{8}$$

$$= 0, \qquad on \quad \bar{A}$$

where A is the set in function space defined by

$$A = \{y_0^T : \int_0^T \phi^2(t,\omega)\,dt < \infty\}.$$

REMARK 1. By the measurability assumption on z we may write $z(t,\omega) = \tilde{\phi}(y_0^t,\omega)$ and (1) becomes a functional equation in y_0^T:

$$y(t,\omega) = \int_0^t \tilde{\phi}(y_0^s,\omega)\,ds + w(t). \tag{9}$$

The statement of Theorem 2 assumes the existence of a solution to this functional equation. Necessary and sufficient conditions for the existence of a solution to (9)

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[4] This result is due to Girsanov ([4] Lemma 1) and Dynkin ([2] Theorem 7.3).

[5] ϕ will be used for z whenever $z(t,\cdot)$ is $\mathscr{F}_t$ measurable.

are unknown (even in the special case where (9) is a stochastic differential equation). However, various sufficient conditions are known for the general case ([5]) and for the particular case where $y(t)$ is Markovian ([10], page 121, also [9], page 77). Note that Theorem 2 does not require the solution of (9) to be unique and if (9) has more than one solution, Theorem 2 may be applied separately to each of the solutions.

REMARK 2. Theorem 2, taken together with the innovations theorem of [7] and [3], can be used to give an alternative proof of the general L.R. formula of [7]–[8]. Conversely, of course, Theorem 2 could have been inferred from the results of [8].

Our final result is Theorem 3, which states that if, as in Theorem 2

$$y(t,\omega) = \int_0^t \tilde{\phi}(y_0{}^s, \omega)\, ds + w(t,\omega) \tag{9}$$

then (note the argument of $\tilde{\phi}$)

$$\int_0^T \tilde{\phi}^2(w_0{}^t, \omega)\, dt < \infty \quad \text{a.s.} \quad \Rightarrow P_w \ll P_y,$$

and

$$\begin{aligned} \frac{dP_w}{dP_y} &= \exp\left\{-\int_0^T \phi(t,\omega)\, dy(t,\omega) + \tfrac{1}{2}\int_0^T \phi^2(t,\omega)\, dt\right\}, && \text{on } A \\ &= 0, && \text{on } \bar{A} \end{aligned}$$

where A is as defined above in Theorem 1.

Finally, we may remark that the conditions on $\phi(\cdot,\cdot)$ in Theorem 2 and Theorem 3 can also be shown to be necessary but the proofs seem to demand a different collection of ideas and therefore will not be given here (however, see [8]).

Relations to previous work. Our results on absolute continuity were partly motivated by a preprint of a paper by Kadota and Shepp [6] in which they gave a different proof of Theorem 1 and Proposition 1 and also establish the discrete-time analogues of the absolute continuity parts of Theorem 2 and Theorem 3. The result of Theorem 1 under the stronger assumption $E\int z^2\, dt < \infty$ had been conjectured by one of us in [7], where it has been suggested that the entropy criteria of Hájek, Perez and others could be used for a proof. In fact, the proof of Kadota and Shepp starts with discrete-time approximations and shows via the entropy that $P_y < P_w$ if $E\int z^2(t)\, dt < \infty$; a "truncation" argument is then used to obtain the weaker condition $\int z^2(t)\, dt < \infty$ a.s. Our proof avoids the discrete-time approximation and uses instead a theorem of Girsanov (see Appendix) and the truncation argument is reused to directly obtain the result under the weaker condition. We should note also that under the assumption that $z(\cdot)$ was uniformly bounded, Girsanov [4] implicitly proved not only absolute continuity but also equivalence (cf. our Proposition 1). Girsanov also remarked ([4] page 296) that it would be important to be able to prove his main theorem, as described in our Appendix I, with the condition (A.3) replaced by the condition that $z(\cdot)$ be square-integrable almost surely. But it is known that this is impossible (cf. McKean [9] page 67): the difficulty is that the a.s. square-integrability of $z(\cdot)$ suffices for absolute continuity (as shown by Theorem 1) but not for equivalence.

2. Derivations. For the convenience of the reader we shall first restate our basic assumptions.

Let $(\Omega, \mathscr{B}, \mathscr{P})$ be a probability space, $w(t, \omega)$ a Wiener process on $(\Omega, \mathscr{B}, \mathscr{P})$, $z(t, \omega)$ a random function on $(\Omega, \mathscr{B}, \mathscr{P})$, both measurable on $[0, T] \times \mathscr{B}$. Let $\mathscr{B}_t$ be a monotone increasing family of sub-σ-fields of $\mathscr{B}$ such that for all $t \in [0, T]$, the paths $z_0{}^t$, $w_0{}^t$ are measurable on $\mathscr{B}_t$ and $\mathscr{B}_t$ is independent of future increments $\{w(t_1)-w(t), w(t_2)-w(t), \cdots, t_i \geqq t\}$ of $w(t)$. We also assume that

$$\textstyle\int_0^T z^2(t, \omega)\, dt < \infty \quad \text{a.s.} \quad \mathscr{P}.$$

These assumptions will be kept throughout this note. Unless otherwise specified all a.s. statements will refer to $\mathscr{P}$. The σ-fields $\mathscr{B}_t$, $\mathscr{B}$ will be assumed to be complete relative to $\mathscr{P}$.

In the proofs of Theorem 1 and Theorem 2, we first "truncate" the process $z(\cdot)$ so that its energy is a.s. uniformly bounded. This truncation enables us, via the result in Lemma 1 below, to apply a theorem of Girsanov; the reader will find it helpful before proceeding to review the presentation of this theorem in the Appendix.

LEMMA 1. *If for some* K

$$\textstyle\int_0^T z^2(t, \omega)\, dt < K \quad \text{a.s.} \quad \mathscr{P}.$$

then

$$E\{\exp[\textstyle\int_0^T z(t, \omega)\, dw(t, \omega) - \frac{1}{2}\int_0^T z^2(t, \omega)\, ds]\} = 1. \tag{10}$$

PROOF: This result is well known for the case where $\sup_{0<t<T} |z(t, \omega)| \leqq c$ (Dynkin [2] Theorem 7.3; Girsanov [4] Lemma 1). We shall use a random time substitution to reduce the present problem to this case. For this, let

$$\tau(t, \omega) \equiv \textstyle\int_0^t z^2(s, \omega)\, ds \tag{11}$$

and define

$$\begin{aligned} \tilde{w}(\tau, \omega) &= \textstyle\int_0^t z(s, \omega)\, dw(s, \omega) \qquad \text{if} \quad \tau \leqq \int_0^T z^2(s, \omega)\, ds; \\ &= \textstyle\int_0^T z(s, \omega)\, dw(s, \omega) \qquad \text{if} \quad \tau > \int_0^T z^2(s, \omega)\, ds. \end{aligned}$$

Then (McKean [9] Section 2.5), $\tilde{w}(\tau, \omega)$ is a Wiener process (Brownian motion) stopped at the time

$$\tau_f(\omega) \equiv \textstyle\int_0^T z^2(s, \omega)\, ds \leqq K\cdot \tag{12}$$

Let $\tilde{\tilde{w}}(\cdot, \omega)$ be an appropriate extension of $\tilde{w}(\cdot, \omega)$ to a nonstopped Wiener process. With this time change, we can write

$$\begin{aligned} &\exp[\textstyle\int_0^T z(s, \omega)\, dw(s, \omega) - \frac{1}{2}\int_0^T z^2(s, \omega)\, ds] \\ &\quad = \exp[\textstyle\int_0^K \chi_f(\tau, \omega)\, d\tilde{\tilde{w}}(\tau, \omega) - \frac{1}{2}\int_0^K \chi_f{}^2(\tau, \omega)\, d\tau] \end{aligned} \tag{13}$$

where

$$\chi_f(\tau,\omega) = 1 \quad \text{if} \quad \tau \leqq \tau_f;$$
$$= 0 \quad \text{otherwise.}$$

Since $\sup|\chi_f(\tau, \omega)| \leqq 1$, our desired result now follows from the previously quoted result for uniformly bounded $z(\cdot)$.

LEMMA 2. *If the process $z(\cdot)$ is completely independent of $w(\cdot)$, then* $\int_0^T z^2(t, \omega)\, dt < \infty$ a.s. $\mathscr{P}$ *implies that*

$$E\{\exp[\int_0^T z(t,\omega)\, dw(t,\omega) - \tfrac{1}{2}\int_0^T z^2(t,\omega)\, dt]\} = 1.$$

PROOF. We note first that if $z(\cdot,\omega)$ were deterministic, then our result follows easily from Lemma 1. (It is also well known in detection theory.) Now we shall reduce the case of random $z(\cdot)$, independent of $w(\cdot)$, to the known $z(\cdot)$ case by conditioning.

Let $\mathscr{B}_z$ be the sub-σ-field of $\mathscr{B}$ induced by $z_0{}^T$ and let

$$\alpha(\omega) = \exp \int_0^T z(t,\omega)\, dw(t,\omega) - \tfrac{1}{2}\int_0^T z^2(t,\omega)\, dt.$$

It follows directly from Fubini's theorem and the definition of conditional expectations that at the point $z_0{}^T = x_0{}^T$

$$E\{\alpha(\omega) \mid \mathscr{B}_z\}(x_0{}^T) = E\{\exp \int_0^T x(t)\, dw(t,\omega) - \tfrac{1}{2}\int_0^T x^2(t)\, dt\} = 1$$

by the deterministic $z(\cdot)$ result just quoted. Therefore

$$E\{\alpha(\omega)\} = E\{E\{\alpha(\omega) \mid \mathscr{B}_z\}\} = E\{1\} = 1.$$

PROPOSITION 1. *Let*

$$y_N(t,\omega) \equiv \int_0^t z_N(s,\omega)\, ds + w(t,\omega)$$

where

$$\int_0^T z_N{}^2(t,\omega)\, dt \leqq N \quad \text{a.s.} \quad \mathscr{P}$$

and let P_{y_N} denote the measure induced by $y_N(\cdot,\cdot)$ on the space of continuous functions. Then $P_{y_N} \sim P_w$, i.e., $P_{y_N} \ll P_w$ and $P_{w_N} \ll P_y$.

PROOF. Let

$$\beta(\omega) \equiv \exp[-\int_0^T z_N(s,\omega)\, dw(s,\omega) - \tfrac{1}{2}\int_0^T z_N{}^2(s,\omega)\, ds]$$

and

$$d\mathscr{P}_0(\omega) \equiv \beta(\omega) \cdot d\mathscr{P}(\omega).$$

Since by Lemma 1, $E[\beta(\omega)] = 1$, $\mathscr{P}_0(\omega)$ is a probability measure and we can apply Girsanov's theorem (set $\Psi = -z_N = -\phi$ and $\Phi = 1$ in the form shown in the Appendix) to obtain the result that

$$w(t,\omega) + \int_0^T z_N(s,\omega)\, ds \equiv y_N(t,\omega)$$

is Wiener under $\mathscr{P}_0$. Furthermore, part (c) of Girsanov's theorem shows that $\mathscr{P}_0 \sim \mathscr{P}$. But the restriction of $\mathscr{P}_0$ to $\mathscr{F}_T{}^N$ (the σ-field induced by $y_N(s)$, $0 \leqq s \leqq T$) is just the Wiener measure on $(\Omega, \mathscr{F}_T{}^N)$ and, by definition, $\mathscr{P}_{y_N}$ is the restriction of $\mathscr{P}$ to $(\Omega, \mathscr{F}_T{}^N)$. Consequently the measures that these restrictions induce on function space are P_w and P_{y_N} respectively. Therefore it follows immediately that $\mathscr{P}_0 \sim \mathscr{P}$ implies $P_{y_N} \sim P_w$.

PROPOSITION 2. *Let* $\{z(t, \omega)\}$ *be statistically independent of* $\{w(t, \omega)\}$. *Then* $P_y \sim P_w$ *if* $\int_0^T z^2(t, \omega)\, dt < \infty$ a.s. $\mathscr{P}$.

PROOF. This follows immediately from Lemma 2 and Girsanov's theorem. (Note that no truncation, as in Proposition 1, is necessary.)

With these preliminary results, we are ready to prove

THEOREM 1. *Let*

$$y(t, \omega) = \int_0^t z(s, \omega)\, ds + w(t, \omega).$$

Then,

$$\int_0^T z^2(s, \omega)\, ds < \infty \quad \text{a.s.} \quad \mathscr{P}.$$

implies that $P_y \ll P_w$.

PROOF. We begin with a "stopped" or "truncated" process $y_N(t, \omega)$ defined by

$$y_N(t, \omega) \equiv \int_0^t z_N(s, \omega)\, ds + w(t)$$

where

$$\begin{aligned} z_N(t, \omega) &\equiv z(t, \omega), \qquad && \text{if } \int_0^t z^2(s, \omega)\, ds \leqq N; \\ &= 0, && \text{otherwise.} \end{aligned}$$

Let $\mathscr{P}_{y_N}$ denote the restriction to $(\Omega, \mathscr{F}_T{}^N)$ of $\mathscr{P}$. Then by Proposition 1 we have $P_{y_N} \sim P_w$. From this fact we shall now show, by contradiction, that under our hypothesis on $z(\cdot)$, we must have $P_y \ll P_w$. For if not, there must be a set $\tilde{B}$ in function space such that $P_w(\tilde{B}) = 0$ but $P_y(\tilde{B}) = \varepsilon > 0$. Let B be the set in the original probability space Ω that is the inverse image under $y_N(\cdot, \cdot)$ of the set $\tilde{B}$ in function space. The existence of $\tilde{B}$ will lead to a contradiction. To establish this, first note that since $\int_0^T z^2(t, \omega)\, dt < \infty$ a.s. $\mathscr{P}$, there must exist a number N so large that the set

$$C \equiv \{\omega : \int_0^T z^2(t, \omega)\, dt \leqq N\}$$

satisfies $\mathscr{P}(B \cap C) \geqq \frac{1}{2}\varepsilon$.

However, we note that on the set B the processes $z(\cdot)$ and $y(\cdot)$ coincide with the truncated processes $z_N(\cdot)$ and $y_N(\cdot)$. Therefore on the set $B \cap C$, it is true since $P_0(B) = P_w(\tilde{B}) = 0$ that

$$0 = \mathscr{P}_0(B \cap C) = \mathscr{P}(B \cap C) < \tfrac{1}{2}\varepsilon.$$

This is the desired contradiction.

THEOREM 2. *If, in addition to the assumptions of Theorem* 1, $\{z(t, \omega)\}$ *is measurable with respect to* $\{\mathscr{F}_t\}$, *say* $z(t, \omega) = \phi(t, \omega)$, *then we can write*

$$\frac{dP_y}{dP_w} = \exp\left\{\int_0^T \phi(t,\omega)\,dy(t,\omega) - \tfrac{1}{2}\int_0^T \phi^2(t,\omega)\,dt\right\} \qquad on \quad A \tag{14a}$$

$$= 0 \qquad on \quad \bar{A}$$

where A is the set in function space defined by

$$A = \{y_0{}^T : \textstyle\int_0^T \phi^2(t,\omega)\,dt < \infty\}. \tag{15}$$

The set A satisfies $P_y(A) = 1$, $P_w(\bar{A}) \geqq 0$. (Note: By our assumption on $z(t, \omega)$, $\phi(t, \omega)$ is a function of $y_0{}^t$ and therefore the right-hand side of (14a) and (15) are well defined on function space.)

PROOF. We shall first obtain the Radon–Nikodym derivative for a suitably stopped version of the problem and then use the martingale convergence theorem to deduce (14). Let

$$\tau(\omega) = \inf\{t(0 \leqq t \leqq T) : \textstyle\int_0^t \phi^2(s,\omega)\,ds \geqq K\} \tag{16}$$

and let

$$\chi_\tau(s,\omega) = 1, \qquad s \leqq \tau$$

$$= 0, \qquad s > \tau.$$

Now define $y_\tau(t, \omega)$ by the relation

$$y_\tau(t,\omega) = \textstyle\int_0^{t\wedge\tau} z(s,\omega)\,ds + \int_0^{t\wedge\tau} dw(s,\omega)$$

$$= \textstyle\int_0^t \chi_\tau(s,\omega)z(s,\omega)\,ds + \int_0^t \chi_\tau(s,\omega)\,dw(s,\omega).$$

[Note that the definition of y_τ differs from that of y_N in the proof of Theorem 1, since here $w(\cdot)$ is also stopped. We choose this stopping rule so as to obtain a nested (with τ) family of sigma-fields generated by the stopped variables $\{y_\tau(t, \omega)\}$.]

We now apply Girsanov's theorem to the stopped process y_τ. By setting

$$\psi(t,\omega) = \chi_\tau(t,\omega)z(t,\omega) = -\phi(t,\omega), \qquad \Phi(t,\omega) = \chi_\tau(t,\omega)$$

in Girsanov's theorem as stated in the Appendix, we can deduce that, under a measure $\mathscr{P}_0$ as defined in the theorem, $y_\tau(t, \omega)$ is a stopped Wiener process $y_\tau(t, \omega) = 0 + \tilde{w}(t, \omega)$ and

$$\alpha(\omega)^{-1} = \frac{d\mathscr{P}}{d\mathscr{P}_0} = \exp\left\{\int_0^{T\wedge\tau} \phi(t,\omega)\,dw(t,\omega) + \tfrac{1}{2}\int_0^{T\wedge\tau} \phi^2(t,\omega)\,dt\right\} \tag{17a}$$

$$= \exp\left\{\int_0^{T\wedge\tau} \phi(t,\omega)\,dy(t,\omega) - \tfrac{1}{2}\int_0^{T\wedge\tau} \phi^2(t,\omega)\,dt\right\}. \tag{17b}$$

Now, let $\mathscr{P}_{y_\tau}$ = the restriction of $\mathscr{P}$ to $\mathscr{F}_\tau = \sigma\{y_\tau(s, \omega), 0 \leqq s \leqq T\}$ and $\mathscr{P}_{W_\tau}$ = the restriction of $\mathscr{P}_0$ to $\mathscr{F}_\tau$, and let P_{y_τ} and P_{W_τ} denote the measures induced on the space of continuous functions by $\mathscr{P}_{y_\tau}$ and $\mathscr{P}_{W_\tau}$ respectively.

$$\frac{dP_{y_\tau}}{dP_{W_\tau}} = E\left\{\frac{d\mathscr{P}}{d\mathscr{P}_0}\middle|\mathscr{F}_\tau\right\} = \exp\left\{\int_0^{T\wedge\tau} \phi(t, \omega)\, dy(t, \omega) - \tfrac{1}{2}\int_0^{T\wedge\tau} \phi^2(t, \omega)\, dt\right\}$$

since $d\mathscr{P}/d\mathscr{P}_0$ as given by (16) is already measurable with respect to $\mathscr{F}_\tau$ and by our assumption on ϕ, the right-hand side of (17b) is defined on function space. Finally, since the Borel fields $\mathscr{F}_t$ are nested and $\lim_{K\to\infty}\mathscr{F}_\tau = \mathscr{F}_T$ (recall the definition of K in (16), and by Theorem 1, $P_y \ll P_w$, we can apply a martingale convergence theorem (Doob [1] Theorem e.w., page 331] and Appendix, page 632) to obtain $dP_y/dP_w = \lim_{K\to\infty} dP_{y_\tau}/dP_{w_\tau}$. But some more care is needed before the exponential formula (14) can be written down.

Let A be as defined by (15). By hypothesis, $P_y(A) = 1$, and on A, under P_y we will have

(18a) $$\lim_{K\to\infty} \textstyle\int_0^{T\wedge\tau} \phi(t, \omega)\, dy(t, \omega) = \int_0^T \phi(t, \omega)\, dy(t, \omega)$$

(18b) $$\lim_{K\to\infty} \textstyle\int_0^{T\wedge\tau} \phi^2(t, \omega)\, dt = \int_0^T \phi^2(t, \omega)\, dt.$$

Therefore we can write, under P_y.

(19a) $$\frac{dP_y}{dP_w} = \exp\left\{\int_0^T \phi(t, \omega)\, dy(t, \omega) - \tfrac{1}{2}\int_0^T \phi^2(t, \omega)\, dt\right\} \qquad \text{on} \quad A$$

and we shall define

(19b) $$\frac{dP_y}{dP_w} = 0 \qquad \text{on} \quad \bar{A}.$$

Since $\bar{A}$ has measure zero under P_y, this completely specifies dP_y/dP_w under P_y. But it is necessary to specify dP_y/dP_w a.e. P_w. First, since dP_y/dP_w is a derivative it integrates to 1 under P_w and therefore it must be finite a.s. P_w. On the set A, dP_y/dP_w as given by (19) is finite under P_w, as may be seen by transforming the stochastic integral to a Wiener process by the random time change used above in Lemma 1 and Theorem 1. Therefore (19) completely describes dP_y/dP_w (up to equivalence under P_w). This completes the proof.

THEOREM 3. *Let*

$$y(t, \omega) = \textstyle\int_0^t \phi(s, \omega)\, ds + w(t, \omega)$$

where, $\phi(s, \omega)$ is a measurable function of $\{y(\tau, \omega), 0 \leqq \tau \leqq s\}$, say, $\phi(s, \omega) = \tilde{\phi}(y_0{}^s(\omega), s)$. If

(20) $$\textstyle\int_0^T \tilde{\phi}^2(w_0{}^t, t)\, dt < \infty \quad \text{a.s.}$$

then

(21a)
$$P_w \ll P_y$$

(21b)
$$\frac{dP_w}{dP_y} = \exp\left\{-\int_0^T \phi(t,\omega)\,dy(t,\omega) + \tfrac{1}{2}\int_0^T \phi^2(t,\omega)\,dt\right\} \qquad on \quad A$$
$$= 0 \qquad on \quad \bar{A}.$$

PROOF. Let τ be a stopping time defined exactly as in the proof of Theorem 2. Then, as in Theorem 2, we can conclude (cf. (16a)) that

$$\frac{dP_{w_\tau}}{dP_{y_\tau}} = \exp\left\{-\int_0^{T\wedge\tau} \phi(t,\omega)\,dy(t,\omega) + \tfrac{1}{2}\int_0^{T\wedge\tau} \phi^2(t,\omega)\,dt\right\}.$$

Let f be a generic point in function space. By the previously noted martingale theorem of Doob, we can conclude that $\lim_{K\to\infty} dP_{w_\tau}/dP_{y_\tau}$, say $\lambda(f)$, exists and satisfies the relation

$$P_w(A) = \textstyle\int_A \lambda(f)\,dP_y + P_w(N \cap A),$$

where $N = \{f: \lambda(f) = 0\}$. Now, if $P_w(N) = 0$ then clearly $P_w \ll P_y$ and $\lambda(\omega)$ can be ibentified with dP_w/dP_y. Following the discussion in the proof of Theorem 2, and by our hypothesis (20), it follows that $P_w(N) = 0$. The fact that λ can be expressed as in (21) now follows as in the discussion at the end of the proof of Theorem 2.

NOTE. If the conditions of Theorem 2 *and* Theorem 3 are satisfied we have $P_y \sim P_w$. However, the following examples, derived by L. A. Shepp, (private communication) show that the conditions of any of these theorems without the conditions of the other are not sufficient for equivalence. Let

$$dy(t) = \phi(t)\,dt + dw(t), \qquad 0 \leqq t \leqq 1$$

where

$$\phi(t) = -(1-t)^{-1}, \qquad \text{if} \quad \inf y(s) > -1, \qquad 0 \leqq s \leqq t$$
$$= 0, \qquad \text{otherwise.}$$

Then under measure P_y, $y(\cdot)$ is certain to dip below -1 in the interval $[0 \leqq t \leqq 1]$, but this is not necessarily so under P_w. Therefore, $P_y \ll P_w$ but *not* vice versa. Note also that $\phi^2(\cdot)$ is a.s. integrable on [0, 1] under P_y but not under P_w.

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APPENDIX

Girsanov's Theorem. A basic tool in our paper is a theorem of Girsanov [4] Theorem 1. For convenience, we restate it here along with a useful adjunct to it (part (c) of the statement below).

Let $(\Omega, \mathscr{B}, \mathscr{P})$ be a probability space and let $\mathscr{B}_t$, $t \in [0, T]$, denote a monotone family of sub-σ-algebras of $\mathscr{B}$. The paths $w_0{}^t$ are assumed to be measurable on $\mathscr{B}_t$

for all t in $[0, T]$, and $\mathscr{B}_t$ is assumed to be independent of future increments of $w(t)$. Let $f(t, \omega)$ be a random function on the probability space and let A_1, A_2, A_3 denote the following properties:

A_1: $f(t, \omega)$ is measurable in both variables and for fixed $t, f(t, \omega)$ is $\mathscr{B}_t$ measurable

A_2: $\int_0^T |f(t,\omega)|\, dt < \infty$ a.s.

A_3: $\int_0^T |f(t,\omega)|^2\, dt < \infty$ a.s.

GIRSANOV'S THEOREM. *Let*

$$y(t,\omega) = \int_0^t \Psi(s,\omega)\, ds + \int_0^t \Phi(s,\omega)\, dw(s,\omega) \tag{A.1}$$

where $\Psi(s, \omega)$ satisfies A_1 and A_2 and $\Phi(s, \omega)$ satisfies A_1 and A_3. Let

$$\alpha(\omega) = \exp \int_0^T \phi(s,\omega)\, dw(s,\omega) - \tfrac{1}{2}\int_0^T \phi^2(s,\omega)\, ds \tag{A.2}$$

where $\phi(s, \omega)$ satisfies A_1 and A_3. Assume that

$$\int \alpha(\omega)\, d\mathscr{P}(\omega) = 1 \tag{A.3}$$

and let

$$\mathscr{P}_0(\omega) = \alpha(\omega)\, d\mathscr{P}(\omega). \tag{A.4}$$

Then, under $\mathscr{P}_0$

(a) $\tilde{w}(t) = w(t) - \int_0^t \phi(s,\omega)\, ds$ *is Wiener, and*

(b) $y(t) = \int_0^t (\Psi(s,\omega) + \phi(s,\omega)\Phi(s,\omega))\, ds + \int_0^t \Phi(s,\omega)\, d\tilde{w}(s,\omega)$. *Furthermore,*

(c) *The measures $\mathscr{P}$ and $\mathscr{P}_0$ on $(\Omega, \mathscr{B})$ are equivalent.*

REMARKS. (a) and (b) are direct restatements of Girsanov's theorem. To prove (c) note that by (A.4), $\mathscr{P}_0 \ll \mathscr{P}$.

To prove $\mathscr{P} \ll \mathscr{P}_0$, assume first that this is false. Then there exists a set A $\mathscr{B}$ such htat $\mathscr{P}_0(A) = 0, \mathscr{P}(A) = \varepsilon > 0$. Therefore by (A.4) we must have $\alpha(\omega) = 0, \omega \in A$. Let $\tau(t, \omega)$ be defined by

$$\tau = \int_0^t \phi^2(s,\omega)\, ds$$

and set

$$\psi(\tau,\omega) = \int_0^{T \wedge \tau} \phi(s,\omega)\, dw(s,\omega).$$

Then $\psi(\tau, \omega)$ is a Brownian motion stopped at $\tau_0(\omega) = \int_0^T \phi^2(s,\omega)\, ds$ (McKean [8] Section 2.5) and since $\phi(s, \omega)$ satisfies A_3, τ_0 is finite,

$$\alpha(\omega) = \exp\left[w_1(\tau_0(\omega)) - \tfrac{1}{2}\tau_0{}^2(\omega)\right]$$

where w_1 is a Brownian motion on $(\Omega, \mathscr{B}, \mathscr{P})$. By the continuity of the Brownian motion $\mathscr{P}(\omega\colon \alpha(\omega) = 0)$. Hence $\mathscr{P}(A) = 0$, which proves (c) by contradiction.

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